Section 2

MECHANICS

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CONTENTS

2a. Fundamental Concepts of Mechanics. Units and Conversion Factors

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Symbols

$\mathbf{a}$	vector acceleration
a_x, a_y, a_z	components of acceleration in rectangular coordinates
a_r, a_θ, a_z	components of acceleration in cylindrical coordinates
a_r, a_θ, a_ϕ	components of acceleration in spherical coordinates
$\mathbf{C}$	elastic coefficient matrix
C_{ij}	elastic coefficient
$\mathbf{D}$	strain tensor
$\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$	base vectors of coordinate system
e_{xx}, e_{xy}, etc.	components of strain tensor
E	Young's modulus
$\mathbf{f}_c$	inertial force (centrifugal or Coriolis)
$\mathbf{F}$	force vector
g	acceleration of gravity
G	shear modulus
$\mathcal{g}$	impulse
I_{xx}, I_{yy}, I_{zz}	moment of inertia
I_{xy}, I_{yz}, I_{xz}	products of inertia
k	bulk modulus
K	kinetic energy
l	depth of liquid
$\mathbf{L}$	moment of momentum
m, M	mass
N	number of particles
p	pressure
r	radius vector
$\mathbf{r}$	position vector
$\mathbf{R}$	position vector of center of mass
$\mathcal{R}$	radius of gyration

s	vector displacement
S	stress tensor
S_{ij}	elastic constant
t	time
T	torque
U	total mechanical energy
v	velocity vector
v_x, v_y, v_z	components of velocity in rectangular coordinates
v_r, v_θ, v_z	components of velocity in cylindrical coordinates
v_r, v_θ, v_ϕ	components of velocity in spherical coordinates
V	volume, potential energy, wave velocity
W	work
x, y, z	rectangular coordinates
X_x, X_y, etc.	components of stress tensor
α	angular acceleration
γ	surface tension
Δ	deformation displacement of a deformable medium
η	viscosity
θ	colatitude in spherical coordinates, azimuth in cylindrical coordinates
Θ	total dilatation
λ	Lamé elastic constant, wavelength in harmonic wave
ξ, η, ζ	components of deformation, displacement of a deformable medium
ρ	density
σ	Poisson's ratio
ϕ	longitude in spherical coordinates, velocity potential
ω	angular velocity
$\omega_x, \omega_y, \omega_z$	components of angular velocity in rectangular coordinates

2a-1. Newtonian Concepts of Mechanics. The science of mechanics deals with the motion of material bodies, which ideally can be considered as made up of point particles. In order to describe the motion of a particle three concepts are needed: *a frame of reference*, *distance*, and *time interval*. These concepts are left undefined as intuitive concepts with sufficiently universal meanings. Distance and time intervals are measured in terms of standards which have a wide range of acceptance, such as the *standard meter* and the *sidereal day*. (The important systems of units are tabulated in Secs. 2a-8 and 2a-9.) The frame of reference consists of a *reference point* and a *coordinate system* (whose origin may be at the reference point); a *reference* event is necessary as well as a frame of reference.

The position of a particle may be specified with respect to the reference point by considering a rectangular coordinate system whose origin is at the reference point. The position of any particle is then given in terms of the distances along the coordinate axes from the origin to the projection on these axes of the point representing the position of the particle.

The location of an event in time, or the time of an event, similarly is expressed in terms of the time interval with respect to the reference event. The terms "time interval" and "time" are usually used interchangeably.

The above concepts are usually referred to as Newtonian; they suffice for classical mechanics.

2a-2. Kinematics—The Space-Time Relationships in the Motions of Point Particles. *Velocity.* Velocity is the rate of change of position with respect to time. Two types of velocity are commonly used, instantaneous and average. Instantaneous velocity is the time rate of change of position calculated pointwise, thus being a derivative. Average velocity is the time rate of change of position calculated as the quotient of a finite distance and the corresponding finite time interval.

Velocity is a vector with components which depend in general on the coordinate

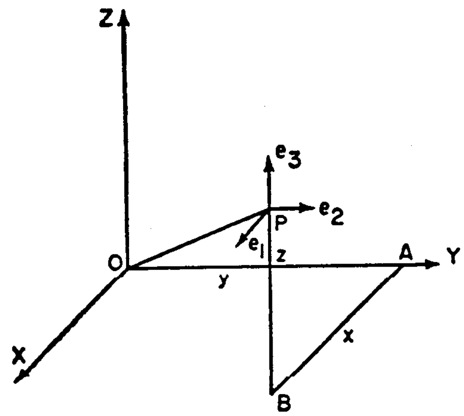

FIG. 2a-1. Base vectors in rectangular coordinates.

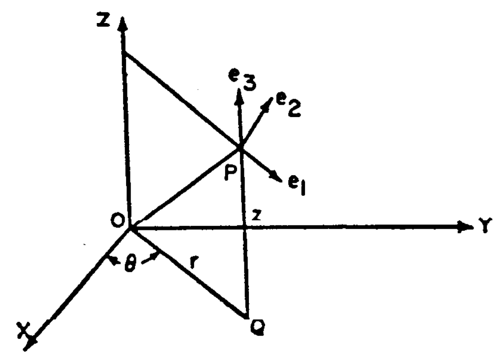

FIG. 2a-2. Base vectors in cylindrical coordinates.

system used. If $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ are base vectors of the coordinate system under consideration, then, for three commonly used systems:

1. Rectangular coordinates (cf. Fig. 2a-1):

$$\mathbf{v} = \mathbf{e}_1 v_x + \mathbf{e}_2 v_y + \mathbf{e}_3 v_z = \mathbf{e}_1 \frac{dx}{dt} + \mathbf{e}_2 \frac{dy}{dt} + \mathbf{e}_3 \frac{dz}{dt} \tag{2a-1}$$

2. Cylindrical coordinates (cf. Fig. 2a-2):

$$\mathbf{v} = \mathbf{e}_1 v_r + \mathbf{e}_2 v_\theta + \mathbf{e}_3 v_z = \mathbf{e}_1 \frac{dr}{dt} + \mathbf{e}_2 r \frac{d\theta}{dt} + \mathbf{e}_3 \frac{dz}{dt} \tag{2a-2}$$

3. Spherical coordinates (cf. Fig. 2a-3):

$$\mathbf{v} = \mathbf{e}_1 v_r + \mathbf{e}_2 v_\theta + \mathbf{e}_3 v_\phi = \mathbf{e}_1 \frac{dr}{dt} + \mathbf{e}_2 r \frac{d\theta}{dt} + \mathbf{e}_3 r \sin\theta \frac{d\phi}{dt} \tag{2a-3}$$

Acceleration. Acceleration is the rate of change of velocity with respect to time. Instantaneous and average acceleration may be defined analogously to instantaneous and average ~~velocities; however, instantaneous acceleration,~~ or the time derivative of velocity (or equivalently the second time derivative of position), is the more commonly used quantity. Acceleration is a vector with components which depend in general on the coordinate system used. If $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ are the unit base vectors of the coordinate system under consideration, then for the commonly used systems:

1. Rectangular coordinates:

$$\mathbf{a} = \mathbf{e}_1 a_x + \mathbf{e}_2 a_y + \mathbf{e}_3 a_z = \mathbf{e}_1 \frac{d^2x}{dt^2} + \mathbf{e}_2 \frac{d^2y}{dt^2} + \mathbf{e}_3 \frac{d^2z}{dt^2} \tag{2a-4}$$

2. Cylindrical coordinates:

$$\mathbf{a} = \mathbf{e}_1 a_r + \mathbf{e}_2 a_\theta + \mathbf{e}_3 a_z = \mathbf{e}_1 \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right] + \mathbf{e}_2 \left[r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right] + \mathbf{e}_3 \frac{d^2z}{dt^2} \tag{2a-5}$$

3. Spherical coordinates:

$$\begin{aligned}\mathbf{a} = \mathbf{e}_1 a_r + \mathbf{e}_2 a_\theta + \mathbf{e}_3 a_\phi = {} & \mathbf{e}_1 \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 - r\sin^2\theta\left(\frac{d\phi}{dt}\right)^2\right] \\ & + \mathbf{e}_2 \left[r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt} - r\sin\theta\cos\theta\left(\frac{d\phi}{dt}\right)^2\right] \\ & + \mathbf{e}_3 \left[r\sin\theta\frac{d^2\phi}{dt^2} + 2r\cos\theta\frac{d\theta}{dt}\frac{d\phi}{dt} + 2\sin\theta\frac{dr}{dt}\frac{d\phi}{dt}\right]\end{aligned} \tag{2a-6}$$

2a-3. Newtonian Dynamics of Particles—Relationship of the Motion of Particles to the Forces Acting upon Them. *Inertial Frames of Reference.* Not all frames of reference are equally useful in describing the motion of a body; of all possible frames there is a set, called "inertial frames of reference," in which particularly simple laws describe the motion of a particle. An intuitive definition of an inertial frame of reference regards such a frame as being one which is "embedded in space" with respect to an observer; more exactly, an inertial frame of reference is one in which an isolated body moves with constant velocity. It may be easily seen from Newton's second law of motion (below) that any inertial frame is transformed to any other by uniform motion in a straight line.

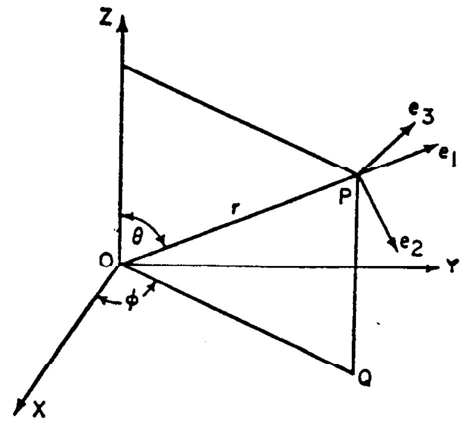

FIG. 2a-3. Base vectors in spherical coordinates.

Definitions of Useful Concepts. MASS. The Newtonian mass of a particle may be defined by considering the acceleration associated with the mutual interaction of this particle with a second, a test particle, when the two form an isolated system. The mass of the first particle is defined as a constant times the ratio of the magnitude of the accelerations of the second and first particles, respectively. The constant depends only on the choice of the second particle, and by mutual consent the constant may arbitrarily be set equal to unity. The second particle then represents the standard unit of mass, and the mass of the first is thus determined by the above-mentioned ratio of accelerations. This method, although having the advantage of yielding an unequivocal definition of mass, is not usually a practicable one and is replaced by other methods (e.g., the balance) in actual determinations. Implicit in this definition is the assumption of additivity of masses, thus enabling the mass of a finite body, as an aggregate of particles, to be determined uniquely.

DENSITY. The density of a substance is defined as the mass per unit volume of the substance, and is calculated from the formula

$$\rho = \frac{m}{V} \tag{2a-7}$$

where ρ is the density, m is the mass, V is the volume occupied by mass m. Density is thus a measure of the volume concentration of mass.

MOMENTUM. The momentum of a particle is defined as the product of its mass and velocity and is therefore a vector quantity.

KINETIC ENERGY. The kinetic energy of a particle is defined as one-half the product of its mass and the square of its velocity, and is a scalar.

FORCE. The force acting upon a particle is assumed as the cause of the acceleration of the particle. It may be defined as that vector function which, in magnitude and direction, equals the time rate of change of momentum of the particle. Thus

$$\mathbf{F} = \frac{d}{dt}(m\mathbf{v}) \tag{2a-8}$$

where $\mathbf{F}$ represents the force, and m and $\mathbf{v}$ are the particle mass and velocity, respectively.

This force depends in general not only on the particle in question but also on the nature of other particles in, and properties of, the system of which the original particle is a part, the mutual separations and velocities of the particles and possibly of the time. Although force has been defined so far only for a particle, the definition may be extended to finite distributions of matter by considering infinitesimal portions as particles and integrating.

Newton's Laws. The dynamics of particles situated in an inertial frame of reference is governed by Newton's three laws of motion. The extension of these laws to a noninertial frame is, in principle, immediately forthcoming by considerations of the accelerations of the noninertial frame with respect to an inertial one; thus Newton's laws govern the dynamics of particles when Newtonian concepts are valid. Newton's laws are as follows:

1. A particle, not under the action of a force, will maintain its velocity unchanged in magnitude and direction.
2. A force acting on a particle causes a change of momentum of the particle, the rate of change of momentum being vectorially equal to the force.
3. If one particle exerts a force on a second, then the second exerts a force, equal in magnitude but opposite in direction, on the first.

Statics. The branch of dynamics which deals with particles undergoing no acceleration is termed "statics." We see from Newton's second law that, in this case,

$$\mathbf{F} = 0 \tag{2a-9}$$

where $\mathbf{F}$ refers to the vectorial sum of all the forces acting on the particle.

Noninertial Dynamics. At times it is convenient to consider the dynamics of a particle in a noninertial frame, e.g., motion relative to rotating or other moving axes. There will then be an apparent force acting on the particle which is the difference between the Newtonian force (that acting in the inertial system) and the inertial force $m\mathbf{a}_0$, where $\mathbf{a}_0$ is the acceleration of the noninertial system with respect to the inertial frame. Symbolically, $\mathbf{F}_d = \mathbf{F} - m\mathbf{a}_0$, where $\mathbf{F}_d$ is the apparent force, and $\mathbf{F}$ is the Newtonian force. We can set $\mathbf{F}_d = m\mathbf{a}_d$, where $\mathbf{a}_d$ is the acceleration of the particle with respect to the noninertial frame.

D'ALEMBERT'S PRINCIPLE. Often it is advantageous to choose a noninertial system such that $\mathbf{F}_d = 0$; the dynamical problem in the noninertial system then reduces to a statical one. That such a noninertial system can be chosen is one statement of D'Alembert's principle.

INERTIAL FORCES—CENTRIPETAL AND CORIOLIS FORCES. The difference between the Newtonian force and the apparent noninertial force can be termed the "inertial force." Centripetal and Coriolis forces are two commonly occurring examples of such inertial forces. The centripetal force is given for a particle of unit mass by

$$\mathbf{f}_c = \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \tag{2a-10}$$

where $\boldsymbol{\omega}$ is the instantaneous angular velocity of the moving axes about the axis of rotation and $\mathbf{r}$ is the position vector of the particle with respect to the moving axes. The Coriolis force is given for a particle of unit mass by

$$\mathbf{f}_c = 2\boldsymbol{\omega} \times \mathbf{v} \tag{2a-11}$$

where $\boldsymbol{\omega}$ has the same meaning as above and $\mathbf{v}$ is the apparent velocity of the particle with respect to the moving axes.

Conservation of Momentum. IMPULSE-MOMENTUM THEOREM. The impulse of a force acting between times t_0 and t_1 is defined by

$$\mathcal{I} = \int_{t_0}^{t_1} \mathbf{F}\, dt \tag{2a-12}$$

From Newton's second law, the impulse of the total force acting on a particle during some time interval is equal to the change in the momentum of the particle during the time interval, i.e.,

$$\mathcal{I} = m\mathbf{v}_1 - m\mathbf{v}_0 \tag{2a-13}$$

CONSERVATION OF MOMENTUM. When the total force acting upon a particle is zero, the momentum of the particle is a constant; this follows directly from the impulse-momentum theorem.

Conservation of Energy. WORK-ENERGY THEOREM. The work done on a particle by a force acting during the displacement of a particle from position P_0 to position P_1 is defined as

$$W = \int_{P_0}^{P_1} \mathbf{F} \cdot d\mathbf{s} \tag{2a-14}$$

where $d\mathbf{s}$ is an infinitesimal displacement along the path of the particle. From Newton's second law the work done by the total force acting on a particle during some displacement of the particle is equal to the change in kinetic energy of the particle:

$$W = \tfrac{1}{2}mv_1^2 - \tfrac{1}{2}mv_0^2 \tag{2a-15}$$

POTENTIAL ENERGY. If the work done by a force acting on a particle does not depend upon the path of the particle, but only on the initial and end points of its motion, we call the force a "conservative force." The condition for a force to be conservative is that its curl shall vanish; i.e.,

$$\nabla \times \mathbf{F} = 0 \tag{2a-16}$$

If the force is conservative, we may define a potential-energy function of position V such that

$$\mathbf{F} = -\nabla V \tag{2a-17}$$

CONSERVATION OF ENERGY. If the total force acting upon a particle is conservative, the sum of the kinetic and potential energies is a constant; this follows from the work-energy theorem and the definition of the potential energy:

$$\tfrac{1}{2}mv^2 + V(x,y,z) = U \tag{2a-18}$$

where U, the total mechanical energy, is a constant.

2a-4. Dynamics of Systems of Particles. In examining the dynamics of a system of point masses, consider N point particles, each of mass m_i, where $i = 1, 2, \ldots, N$. The total force acting on m_i due to m_j is $\mathbf{F}_{ij}$; in addition, a total external force $\mathbf{F}_i$ acts on m_i. At any time t, m_i has a position $\mathbf{r}_i$, a velocity $\dot{\mathbf{r}}_i$, and an acceleration $\ddot{\mathbf{r}}_i$, all relative to some inertial frame. (The dots denote differentiation with respect to time.)

Definition of Useful Concepts. CENTER OF MASS. The position of the center of mass of the above system is given by

$$\mathbf{R} = \frac{\sum_{i=1}^{N} m_i\mathbf{r}_i}{\sum_{i=1}^{N} m_i} \tag{2a-19}$$

MOMENT OF MOMENTUM. The moment of momentum of the ith particle in the above system is defined as

$$\mathbf{L}_i = \mathbf{r}_i \times m_i\dot{\mathbf{r}}_i \tag{2a-20}$$

The total moment of momentum of the system is

$$\mathbf{L} = \sum_{i=1}^{N} \mathbf{L}_i = \sum_{i=1}^{N} m_i(\mathbf{r}_i \times \dot{\mathbf{r}}_i) \tag{2a-21}$$

If the collection of particles is a rigid body, the moment of momentum is called the "angular momentum" (cf. Sec. 2a-5).

TORQUE (MOMENT OF FORCE). The torque due to a force $\mathbf{F}_i$ acting on the ith particle in the above system is defined as

$$\mathbf{T}_i = \mathbf{r}_i \times \mathbf{F}_i \tag{2a-22}$$

The total torque acting on the system is $\mathbf{T} = \sum_{i=1}^{N} \mathbf{T}_i$. (The force $\mathbf{F}_i$ includes forces externally applied to the particle, as well as internal forces of interaction among the particles of the system.)

Application of Newton's Laws. We may apply Newton's second law to each particle of the system, and obtain

$$m_i\ddot{\mathbf{r}}_i = \mathbf{F}_i{}^i + \mathbf{F}_i{}^e \tag{2a-23}$$

where $\mathbf{F}_i{}^i = \Sigma_{j \neq 1}\mathbf{F}_{ij}$ is the total internal force acting on m_i (due to all other particles), and $\mathbf{F}_i{}^e$ is the external force on the ith particle.

If we sum over all particles of the system, we obtain, by use of Newton's third law,

$$\sum_{i=1}^{N} \mathbf{F}_i{}^i = 0 \tag{2a-24}$$

MOTION OF THE CENTER OF MASS. The analog of Newton's second law for the entire system is therefore

$$M\ddot{\mathbf{R}} = \sum_{i=1}^{N} \mathbf{F}_i{}^e \tag{2a-25}$$

where $M = \sum_{i=1}^{N} m_i$ is the total mass of the system, $\ddot{\mathbf{R}}$ is the acceleration of the center of mass of the system, and $\Sigma_i\mathbf{F}_i{}^e$ is the total external force.

MOMENT OF MOMENTUM AND TORQUE. By forming the cross product of both sides of Eq. (2a-23) with $\mathbf{r}_i$ and summing over all particles we can show that

$$\frac{d}{dt}\sum [\mathbf{r}_i \times (m_i\dot{\mathbf{r}}_i)] = \sum \mathbf{T}_i{}^e = \mathbf{T}^e \tag{2a-26}$$

provided that the internal force $\mathbf{F}_{ij}$ acts along the straight line connecting the particles i and j in each case.

In particular, if $\mathbf{r}_{ic}$ is the position of the ith particle with respect to the center of mass, so that

$$\mathbf{r}_{ic} = \mathbf{r}_i - \mathbf{R}$$

it follows from Eq. (2a-26) that

$$\frac{d}{dt}\sum_{i=1}^{N} \mathbf{r}_{ic} \times (m_i\dot{\mathbf{r}}_{ic}) = \sum_{i=1}^{N} \mathbf{r}_{ic} \times \mathbf{F}^e \tag{2a-27}$$

That is, the time rate of change of the moment of momentum is equal to the total external torque when both are taken with respect to the center of mass. The above equation is also true if the center of mass is replaced by any point moving with the velocity of the center of mass, which may, of course, also be at rest.

Conservation of Momentum. It follows from Eqs. (2a-25) and (2a-26) that:

1. If the total external force is zero, the linear momentum of the center of mass is constant.

2. If the total external torque about a fixed point, or one moving with velocity of the center of mass, is zero, the moment of momentum about that point is constant.

Conservation of Energy. WORK-ENERGY THEOREM. The total work done by the external and internal forces acting on the system is equal to the change in the total kinetic energy of the system (the sum of the kinetic energies of all particles)

$$\frac{1}{2} \sum_{i=1}^{N} m_i(\mathbf{v}_i''^2 - \mathbf{v}_i'^2) = \sum_{1=1}^{N} \int_{r_i'}^{r_i''} (\mathbf{F}_i^e + \mathbf{F}_i^i) \cdot d\mathbf{r}_i \qquad (2a\text{-}28)$$

where $\mathbf{v}_i'$, $\mathbf{v}_i''$ are the velocities of the ith particle at position $\mathbf{r}_i'$ and $\mathbf{r}_i''$, respectively, and $\mathbf{F}_i^i = \Sigma_{j\neq 1}\mathbf{F}_{ij}$ is the total internal force acting on the ith particle.

CONSERVATION OF ENERGY. If the internal and external forces are conservative, so that they can be derived from potentials,

$$\mathbf{F}_i^i = -\nabla V_i^i \quad \text{and} \quad \mathbf{F}_i^e = -\nabla V_i^e \qquad (2a\text{-}29)$$

then the sum of the kinetic and potential energies of all the particles is a constant

$$\sum_{i=1}^{N} (\tfrac{1}{2}m_i v_i^2 + V_i^i + V_i^e) = U \qquad (2a\text{-}30)$$

where U is the total energy of the system.

2a-5. Dynamics of Rigid Bodies. *Definitions of Kinematical Concepts.* A rigid body is an aggregate of particles the distance between any two of which remains constant. The position of a rigid body in any frame of reference is completely determined by fixing the position of three noncollinear points. This means that the number of degrees of freedom of the rigid body is six. There are two principal types of motion of a rigid body: (1) *translation*, in which all particles move with the same velocity and acceleration in parallel paths, and (2) *rotation*, in which some point or line of points (axis) remains fixed in space. Every motion of a rigid body can be considered as a combination of translations and rotations.

The instantaneous angular velocity $\boldsymbol{\omega}$ is the primary quantity descriptive of the kinematics of a rigid body. This is a vector lying along the instantaneous axis of rotation and having the magnitude such that its cross product with the position vector $\mathbf{r}_P$ of any point P of the rigid body relative to an origin on the axis yields the velocity of the point P. Symbolically

$$\mathbf{v}_P = \dot{\mathbf{r}}_P = \boldsymbol{\omega} \times \mathbf{r}_P \qquad (2a\text{-}31)$$

The angular velocity can always be resolved into rectangular components ω_x, ω_y, ω_z, i.e.,

$$\boldsymbol{\omega} = \mathbf{i}\omega_x + \mathbf{j}\omega_y + \mathbf{k}\omega_z \qquad (2a\text{-}32)$$

Angular acceleration is the time rate of change of angular velocity, i.e. (to use the dot notation),

$$\boldsymbol{\alpha} = \dot{\boldsymbol{\omega}} \qquad (2a\text{-}33)$$

Dynamical Concepts and Equations of Motion. The total moment of momentum **L** of the rigid body with respect to some fixed origin of coordinates either inside or outside the body [cf. Eq. (2a-20)] is called the angular momentum of the rigid body about the origin. By expansion of the summand in Eq. (2a-21) after employing Eq. (2a-31) there results

$$\begin{aligned} \mathbf{L} &= \mathbf{i}(\omega_x I_{xx} - \omega_y I_{xy} - \omega_z I_{xz}) \\ &\quad + \mathbf{j}(-\omega_x I_{yx} + \omega_y I_{yy} - \omega_z I_{yz}) \\ &\quad + \mathbf{k}(-\omega_x I_{zx} - \omega_y I_{zy} + \omega_z I_{zz}) \end{aligned} \tag{2a-34}$$

where I_{xx}, I_{yy}, I_{zz} are called the "moments of inertia" of the rigid body about the x, y, z axes, respectively, and I_{xy}, I_{yz}, I_{zx}, etc., are called "products of inertia." We have

$$\begin{aligned} I_{xx} &= \Sigma m_i(y_i^2 + z_i^2) \qquad \text{etc.} \\ I_{xy} &= \Sigma m_i x_i y_i \qquad \text{etc.} \end{aligned} \tag{2a-35}$$

By proper choice of axes (called "principal" axes) the products of inertia can be made to vanish. If we write

$$I_{xx} = MR^2 \tag{2a-36}$$

where

$$R^2 = \frac{\Sigma m_i(y_i^2 + z_i^2)}{M} \tag{2a-37}$$

and M is the total mass of the rigid body, R is termed the "radius of gyration" about the x axis.

The fundamental equation of motion (Newton's second law) of the rigid body about a *fixed* origin is

$$\dot{\mathbf{L}} = \mathbf{T} \tag{2a-38}$$

where **T** is the total torque about the instantaneous axis through the fixed origin. If the fixed origin is chosen as the center of mass, the total motion is obtained by superposing the translational motion of the center of mass on the rotational motion about the center of mass.

Static Equilibrium. A rigid body is in translational equilibrium if its center of mass moves with constant velocity in an inertial frame. It is in rotational equilibrium about any point if the resultant torque about the point vanishes. This means $\dot{\mathbf{L}} = 0$ and corresponds to conservation of angular momentum. The behavior of a rigid body under these conditions is the subject matter of rigid statics.

Moving Axes. Euler's Equation. For axes fixed in space, ω and the moments and products of inertia in general change with time as the rigid body moves. Simplification often results by using axes *fixed in the body*, since then I_{xx}, I_{xy}, etc., remain constant. Then, for motion about a fixed point the axes rotate, and we have

$$\dot{\mathbf{L}} = \mathbf{i}\dot{L}_x + \mathbf{j}\dot{L}_y + \mathbf{k}\dot{L}_z + \boldsymbol{\omega} \times \mathbf{L} \tag{2a-39}$$

where L_x, L_y, L_z are the components of angular momentum about the moving axes and $\boldsymbol{\omega}$ is the instantaneous angular velocity of the body about the instantaneous axis of rotation. If we choose principal axes the equation of motion (2a-38) becomes

$$\begin{aligned} \mathbf{T} &= \mathbf{i}[I_{xx}\dot{\omega}_x + (I_{zz} - I_{yy})\omega_y\omega_z] \\ &\quad + \mathbf{j}[I_{yy}\dot{\omega}_y + (I_{xx} - I_{zz})\omega_z\omega_x] \\ &\quad + \mathbf{k}[I_{zz}\dot{\omega}_z + (I_{yy} - I_{xx})\omega_x\omega_y] \end{aligned} \tag{2a-40}$$

This is *Euler's equation.* The three component equations to which it reduces are usually called Euler's equations.

Kinetic Energy. Work-Energy Theorem. If a rigid body has one point fixed in space and the angular momentum about this point is **L**, while the angular velocity about an instantaneous axis through the point is $\boldsymbol{\omega}$, the kinetic energy of rotational motion is

$$K = \tfrac{1}{2}\boldsymbol{\omega} \cdot \mathbf{L} \tag{2a 41}$$

The work done by the resultant torque T about the fixed point in time dt is

$$dW = \mathbf{T} \cdot \boldsymbol{\omega}\, dt \tag{2a-42}$$

measured with respect to axes fixed in the body. Since

$$\boldsymbol{\omega} \cdot d\mathbf{L} = dK \tag{2a-43}$$

it follows that the work done by the resultant torque in any time interval is equal to the change in kinetic energy of rotation during this same interval.

Total Energy. The total kinetic energy of a rigid body is the sum of the kinetic energy of translation of the center of mass (assuming all the mass to be concentrated there) and the kinetic energy of rotation about the center of mass. The total potential energy is the sum of the potential energy of the center of mass (with all the mass concentrated there) due to the external forces acting on the body and the potential energy of all the particles of the body due to the internal forces of cohesion that hold the body together. If the body remains really rigid throughout its motion, the last-named potential energy remains constant. With this understanding, the law of conservation of energy of a rigid body is phrased as precisely as that in the case of a particle.

2a-6. Dynamics of Deformable Media. *General Concepts of Strain and Stress.* Whenever an extended medium moves in such a way that the distance between any two particles constituting the medium changes, the medium is said to be *deformed.* Deformations are of two general types: (1) dilatational or extensional, in which a change in the density of the medium takes place (change in the size, if the medium is finite) and (2) shear, in which a change in the shape alone takes place. The corresponding *fractional* deformations (nondimensional quantities) are termed *strains.* Thus the *dilatational strain* is the negative of the change in density divided by the mean density. The *extensional strain* (in the case of a rod, string, or other linear medium) is the change in length divided by the mean length. The *shear strain* is the difference in displacement of two parallel planes in the medium divided by the perpendicular distance between them.

When a medium is deformed by the application of external forces, the dynamics of the deformation is best described in terms of internal *stresses* which are assumed to change with the deformation. A stress is a force per unit area with which the part of the medium on one side of an imaginary surface acts on the part on the other side. If the force is normal to the surface, the stress is dilatational; if the force is parallel to the surface, the stress is a shear. The stresses associated with deformations are strictly *excess* stresses (i.e., the change in stress produced by the application of the external force). The adjective is normally omitted.

Elastic Media. Hooke's Law. If when the deforming forces are removed a medium reverts to its original condition, it is said to be *elastic.* In such media the ratio of stress to strain is approximately a constant for a certain range of stress variation. This is Hooke's law. For all solid media the imposition of a sufficiently large deforming force leads to a breakdown of this linear relation; i.e., they possess an elastic limit (cf. Sec. 2e). Indeed even larger deforming forces may cause the solid to flow (strain dependent on time) and it becomes *plastic.* Even elastic substances do not always return *immediately* to their original condition after the removal of the deforming force (elastic lag or relaxation). Fluids can experience change of state under sufficiently high stresses.

For an elastic medium for which Hooke's law holds it is possible to define elastic moduli, i.e., ratios of stress to strain. Thus,

$$\frac{\text{Compressional stress}}{\text{Volume strain}} = k = \text{bulk modulus or modulus of volume elasticity}$$
$$\frac{\text{Tensile stress}}{\text{Linear strain}} = E = \text{Young's modulus}$$
$$\frac{\text{Shearing stress}}{\text{Shear strain}} = G = \text{shear modulus or rigidity}$$

The deformation of a homogeneous isotropic elastic medium can be completely described in terms of these three moduli. A fourth, Poisson's ratio σ, is usually added. This is the reciprocal of the ratio of linear extensional strain in a wire or rod to the concomitant lateral contractional strain. The following relations hold among the moduli:

$$E = 3k(1 - 2\sigma) = 2G(1 + \sigma)$$
$$E = \frac{9kG}{G + 3k} \tag{2a-44}$$

Evidently for such media

$$-1 < \sigma < \tfrac{1}{2} \tag{2a-45}$$

General Stress and Strain Expressions for an Arbitrary Medium. If the displacement from its equilibrium position of any particle of a deformable medium is denoted by the vector

$$\mathbf{\Delta} = \mathbf{i}\xi + \mathbf{j}\eta + \mathbf{k}\zeta \tag{2a-46}$$

where the displacement components ξ, η, ζ are in general functions of both space and time, the effective *strain* is denoted by the covariant tensor of the second order written in matrix form as follows:

$$\mathbf{D} = \begin{Vmatrix} \partial\xi/\partial x, \ \tfrac{1}{2}(\partial\eta/\partial x + \partial\xi/\partial y), \ \tfrac{1}{2}(\partial\zeta/\partial x + \partial\xi/\partial z) \\ \tfrac{1}{2}(\partial\eta/\partial x + \partial\xi/\partial y), \ \partial\eta/\partial y, \ \tfrac{1}{2}(\partial\eta/\partial z + \partial\zeta/\partial y) \\ \tfrac{1}{2}(\partial\zeta/\partial x + \partial\xi/\partial z), \ \tfrac{1}{2}(\partial\zeta/\partial y + \partial\eta/\partial z), \ \partial\zeta/\partial z \end{Vmatrix} \tag{2a-47}$$

This is often written in the abbreviated symbolic form

$$D = \begin{Vmatrix} e_{xx}, \ \tfrac{1}{2}e_{xy}, \ \tfrac{1}{2}e_{xz} \\ \tfrac{1}{2}e_{xy}, \ e_{yy}, \ \tfrac{1}{2}e_{yz} \\ \tfrac{1}{2}e_{xz}, \ \tfrac{1}{2}e_{yz}, \ e_{zz} \end{Vmatrix} \tag{2a-48}$$

The diagonal elements in this matrix are dilatational strain components, whereas the nondiagonal elements are shear strain components.

The total stress in a deformable medium is most adequately expressed in terms of the stress tensor S which is represented by the following matrix:

$$\mathsf{S} = \begin{Vmatrix} X_x, \ X_y, \ X_z \\ Y_x, \ Y_y, \ Y_z \\ Z_x, \ Z_y, \ Z_z \end{Vmatrix} \tag{2a-49}$$

Here X_x is the tensile stress in the x direction on the surface normal to the x axis, X_y is the shear stress in the y direction on the surface normal to the x axis; X_z is the shear stress in the z direction on the surface normal to the x axis, etc. It should be noted that the stress tensor is symmetrical, i.e., $X_y = Y_x$, etc. The same is true of the strain tensor ($e_{zx} = e_{xz}$, etc.).

Hooke's Law in Tensor Form for a Homogeneous, Isotropic Elastic Medium. For this case Hooke's law takes the form

$$\mathbf{S} = 2G\mathbf{D} + \lambda\mathbf{D}' \qquad \text{(2a-50)}$$

where G is still the shear modulus, and $\lambda = k - 2G/3$. $\mathbf{D}'$ is the diagonal tensor

$$\mathbf{D}' = \begin{Vmatrix} \Theta & 0 & 0 \\ 0 & \Theta & 0 \\ 0 & 0 & \Theta \end{Vmatrix} \qquad \text{(2a-51)}$$

with

$$\Theta = e_{xx} + e_{yy} + e_{zz} \qquad \text{(2a-52)}$$

Hooke's Law for an Arbitrary Crystalline Medium. If the medium is a crystal with different properties in different directions, Hooke's law takes the form of the following linear equations expressing the strain components in terms of the stress components.

$$\begin{aligned} e_{xx} &= S_{11}X_x + S_{12}Y_y + S_{13}Z_z + S_{14}Y_z + S_{15}Z_x + S_{16}X_y \\ e_{yy} &= S_{21}X_x + S_{22}Y_y + S_{23}Z_z + S_{24}Y_z + S_{25}Z_x + S_{26}X_y \\ e_{zz} &= S_{31}X_x + S_{32}Y_y + S_{33}Z_z + S_{34}Y_z + S_{35}Z_x + S_{36}X_y \\ e_{yz} &= S_{41}X_x + S_{42}Y_y + S_{43}Z_z + S_{44}Y_z + S_{45}Z_x + S_{46}X_y \\ e_{zx} &= S_{51}X_x + S_{52}Y_y + S_{53}Z_z + S_{54}Y_z + S_{55}Z_x + S_{56}X_y \\ e_{xy} &= S_{61}X_x + S_{62}Y_y + S_{63}Z_z + S_{64}Y_z + S_{65}Z_x + S_{66}X_y \end{aligned} \qquad \text{(2a-53)}$$

The 36 coefficients $S_{11}, S_{12}, \ldots, S_{ij}, \ldots, S_{66}$ are called the "elastic constants." If the above linear equations are solved for the stress components in terms of the strain components, the corresponding coefficients C_{ij} are called "elastic coefficients." It can be shown that, for any ij, $C_{ij} = C_{ji}$ and $S_{ij} = S_{ji}$.

For a cubic crystal the elastic coefficient matrix reduces to

$$\mathbf{C} = \begin{Vmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{12} & C_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{44} \end{Vmatrix} \qquad \text{(2a-54)}$$

Moreover for a cubic crystal $C_{44} = 1/S_{44}$. The bulk modulus in this case is given by

$$k = \frac{C_{11} + 2C_{12}}{3} \qquad \text{(2a-55)}$$

Equation of Motion of a Deformed Homogeneous Isotropic Elastic Medium. The equation of motion of such a medium of density ρ, in which the displacement from equilibrium is the vector $\mathbf{\Delta}$, takes the form

$$\rho\ddot{\mathbf{\Delta}} = \left(k + \frac{4G}{3}\right)\nabla\nabla\cdot\mathbf{\Delta} - G\nabla\times\nabla\mathbf{\Delta} \qquad \text{(2a-56)}$$

If $\nabla\times\mathbf{\Delta} = 0$ this is the equation of *irrotational* waves traveling with velocity

$$V_i = \sqrt{\frac{k + 4G/3}{\rho}} \qquad \text{(2a-57)}$$

If $\nabla\cdot\mathbf{\Delta} = 0$, this is the equation of solenoidal waves traveling with velocity

$$V_s = \sqrt{\frac{G}{\rho}} \qquad \text{(2a-58)}$$

2a-7. Fluid Dynamics. *General Concepts. Fluids in Equilibrium.* A perfect fluid is a deformable medium in which deforming forces give rise only to dilatations and never to shears. This is an ideal concept and is realized only approximately for actual fluids. Gases manifest the property more nearly than liquids, though both are normally considered to be fluids. Liquids can present under many circumstances the phenomenon of a free surface.

The dilatational stress in the case of a fluid is termed the *pressure*, which is the force per unit area directed *against* any surface imagined to exist in the fluid. A perfect fluid in equilibrium under the influence of an external force F acting on unit mass is subject to the relation

$$\rho \mathbf{F} = \nabla p \tag{2a-59}$$

where p is the pressure (here treated for simplicity as a scalar since it acts *normally* to every surface when the fluid is in equilibrium) and ρ the density, all quantities being considered as functions of space alone. The solution of this equation for given $\mathbf{F}$ gives p as a function of position in space and yields Pascal's law of the transmissibility of pressure in a fluid in equilibrium. From this also follows at once the principle of Archimedes that any fluid in equilibrium exerts on a body immersed in it a buoyant force equal in magnitude to the weight of the fluid displaced by the body and directed upward through the center of gravity.

Flow Concepts. Equation of Continuity. In the Eulerian system to which this review is confined the flow velocity of a fluid is the vector $\mathbf{v}$ whose magnitude at any point and at any time is the volume flow per unit time per unit area placed normal to the direction of flow, the latter being the direction of $\mathbf{v}$. This quantity is a function of both space and time. In any continuous indestructible fluid of density ρ containing no sources or sinks $\mathbf{v}$ obeys the so-called equation of continuity

$$\nabla \cdot (\rho \mathbf{v}) = -\dot{\rho} \tag{2a-60}$$

where it is to be noted that ρ also is a function of space and time. For a homogeneous incompressible fluid this equation reduces to

$$\nabla \cdot \mathbf{v} = 0 \tag{2a-61}$$

i.e., $\mathbf{v}$ is a solenoidal vector. If further $\mathbf{v}$ is irrotational, so that $\nabla \times \mathbf{v} = 0$, it follows that

$$\mathbf{v} = \nabla \phi \tag{2a-62}$$

where ϕ is a scalar potential, called the "velocity potential," and the equation of continuity reduces to Laplace's equation

$$\nabla^2 \phi = 0 \tag{2a-63}$$

Equation of Motion. Bernoulli's Principle. The vector equation of motion of a compressible fluid of density ρ subject to an external force $\mathbf{F}$ is

$$\dot{\mathbf{v}} + \mathbf{v} \cdot \nabla \mathbf{v} = \mathbf{F} - \frac{\nabla p}{\rho} \tag{2a-64}$$

where p is the pressure.

For irrotational flow in a conservative force field ($\mathbf{F} = -\nabla V$) it follows from the equation of motion that

$$\tfrac{1}{2}\rho v^2 + \rho V + p = \text{const} \tag{2a-65}$$

which is the principle of Bernoulli. It can also be shown that, even if the flow is not irrotational, as long as it is *steady* and in streamlines, so that $\mathbf{v}$ does not depend on the time, the above equation of Bernoulli will still hold as one proceeds along any given streamline, though the constant will in general be different for different streamlines.

Viscous Fluids. In contrast to a perfect fluid in which no shearing strains can exist, a viscous fluid is one in which the part of the medium flowing in one layer exerts a tangential or shearing stress on that flowing in the same direction in an adjacent layer. In the simplest type of viscous flow the tangential force is proportional to the velocity gradient normal to the layer and the coefficient of proportionality is called the viscosity η. Specifically

$$\eta = \frac{\text{shearing stress}}{\text{velocity gradient normal to flow}} \qquad \text{(2a-66)}$$

The analogy between this relation and that defining the shear modulus for an elastic medium is obvious, the difference being that here the denominator is the rate of *change* of shear strain instead of the strain itself. The suggestion is immediate that the discussion of viscous flow can develop along the lines of the analysis of the behavior of deformable media in general (cf. Sec. 2a-6). This is indeed the case; it makes pressure appear as a tensor (analogous to the stress tensor). See also Secs. 2m, 2r, 2u.

A solid moving through a viscous fluid encounters increased resistance because of the viscosity. The simplest case is that in which a sphere of radius a moves through a fluid of vicosity η with *constant* velocity v. The resisting force is then given by Stokes' law

$$F = 6\pi\eta a v \qquad \text{(2a-67)}$$

Surface Tension in Liquids. This is the force per unit length γ in the surface separating a liquid from the material surrounding it. Details concerning this as well as numerical values will be found in Sec. 2n.

Surface Waves in Liquids. When the free surface of a liquid is deformed, the forces acting on the deformed elements are primarily surface tension and gravity. The velocity of the resulting surface wave, if it is harmonic and has wavelength λ, is

$$V = \sqrt{\left(\frac{g\lambda}{2\pi} + \frac{2\pi\gamma}{\rho\lambda}\right)\tanh\frac{2\pi l}{\lambda}} \qquad \text{(2a-68)}$$

where g is the acceleration of gravity, ρ the density, γ the surface tension, and l the depth of the liquid. For a relatively shallow liquid, for which $l \ll \lambda$, and the surface tension not very large, we have

$$V \doteq \sqrt{gl} \qquad \text{(2a-69)}$$

If the liquid is relatively deep, or $l \gg \lambda$,

$$V \doteq \sqrt{\frac{g\lambda}{2\pi} + \frac{2\pi\gamma}{\rho\lambda}} \qquad \text{(2a-70)}$$

For long waves

$$V \doteq \sqrt{\frac{g\lambda}{2\pi}}$$

while for ripples (small λ), surface tension predominates and

$$V \doteq \sqrt{\frac{2\pi\gamma}{\rho\lambda}}$$

Compressional Waves in Fluids. The combination of the equation of motion (2a-64), the equation of continuity (2a-60), and the equation of state of the fluid, i.e., the relation connecting change in density with change in pressure, leads to the wave equation for compressional waves traveling with velocity

$$V = \sqrt{\frac{dp}{d\rho}} \qquad (2a\text{-}71)$$

The values of V for gases and liquids will be found in Sec. 3.

2a-8. Fundamental Units (mks System). These units are defined as follows:

Meter. Unit of length. By international agreement (Oct. 14, 1960) defined to be 1,650,763.73 wavelengths of the orange-red line of krypton 86. This replaces the definition in terms of the platinum-iridium meter bar in Paris.

Kilogram. Unit of mass. Defined to be the mass of a certain solid cylinder of platinum-iridium alloy preserved at the International Bureau of Weights and Measures in Paris.

Second. Unit of time. By earlier international agreement (October 14, 1960) defined to be 1/31,556,925.9747 of the tropical year 1900. (The *tropical year* is defined as the interval of time between two successive passages of the sun through the vernal equinox.) A more recent international conference (1964) adopted provisionally a new definition of the second as the time corresponding to 9.192631770 × 10^9 oscillations of the cesium atom in the so-called atomic clock.

2a-9. Supplementary Fundamental Units (cgs and English Systems). Definitions of these units follow:

Centimeter. Defined to be $\frac{1}{100}$ meter.

Gram. Defined to be 1/1,000 kilogram.

Second. Same as the second defined in Sec. 2a-8.

International Yard. Defined by agreement between the United States and the British Commonwealth (1959) to be 0.9144 meter.

International Pound. Defined by agreement between the United States and the British Commonwealth (1959) to be 0.45359237 kilogram.

2a-10. Angular Units. These units are defined as follows:

Degree. Angle subtended at the center by a circular arc which is $\frac{1}{360}$ of the circumference.

Minute of Arc. $\frac{1}{60}$ of a degree.

Second of Arc. $\frac{1}{60}$ of a minute of arc.

Radian. Angle subtended at the center by a circular arc which is equal in length to the radius of the circle.

Steradian. Solid angle subtended at the center by $1/4\pi$ of the surface area of a sphere of unit radius.

2a-11. Derived Units. These units are defined as follows:

Atmosphere. Pressure exerted by air at mean sea level under standard conditions = 1.013250 × 10^6 dynes/cm².

British Thermal Unit (Mean). Energy required to raise temperature of 1 lb mass of water 1°F (averaged from 32 to 212°F).

Calorie (Mean). Energy required to raise 1 g mass of water 1°C (averaged from 0 to 100°C).

Centimeters of Hg at 0°C. Pressure exerted by column of Hg of stated height at 0°C.

Dyne. Force necessary to give 1 g mass acceleration of 1 cm/sec².

Erg. Work done by force of 1 dyne moving a particle a distance of 1 cm.

Feet of Water at 4°C. Pressure exerted by column of water of stated height at 4°C.

Kilowatthour. Work done in 1 hr at power level or rate of 10^3 watts.

Newton. Force necessary to give 1 kg mass acceleration of 1 m/sec².

Poundal. Force necessary to give 1 lb mass acceleration of 1 ft/sec².

Watt. Rate of doing work, or power expended, in the amount of 10^7 ergs/sec.

TABLE 2a-1. UNITS AND CONVERSION FACTORS, LENGTH

	Angstrom	Centimeter	Fathom	Foot	Inch (U.S.)	Kilo-meter	Light-year
Angstrom	1	10^{-8}	......	3.281×10^{-10}	3.937×10^{-9}	10^{-13}	
Centimeter	10^{8}	1	......	3.281×10^{-2}	0.3937	10^{-5}	
Fathom	....	....	1	6	72		
Foot	....	30.48	0.1667	1	12		
Inch (U.S.)	2.540×10^{8}	2.540	......	8.333×10^{-2}	1		
Kilometer	....	10^{5}	......	3.281×10^{3}	....	1	1.057×10^{-13}
Light-year	....	....	......	....	....	9.46×10^{12}	1
Meter	10^{10}	10^{2}	0.5468	3.281	39.37	10^{-3}	
Micron	10^{4}	10^{-4}	......	....	3.937×10^{-5}		
Mil	....	2.540×10^{-3}	......	....	10^{-3}		
Mile (statute)	....	....	......	5.280×10^{3}	6.336×10^{4}	1.609	1.69×10^{-13}
Millimeter	....	10^{-1}	......	....	3.937×10^{-2}		
Millimicron	10	10^{-7}					
Yard (U.S.)	....	91.44	......	3	36		

	Meter	Micron	Mil	Mile (statute)	Millimeter	Milli-micron	Yard (U.S.)
Angstrom	10^{-10}	10^{-4}	3.937×10^{-9}	....	10^{-7}	10^{-1}	1.094×10^{-10}
Centimeter	10^{-2}	10^{4}	3.937×10^{2}	....	10	10^{7}	1.094×10^{-2}
Fathom	1.829	.....	....	....	....	...	2
Foot	0.3048	.....	....	1.894×10^{-4}	....	...	0.3333
Inch (U.S.)	....	.....	10^{3}	1.578×10^{-5}	25.40	...	2.778×10^{-2}
Kilometer	10^{3}	.....	....	0.6214	....	...	1.094×10^{3}
Light-year	....	.....	....	5.9×10^{12}			
Meter	1	.....		6.214×10^{-4}	....	10^{9}	1.094
Micron	10^{-6}	1	3.937×10^{-2}	....	10^{-3}	10^{3}	
Mil	....	25.40	1	....	2.450×10^{2}		
Mile (statute)	1.609×10^{3}	.....	....	1	....	...	1.760×10^{3}
Millimeter	10^{-3}	10^{3}	39.37	....	1		
Millimicron	10^{-9}	10^{-3}	....	....	....	1	
Yard (U.S.)	0.9144	.....	....	5.682×10^{-4}	....	...	1

TABLE 2a-2. UNITS AND CONVERSION FACTORS, AREA

	Circular mil	Square centimeter	Square foot (U.S.)	Square inch (U.S.)	Square kilometer	Square meter	Square mile	Square millimeter	Square yard
Circular mil	1	5.067×10^{-6}	......	7.854×10^{-7}	......	......	......	5.067×10^{-4}	......
Square centimeter	1.974×10^{5}	1	1.076×10^{-3}	0.1550	......	10^{-4}	......	10^{2}	1.196×10^{-4}
Square foot (U.S.)	......	9.290×10^{2}	1	1.44×10^{2}	......	9.290×10^{-2}	3.587×10^{-8}	......	0.1111
Square inch (U.S.)	1.273×10^{6}	6.452	6.944×10^{-3}	1	......	6.452×10^{-4}	......	6.452×10^{2}	7.716×10^{-4}
Square kilometer	......	......	1.076×10^{7}	......	1	10^{6}	0.3861	......	1.196×10^{6}
Square meter	......	10^{4}	10.76	1.550×10^{3}	10^{-6}	1	3.861×10^{-7}	10^{6}	1.196
Square mile	......	......	2.788×10^{7}	......	2.590	2.590×10^{6}	1	......	3.098×10^{6}
Square millimeter	1.974×10^{3}	10^{-2}	......	1.550×10^{-3}	......	10^{-6}	......	1	......
Square yard	......	8.361×10^{3}	9	1.296×10^{3}	......	0.8361	3.228×10^{-7}	......	1

TABLE 2a-3. UNITS AND CONVERSION FACTORS, VOLUME

	Cubic centimeter	Cubic foot	Cubic inch	Cubic meter	Cubic yard	Fluid ounce (U.S.)	Gallon (U.S.)	Liter*	Pint, dry (U.S.)	Pint, liquid (U.S.)	Quart, dry (U.S.)	Quart, liquid (U.S.)
Cubic centimeter	1	3.531×10^{-5}	6.102×10^{-2}	10^{-6}	1.308×10^{-6}	3.381×10^{-2}	2.642×10^{-4}	9.9997×10^{-4}	1.816×10^{-3}	2.113×10^{-3}	9.081×10^{-4}	1.057×10^{-3}
Cubic foot	2.832×10^{4}	1	1.728×10^{3}	2.832×10^{-2}	3.704×10^{-2}	......	7.481	28.32	......	59.84	25.71	29.92
Cubic inch	16.39	5.787×10^{-4}	1	1.639×10^{-5}	2.143×10^{-5}	0.5541	4.329×10^{-3}	1.639×10^{-2}	2.976×10^{-2}	......	1.488×10^{-2}	1.732×10^{-2}
Cubic meter	10^{6}	35.31	6.102×10^{4}	1	1.308	......	2.642×10^{2}	9.9997×10^{2}	......	2.113×10^{3}	......	1.057×10^{3}
Cubic yard	7.646×10^{5}	27	4.666×10^{4}	0.7646	1	......	2.020×10^{2}	7.645×10^{2}	......	1.616×10^{3}	......	8.079×10^{2}
Fluid ounce (U.S.)	29.57	......	1.805	......	......	1	7.813×10^{-3}	2.957×10^{-2}	......	6.250×10^{-2}	......	3.125×10^{-2}
Gallon (U.S.)	3.785×10^{3}	0.1337	2.310×10^{2}	3.785×10^{-3}	4.951×10^{-3}	1.280×10^{2}	1	3.785	......	8	......	4
Liter*	1.000×10^{3}	3.532×10^{-2}	61.03	1.000×10^{-3}	1.308×10^{-3}	33.81	0.2642	1	1.816	2.113	0.9081	1.057
Pint, dry (U.S.)	5.506×10^{2}	......	33.60	......	......	......	......	0.5506	1	......	0.5000	......
Pint, liquid (U.S.)	4.732×10^{2}	1.671×10^{-2}	28.88	......	6.188×10^{-4}	16	0.1250	0.4732	......	1	......	0.5000
Quart, dry (U.S.)	1.101×10^{3}	3.889×10^{-2}	67.20	......	......	......	......	1.101	2	......	1	......
Quart, liquid (U.S.)	9.464×10^{2}	3.342×10^{-2}	57.75	......	......	32	0.2500	0.9463	......	2	......	1

* 1 milliliter = 1.000027 cubic centimeters.