2t. Compressible Flow of Gases

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Symbols

A	cross-sectional area
a	local sound velocity
C_p	pressure coefficient $(p - p_\infty)/\frac{1}{2}\rho_\infty U^2$
c_v	specific heat at constant volume
c_p	specific heat at constant pressure
E	internal energy per unit mass
M	Mach number
M_∞	free-stream Mach number
m	mass flow
p	pressure
p_∞	free-stream pressure
Q	external heat production rate per unit mass
q	resultant velocity of flow
R	gas constant
t	time
U	free-stream velocity
u, v, w	velocity components of fluid flow
X, Y, Z	rectangular components of external body force
x, y, z	rectangular coordinates
γ	ratio of specific heat at constant pressure to that at constant volume
ρ	density
ρ_∞	free-stream density
ϕ	velocity potential
ϕ_x	$\partial\phi/\partial x$
ϕ_{xx}	$\partial^2\phi/\partial x^2$
ϕ_{xy}	$\partial^2\phi/\partial x\,\partial y$
χ	similarity parameter
ψ	stream function

2t-1. Basic Equations in Rectangular Coordinates. The basic equations of motion for a compressible inviscid gas may be written as follows.

Momentum Equation. By applying Newton's laws of motion the Euler momentum equation may be derived in the form

$$\begin{aligned}
\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} &= \frac{-1}{\rho}\frac{\partial p}{\partial x} + X \\
\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} &= \frac{-1}{\rho}\frac{\partial p}{\partial y} + Y \qquad \text{(2t-1)} \\
\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} &= \frac{-1}{\rho}\frac{\partial p}{\partial z} + Z
\end{aligned}$$

where x, y, z = rectangular coordinates
t = time
u, v, w = velocity components in direction of x, y, and z axes, respectively
p = pressure
ρ = density
X, Y, Z = rectangular components of external body force

Continuity Equation. The assumption that the gas is a continuous medium is expressed by the equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0 \tag{2t-2}$$

Energy Equation. The relationship between the kinetic and internal energy and the work done on the fluid by pressure and external forces is expressed by the equation

$$\rho \frac{DE}{Dt} + \rho \frac{D}{Dt}\left(\frac{1}{2} q^2\right) = \rho Q + \rho(uX + vY + wZ) - \frac{\partial}{\partial x}(pu) - \frac{\partial}{\partial y}(pv) - \frac{\partial}{\partial z}(pw) \tag{2t-3}$$

where $\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$
E = internal energy per unit mass = $\int c_v \, dT$
$q^2 = u^2 + v^2 + w^2$
Q = external-heat-production rate per unit mass
c_v = specific heat at constant volume

Equation of State. For a complete specification of a flow it is necessary to give an equation of state. This commonly takes the form

$$p = f(\rho, T)$$

Many gases obey the equation of state of a perfect gas

$$p = \rho R T$$

under a great variety of conditions. In this equation R is a constant which depends on the particular gas. If the specific heat can be assumed constant, the gas is said to be calorically perfect and

$$E = c_v T$$

where T is the temperature on the absolute scale.

A specific case of great importance is that of isentropic flow. If the entropy is constant throughout the flow, the equation of state can be written as

$$p = K\rho^{\gamma}$$

where K is a constant and γ is the ratio of the specific heat at constant pressure c_p to that at constant volume c_v. Now the flow is completely determined by the momentum equations, the continuity equation, and the equation of state. Many practical flow problems are essentially cases of isentropic flow.

2t-2. Dynamic Similarity and Definition of Basic Flow Parameters. In the testing of models it is necessary to maintain a proper scaling of certain dynamic parameters in addition to the geometric scaling. For compressible inviscid flow with no heat sources and in which body forces are neglected, the only dynamic dimensionless parameter is the Mach number.

Definition of Mach Number. The local Mach number is defined as the ratio of the local flow velocity q to the local sound velocity a; i.e.,

$$M = \frac{q}{a} \tag{2t-4}$$

Thus in a nonuniform flow the Mach number will vary from point to point. The size of the Mach number indicates whether the flow is subsonic, $M < 1$; transonic, $M \simeq 1$; or supersonic, $M > 1$. The term hypersonic is often used to describe flows where $M > 5$.

Dynamic Similarity. If the same gas flows around two geometrically similar bodies, it might be expected that under the right conditions the streamline pattern would be similar. This is true if the Mach numbers of the two flows are equal. It then follows that all other dimensionless coefficients such as drag coefficient, lift coefficient, pressure coefficient, etc., are also equal.

In determining the Mach number in a flow it is necessary to know not only the flow velocity but the sound velocity as well. For a perfect gas the sound velocity is proportional to the square root of the temperature; i.e.,

$$a = \sqrt{\gamma RT}$$

Table 2t-1 is based on this relationship.

2t-3. Basic Idea of One-dimensional Flow. In many cases, as in a pipe of slowly varying cross section, it is possible to make the assumption of constant flow properties across any cross section perpendicular to the pipe axis. Although strictly speaking there are no one-dimensional flows, because of viscous effects on the boundaries, it is still possible to get much valuable information of a practical nature from the assumptions.

TABLE 2t-1. VARIATION OF VELOCITY OF SOUND WITH TEMPERATURE

T, °K	a, fps	a, m/sec
150	805	246
160	832	254
170	857	261
180	882	269
190	907	276
200	930	283
210	953	290
220	975	297
230	997	304
240	1,019	311
250	1,040	317
260	1,060	323
270	1,081	329
280	1,100	335
290	1,120	341
300	1,139	347
310	1,158	353
320	1,176	359
330	1,195	364
340	1,213	370
350	1,230	375

Basic Equations. On the assumption of isentropic flow the equations of motion are

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \qquad \text{(momentum)} \qquad (2t\text{-}5)$$

$$\frac{\partial \rho}{\partial t} + \frac{1}{A}\frac{\partial}{\partial x}(\rho u A) = 0 \qquad \text{(continuity)} \qquad (2t\text{-}6)$$

where A is the cross-sectional area. For unsteady one-dimensional flow in general and in particular for an excellent treatment of flow in pipes of constant area see ref. 3. The above equations also cover the case of cylindrical and spherically symmetric flow; i.e.,

$$\frac{1}{A}\frac{\partial A}{\partial x} = \frac{1}{x} \qquad \text{(for cylindrical flow)}$$

$$\frac{1}{A}\frac{\partial A}{\partial x} = \frac{2}{x} \qquad \text{(for spherically symmetric flow)}$$

In the important case of steady flow the equations can be integrated to give

$$\frac{\gamma}{\gamma - 1}\frac{p}{\rho} + \frac{1}{2}u^2 = \text{const} \tag{2t-7}$$

$$\rho u A = m = \text{const} \tag{2t-8}$$

where m is the mass flow. By taking logarithmic derivatives and remembering the definition of the Mach number M, the continuity equation may be written

$$\frac{du}{u}(1 - M^2) + \frac{dA}{A} = 0 \tag{2t-9}$$

Thus, if $du \neq 0$ and $M = 1$, we see that $dA = 0$. In other words, the Mach number becomes equal to unity only in a section of the pipe where the area is a minimum. This fact is of prime importance in the design of supersonic wind tunnels.

The dependence of the various flow variables on the Mach number for steady one-dimensional isentropic flow is given in Table 2t-2.

2t-4. Two-dimensional and Axially Symmetric Flow. Many important types of flow belong to the class of two-dimensional or axially symmetric flows. These include flows past wedges, cones, bodies of revolution, etc. The important distinctions to be made are those between subsonic and supersonic flow. Purely subsonic flow is qualitatively quite similar to incompressible flow, while supersonic flow exhibits many startlingly different properties. Among these are the appearance of shock waves (see Sec. 2v) and the existence of wavefronts. A general discussion of the above topics can be found in refs. 2, 3, and 6.

The greater bulk of the literature on two-dimensional and axially symmetric flow is concerned with steady flow. The unsteady cases are usually extremely difficult to solve.

Velocity Potential and Stream Function. In cases of irrotational or steady flow it is convenient to introduce the velocity potential or the stream function. This reduces the number of equations to one. The velocity potential exists whenever there is a state of steady or unsteady irrotational flow; i.e., the velocity components satisfy the equations

$$\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = 0 \qquad \frac{\partial w}{\partial x} - \frac{\partial u}{\partial z} = 0 \qquad \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$$

Then the velocity components u, v, w can be expressed as the components of the gradient of the velocity potential ϕ. Thus

$$u = \frac{\partial \phi}{\partial x} \qquad v = \frac{\partial \phi}{\partial y} \qquad w = \frac{\partial \phi}{\partial z} \tag{2t-10}$$

For steady isentropic flow the equations of motion reduce to the single equation for ϕ,

$$\phi_{xx}\left(1 - \frac{\phi_x^2}{a^2}\right) + \phi_{yy}\left(1 - \frac{\phi_y^2}{a^2}\right) + \phi_{zz}\left(1 - \frac{\phi_z^2}{a^2}\right) - 2\phi_{yz}\frac{\phi_{yz}\phi}{a^2} - 2\phi_{xz}\frac{\phi_x\phi_z}{a^2} - 2\phi_{xy}\frac{\phi_x\phi_y}{a^2} = 0 \tag{2t-11}$$

where $$a^2 = \frac{\gamma - 1}{2}(q_{\max}^2 - \phi_x^2 - \phi_y^2 - \phi_z^2)$$

TABLE 2t-2. DEPENDENCE OF FLOW VARIABLES ON MACH NUMBER FOR ONE-DIMENSIONAL ISENTROPIC FLOW*

M	p/p_0	u/a_0	A/A^*	$\rho u^2/2p_0$	$\rho u/\rho_0 a_0$	ρ/ρ_0	T/T_0	a/a_0
0.0	1.00000	0.00000	∞	0.00000	0.00000	1.00000	1.00000	1.00000
0.1	0.99303	0.09990	5.822	0.00695	0.09940	0.99502	0.99800	0.99900
0.2	0.97250	0.19920	2.9635	0.02723	0.19528	0.98028	0.99206	0.99602
0.3	0.93947	0.29734	2.0351	0.05919	0.28437	0.95638	0.98232	0.99112
0.4	0.89561	0.39375	1.5901	0.10031	0.30393	0.92427	0.96899	0.98437
0.5	0.84302	0.48795	1.3398	0.14753	0.43192	0.88517	0.95238	0.97590
0.6	0.78400	0.57950	1.1882	0.19757	0.48704	0.84045	0.93284	0.96583
0.7	0.72093	0.66803	1.0944	0.24728	0.52880	0.79161	0.91075	0.95433
0.8	0.65602	0.75324	1.0382	0.29390	0.55739	0.73999	0.88652	0.94155
0.9	0.59126	0.83491	1.0089	0.33524	0.57362	0.68704	0.86059	0.92768
1.0	0.52828	0.91287	1.00000	0.36980	0.57870	0.63394	0.83333	0.91287
1.1	0.46835	0.98703	1.0079	0.39670	0.57415	0.58170	0.80515	0.89730
1.2	0.41238	1.0574	1.0304	0.41568	0.56161	0.53114	0.77640	0.88113
1.3	0.36091	1.1239	1.0663	0.42696	0.54272	0.48290	0.74738	0.86451
1.4	0.31424	1.1866	1.1149	0.43114	0.51905	0.43742	0.71839	0.84758
1.5	0.27240	1.2457	1.1762	0.42903	0.49203	0.39484	0.68966	0.83045
1.6	0.23527	1.3012	1.2502	0.42161	0.46288	0.35573	0.66138	0.81325
1.7	0.20259	1.3533	1.3376	0.40985	0.43264	0.31969	0.63371	0.79606
1.8	0.17404	1.4023	1.4390	0.39476	0.40216	0.28684	0.60680	0.77904
1.9	0.14924	1.4479	1.5553	0.37713	0.37210	0.25699	0.58072	0.76205
2.0	0.12780	1.4907	1.6875	0.35785	0.34294	0.23005	0.55556	0.74535
2.1	0.10935	1.5308	1.8369	0.33757	0.31504	0.20580	0.53135	0.72894
2.2	0.09352	1.5682	2.0050	0.31685	0.28863	0.18405	0.50813	0.71283
2.3	0.07997	1.6033	2.1931	0.29614	0.26387	0.16458	0.48591	0.69707
2.4	0.06840	1.6360	2.4031	0.27579	0.24082	0.14719	0.46468	0.68168
2.5	0.05853	1.6667	2.6367	0.25606	0.21948	0.13169	0.44444	0.66667
2.6	0.05012	1.6953	2.8960	0.23715	0.19983	0.11788	0.42517	0.65205
2.7	0.04295	1.7222	3.1830	0.21917	0.18181	0.10557	0.40683	0.63784
2.8	0.03685	1.7473	3.5001	0.20222	0.16534	0.09463	0.38941	0.62403
2.9	0.03165	1.7708	3.8498	0.18633	0.15032	0.08489	0.37286	0.61062
3.0	0.02722	1.7928	4.2346	0.17151	0.13666	0.07623	0.35714	0.59761
3.1	0.02345	1.8135	4.6573	0.15774	0.12426	0.06852	0.34223	0.58501
3.2	0.02023	1.8329	5.1210	0.14499	0.11301	0.06165	0.32808	0.57279
3.3	0.01748	1.8511	5.6287	0.13322	0.10281	0.05554	0.31466	0.56095
3.4	0.01512	1.8682	6.184	0.12239	0.09359	0.05009	0.30193	0.54948
3.5	0.01311	1.8843	6.790	0.11243	0.08523	0.04523	0.28986	0.53838
3.6	0.01138	1.8995	7.450	0.10328	0.07768	0.04089	0.27840	0.52763
3.7	0.00990	1.9137	8.169	0.09490	0.07084	0.03702	0.26752	0.51723
3.8	0.00863	1.9272	8.951	0.08722	0.06466	0.03355	0.25720	0.50715
3.9	0.00753	1.9398	9.799	0.08019	0.05906	0.03044	0.24740	0.49740
4.0	0.00659	1.9518	10.72	0.07379	0.05399	0.02766	0.23810	0.48795
4.1	0.00577	1.9631	11.71	0.06788	0.04940	0.02516	0.22925	0.47880
4.2	0.00506	1.9738	12.79	0.06250	0.04524	0.02292	0.22084	0.46994
4.3	0.00445	1.9839	13.95	0.05759	0.04147	0.02090	0.21286	0.46136
4.4	0.00392	1.9934	15.21	0.05309	0.03805	0.01909	0.20525	0.45305
4.5	0.00346	2.0025	16.56	0.04898	0.03494	0.01745	0.19802	0.44499
4.6	0.00305	2.0111	18.02	0.04521	0.03212	0.01597	0.19113	0.43719
4.7	0.00270	2.0192	19.58	0.04177	0.02955	0.01464	0.18457	0.42962
4.8	0.00239	2.0269	21.26	0.03862	0.02722	0.01343	0.17832	0.42228
4.9	0.00213	2.0343	23.07	0.03572	0.02509	0.01233	0.17235	0.41516
5.0	0.00189	2.0412	25.00	0.03308	0.02315	0.01134	0.16667	0.40825

* A more complete table may be found in refs. 4, 5, and 7.

and q_{max} is the velocity with which the gas flows into a vacuum. Other forms of this equation in different numbers of dimensions and for unsteady flow can be found in ref. 2.

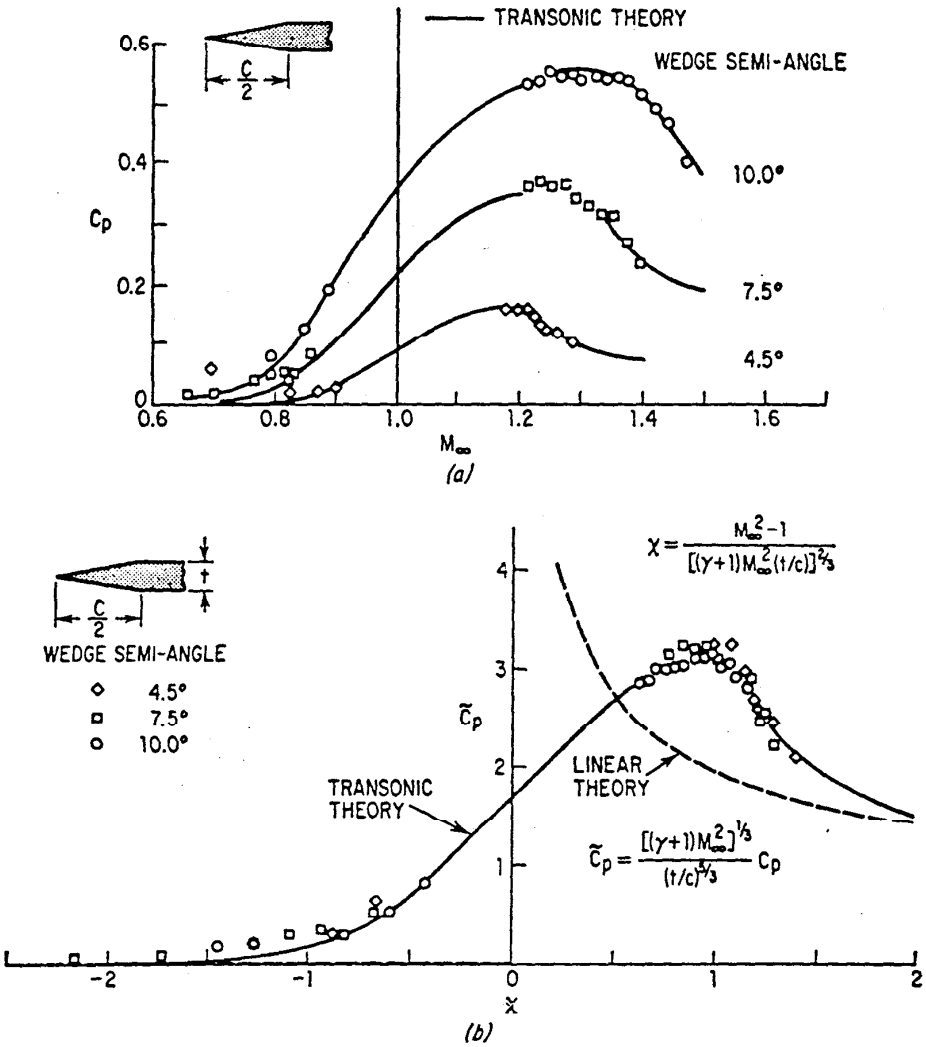

FIG. 2t-1. Comparison of the extended transonic similarity law with experiment. (*a*) Plotted in conventional coordinates. (*b*) Plotted in transonic similarity coordinates. (*After J. R. Spreiter, NACA; taken from ref.* 6.)

In compressible flow a stream function ψ exists only for steady two-dimensional or axially symmetric flow. The introduction of the function ψ causes the continuity equation to be satisfied identically. In two dimensions

$$u = \frac{1}{\rho}\psi_y \qquad v = -\frac{1}{\rho}\psi_x \tag{2t-12}$$

If cylindrical coordinates (x, r, θ) are used and the flow is independent of θ, then the function ψ may be defined by

$$u = \frac{1}{\rho r}\psi_r \qquad v = -\frac{1}{\rho r}\psi_x \tag{2t-13}$$

Note that u and v are now the velocity components in the x and r directions and $r = \sqrt{y^2 + z^2}$. Further details are given in ref. 2.

Equations of Small-perturbation Theory. For many slender or flat two- and three-dimensional bodies it may be assumed that the flow is disturbed very little from uniform flow. Thus if the free-stream velocity U is parallel to the x coordinate and M_∞ is the free-stream Mach number, the velocity components can be written in the form

$$u = U + \phi_x \qquad v = \phi_y \qquad w = \phi_z \tag{2t-14}$$

Here ϕ is called the disturbance potential. When Eqs. (2t-14) are put into Eq. (2t-11) and all nonlinear terms are neglected, the equation

$$(1 - M_\infty^2)\phi_{xx} + \phi_{yy} + \phi_{zz} = 0 \tag{2t-15}$$

is obtained. This equation holds for subsonic and moderate supersonic flows.

If M_∞ is very close to 1, Eq. (2t-15) is no longer valid and must be replaced by the equation

$$(1 - M_\infty^2)\phi_{xx} + \phi_{yy} + \phi_{zz} = \frac{M_\infty^2(\gamma + 1)}{U}\phi_x\phi_{xx} \tag{2t-16}$$

Similarity Rules. In many flows where the velocity perturbations are small, it is possible to show that the pressure, lift, drag, etc., depend on the various flow parameters in a simple manner. For example, in two-dimensional flow the pressure coefficient

$$C_p = \frac{p - p_\infty}{\frac{1}{2}\rho_\infty U^2}$$

is related to M_∞ and the thickness ratio τ by the formula

$$\frac{C_p[(\gamma + 1)M_\infty^2]^{\frac{1}{3}}}{\tau^{\frac{2}{3}}} = F\left(\frac{1 - M_\infty^2}{[\tau(\gamma + 1)M_\infty^2]^{\frac{2}{3}}}\right) = F(\chi)$$

This holds for subsonic, transonic, and supersonic flow. For hypersonic flow the similarity parameter is $K = M_\infty\tau$. Van Dyke in ref. 8 showed that the parameter $K' = \sqrt{M_\infty^2 - 1}\,\tau$ could be used as a unified similarity parameter. More information may be found in ref. 9.

The well-known Prandtl-Glauert rule can be found as a special case of the above formula. Further details can be found in ref. 6. The power of similarity rules is shown in Fig. 2t-1, where data and theory for flow past different wedges can be directly compared if plotted in terms of the similarity parameter χ.

References

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