2c. Centers of Mass and Moments of Inertia

R. BRUCE LINDSAY

Brown University

TABLE 2c-1. CENTERS OF MASS*

Body	*Center of Mass*
1. Uniform circular wire of radius R, subtending angle 2θ at center	On axis of symmetry distant $(R \sin \theta)/\theta$ from center
2. Uniform triangular sheet	At intersection of the medians
3. Uniform rectangular sheet	At intersection of the diagonals
4. Uniform quadrilateral sheet	From each vertex lay off segments equal to $\frac{1}{3}$ the length of the corresponding sides meeting at this vertex. Draw extended lines through the ends of the segments associated with each vertex, respectively. These intersect to form a parallelogram. The intersection of the diagonals of this parallelogram is the center of mass of the quadrilateral
5. Uniform circular sector sheet of radius R subtending angle 2θ at center of circular arc	On axis of symmetry distant $(2R \sin \theta)/3\theta$ from center
6. Uniform circular segment sheet of radius R, subtending angle 2θ at center of circular arc and length of chord equal to $l = 2R \sin \theta$	On axis of symmetry distant $l^3/12A$ from center, where A = area of segment $= \frac{R^2(2\theta - \sin 2\theta)}{2}$
7. Uniform semielliptical sheet, major and minor axes of equivalent ellipse equal to $2a$ and $2b$, respectively	On axis of symmetry distant $4a/3\pi$ from center of equivalent ellipse if the semiellipse is bounded by minor axis. The distance is $4b/3\pi$ if the semiellipse is bounded by the major axis
8. Uniform quarter-elliptical sheet, major and minor axes of equivalent ellipse equal to $2a$ and $2b$, respectively	At point $4b/3\pi$ above major axis and $4a/3\pi$ above minor axis
9. Uniform parabolic sheet segment. Chord $= 2l$ perpendicular to axis of symmetry distant h from vertex	On axis of symmetry distant $3h/5$ from vertex
10. Right rectangular pyramid (rectangular base with sides a and b and with height h)	On axis of symmetry distant $h/4$ from base
11. Pyramid (general)	On line joining apex with center of symmetry of base at distance three-quarters of its length from apex

* For definition see Sec. 2a-4. All bodies cited are homogeneous rigid bodies.

TABLE 2c-1. CENTERS OF MASS (*Continued*)

Body	*Center of Mass*
12. Frustum of pyramid with area of larger base S and smaller base s, and altitude h	On line joining apex of corresponding pyramid with center of symmetry of larger base and distant $\frac{h(S + 2\sqrt{Ss} + 3s)}{4(S + \sqrt{Ss} + s)}$ from the larger base
13. Right circular cone (height h)	On axis of symmetry distant $h/4$ from base
14. Frustum of right circular cone (altitude h, radii of larger and smaller bases R and r, respectively)	On axis of symmetry distant $\frac{h[(R + r)^2 + 2r^2]}{4[(R + r)^2 - Rr]}$ from the base
15. Cone (general)	On line joining apex with centroid of base at distance three-quarters of its length from apex
16. Frustum of cone with altitude h and radii of larger and smaller bases R and r, respectively	On line joining apex of corresponding cone with centroid of larger base and distant $\frac{h[(R + r)^2 + 2r^2]}{4[(R + r)^2 - Rr]}$ from the larger base
17. Spherical sector of radius R, with plane vertex angle equal to 2θ	On axis of symmetry distant $\frac{3R}{8}(1 + \cos\theta)$ from the vertex
18. Solid hemisphere of radius R	On axis of symmetry distant $3R/8$ from center of corresponding sphere
19. Spherical segment of radius R and maximum height from base equal to h	On axis of symmetry distant $\frac{h(4R - h)}{4(3R - h)}$ above the base of the segment
20. Octant of ellipsoid with semiaxes a, b, c, respectively, and center of corresponding ellipsoid at origin of system of rectangular coordinates	Point with coordinates $\bar{x} = \frac{3a}{8} \quad \bar{y} = \frac{3b}{8} \quad \bar{z} = \frac{3c}{8}$
21. Paraboloid of revolution with altitude h and radius of circular base equal to R	On axis of symmetry distant $h/3$ from the base
22. Uniform hemispherical shell of radius R (excluding base)	On axis of symmetry distant $R/2$ from center of corresponding sphere
23. Conical shell (excluding base)	On line joining the apex with the center of symmetry of the base at distance two-thirds its length from the apex

TABLE 2c-2. MOMENTS OF INERTIA*

Body	Axis	Moment of inertia
Uniform rectangular sheet of sides a and b	Through the center parallel to b	$m\frac{a^2}{12}$
Uniform rectangular sheet of sides a and b	Through the center perpendicular to the sheet	$m\frac{a^2+b^2}{12}$
Uniform circular sheet of radius r	Normal to the plate through the center	$m\frac{r^2}{2}$
Uniform circular sheet of radius r	Along any diameter	$m\frac{r^2}{4}$
Uniform circular ring, radii r_1 and r_2	Through center normal to plane of ring	$m\frac{r_1^2+r_2^2}{2}$
Uniform circular ring, radii r_1 and r_2	A diameter	$m\frac{r_1^2+r_2^2}{4}$
Uniform thin spherical shell, mean radius r	A diameter	$m\frac{2r^2}{3}$
Uniform cylindrical shell, radius r, length l	Longitudinal axis	mr^2
Right circular cylinder of radius r, length l	Longitudinal axis	$m\frac{r^2}{2}$
Right circular cone, altitude h, radius of base r	Axis of the figure	$m\frac{3}{10}r^2$
Spheroid of revolution, equatorial radius r	Polar axis	$m\frac{2r^2}{5}$
Ellipsoid, axes $2a$, $2b$, $2c$....	Axis $2a$	$m\frac{(b^2+c^2)}{5}$
Uniform thin rod.....	Normal to the length, at one end	$m\frac{l^2}{3}$
Uniform thin rod.....	Normal to the length, at the center	$m\frac{l^2}{12}$
Rectangular prism, dimensions $2a$, $2b$, $2c$	Axis $2a$	$m\frac{(b^2+c^2)}{3}$
Sphere, radius r.....	A diameter	$m\frac{2}{5}r^2$
Rectangular parallelepiped, edges a, b, and c	Through center perpendicular to face ab (parallel to edge c)	$m\frac{a^2+b^2}{12}$
Right circular cylinder of radius r, length l	Through center perpendicular to the axis of the figure	$m\left(\frac{r^2}{4}+\frac{l^2}{12}\right)$
Spherical shell, external radius r_1, internal radius r_2	A diameter	$m\frac{2}{5}\frac{(r_1^5-r_2^5)}{(r_1^3-r_2^3)}$
Hollow circular cylinder, length l, external radius r_1, internal radius r_2	Longitudinal axis	$m\frac{(r_1^2+r_2^2)}{2}$
Hollow circular cylinder, length l, radii r_1 and r_2	Transverse diameter	$m\left(\frac{r_1^2+r_2^2}{4}+\frac{l^2}{12}\right)$

* For definitions see Sec. 2a-5; m = mass of body. All bodies are homogeneous.

TABLE 2c-2. MOMENTS OF INERTIA (*Continued*)

Body	Axis	Moment of inertia
Hollow circular cylinder, length l, very thin, mean radius r	Transverse diameter	$m\left(\frac{r^2}{2}+\frac{l^2}{12}\right)$
Right elliptical cylinder, length $2a$, transverse axes $2b$, $2c$	Longitudinal axis $2a$ through center of mass	$m\frac{(b^2+c^2)}{4}$
Right elliptical cylinder, length $2a$, transverse axes $2b$, $2c$	Transverse axis $2b$ through center of mass	$m\left(\frac{c^2}{4}+\frac{a^2}{3}\right)$
Frustum of right circular cone with radii of larger and smaller bases, equal to R and r, respectively	Axis of symmetry	$\frac{3m(R^5-r^5)}{10(R^3-r^3)}$
Right circular cone, radius of base r, altitude h	Perpendicular to axis of symmetry, through center of mass	$\frac{3m}{20}\left(r^2+\frac{h^2}{4}\right)$
Solid hemisphere of radius r	Axis of symmetry	$\frac{2mr^2}{5}$
Spherical sector of radius r, with plane angle at vertex $= 2\theta$	Axis of symmetry through vertex	$\frac{mr^2(1-\cos\theta)(2+\cos\theta)}{5}$
Spherical segment of radius r and maximum height h	Axis of symmetry perpendicular to base	$m\left(r^2-\frac{3rh}{4}+\frac{3h^2}{20}\right)\frac{2h}{3r-h}$
Torus or anchor ring mean radius R, radius of circular cross section r	Axis of symmetry perpendicular to plane of ring	$\frac{m(4R^2+3r^2)}{4}$
Torus mean radius R, radius of circular cross section r	Axis of symmetry in plane of ring	$\frac{m(4R^2+5r^2)}{8}$