2u. Laminar and Turbulent Flow of Gases

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Symbols

C_D	drag coefficient
c_f	skin-friction coefficient
c_p	specific heat at constant pressure
d	pipe diameter
E	internal energy per unit mass
G	mass rate of flow per unit cross-sectional area of pipe
G_r	Grashof number
g	acceleration of gravity
K_N	Nusselt number
k	coefficient of heat conductivity, surface roughness
L	reference length (for Reynolds number)
P_r	Prandtl number
p	pressure
Q	external-heat-production rate per unit mass
R	gas constant, Reynolds number
r	pipe radius
r/k	surface-roughness factor
S_t	Stanton number
T	absolute temperature
T_e	adiabatic wall temperature
T_w	wall temperature
T_∞	free-stream temperature
t	time
u, v, w	velocity components of fluid flow
u	free-stream velocity
X, Y, Z	rectangular components of external body force
x, y, z	rectangular coordinates
δ	boundary-layer thickness
θ	momentum thickness
μ	coefficient of viscosity
ν	kinematic viscosity
ρ	density
τ_w	wall shear stress per unit area

2u-1. Equations of Motion. The study of the motion of any real gas or fluid must of necessity take into consideration the effects of viscosity. The transfer of momentum due to viscosity and the transformation of kinetic energy into heat must be considered in formulating the equations of motion. The following equations govern the motion of a viscous, compressible, heat-conducting gas. The viscosity and heat conductivity are assumed to be functions of the temperature only.

Momentum Equations. In rectangular coordinates, the momentum equations can be written as

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = \rho X + \frac{\partial}{\partial x}\left[\frac{4}{3}\mu\frac{\partial u}{\partial x} - \frac{2}{3}\mu\left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right]$$
$$+ \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\right] - \frac{\partial p}{\partial x}$$
$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = \rho Y + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right]$$
$$+ \frac{\partial}{\partial y}\left[\frac{4}{3}\mu\frac{\partial v}{\partial y} - \frac{2}{3}\mu\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right)\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right] - \frac{\partial p}{\partial y}$$
$$\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = \rho Z + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\right]$$
$$+ \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)\right] + \frac{\partial}{\partial z}\left[\frac{4}{3}\mu\frac{\partial w}{\partial z} - \frac{2}{3}\mu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\right] - \frac{\partial p}{\partial z} \quad \text{(2u-1)}$$

where μ is the coefficient of viscosity and the other terms are as defined in Sec. 2t.

Continuity Equation. The equation of continuity is

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0 \quad \text{(2u-2)}$$

Energy Equation. By using the first law of thermodynamics and by considering that heat conduction may take place in the gas, the following energy equation may be written

$$\rho\left(\frac{\partial E}{\partial t} + u\frac{\partial E}{\partial x} + v\frac{\partial E}{\partial y} + w\frac{\partial E}{\partial z}\right) + p\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)$$
$$= \rho Q + \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) + 2\mu\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right]$$
$$- \frac{2}{3}\mu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)^2 + \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 + \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2 + \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)^2 \quad \text{(2u-3)}$$

where k = heat-conductivity coefficient
E = internal energy per unit mass
Q = external-heat-production rate per unit mass
T = absolute temperature

The coefficients μ and k may be functions of the temperature T.

Equation of State. For a perfect gas the equation of state is

$$p = \rho RT \quad \text{(2u-4)}$$

Stream Function. For a steady flow in two dimensions or for axially symmetric flow a stream function may be defined as in Sec. 2t. It has great utility in boundary-layer work (see ref. 3).

2u-2. Definitions of Basic Parameters. The basic dimensionless parameters of a viscous, compressible, heat-conducting gas are usually considered to be the Mach number, the Reynolds number, the Prandtl number, and the Grashof number (see ref. 2). The Mach number has been defined in Sec. 2t. The other three parameters may be defined as follows:

Reynolds Number. In a flow with reference velocity u and reference length L, the Reynolds number R is defined as

$$R = \frac{uL}{\nu} \tag{2u-5}$$

where $\nu = \mu/\rho$ is the kinematic viscosity. Two viscous flows may not be dynamically similar unless their respective Reynolds numbers are the same.

Prandtl Number. The Prandtl number is defined as

$$P_r = \frac{\mu c_p}{k} \tag{2u-6}$$

where c_p is the specific heat at constant pressure. The Prandtl number depends only on the material properties of the gas.

The Prandtl number is primarily a function of the temperature only. For small temperature changes it is often assumed to be constant (see ref. 2). The variation of P_r with temperature is shown in Tables 2u-1 and 2u-2 for air and for molecular hydrogen H_2.

Grashof Number. The Grashof number may be defined as

$$G_r = \frac{L^3 g(T_1 - T_0)}{\nu^2 T_0} \tag{2u-7}$$

where g is the acceleration of gravity and T_1 and T_2 are two reference temperatures. The Grashof number is important in the study of flows with free convection, e.g., the flow of gas above a heated plate.

2u-3. Exact Solutions. Because of the extreme complexity of the equations of motion, few exact solutions have been found. Nearly all of these are limited to the incompressible steady flow case, with zero heat transfer through the walls bounding the flow. Since gases often behave as if they were nearly incompressible, these solutions may have practical importance.

Pipe Flow. The exact incompressible solution for two-dimensional or axially symmetric steady flow through a pipe of constant cross section is characterized by a parabolic velocity distribution. In the two-dimensional case the complete solution is given by

$$\begin{aligned} u &= -\frac{1}{2\mu} z(h - z) \frac{\partial p}{\partial x} \\ v &= w = 0 \\ \frac{\partial p}{\partial x} &= \text{const} \qquad \frac{\partial p}{\partial y} = \frac{\partial p}{\partial z} = 0 \end{aligned} \tag{2u-8}$$

where the boundaries are at $z = 0$ and $z = h$. In the case of flow through a circular pipe, the theoretical solution has been shown to coincide almost exactly with experiment for laminar flow.

Other Exact Solutions. There are a number of other exact solutions for the incompressible case such as steady flow between concentric cylinders and flow through tubes of noncircular cross section. These can be found by consulting refs. 1 and 3. Hamel (ref. 5) has found a number of nontrivial exact solutions.

2u-4. Boundary Layers. When the Reynolds number of the flow is large, most of the viscous effects take place in the immediate vicinity of the boundaries. The outer flow may then be considered determined by the inviscid flow equations while in the boundary layer certain simplifications of the equation of motion may be made. For the case of two-dimensional flow past flat or slowly curving surfaces the pressure may be assumed to be completely determined by the outer flow.

If the viscous effects are confined to a thin region next to a boundary, it then turns out that most of the viscous terms in Eqs. (2u-1) and (2u-3) can be neglected. The simplified equations are much easier to treat than the full equations.

TABLE 2u-1. PRANDTL NUMBER P_r FOR AIR

T, °K	P_r	T, °K	P_r
100	0.770	560	0.680
120	0.766	580	0.680
140	0.761	600	0.680
160	0.754	620	0.681
180	0.746	640	0.682
200	0.739	660	0.682
220	0.732	680	0.683
240	0.725	700	0.684
260	0.719	720	0.685
280	0.713	740	0.686
300	0.708	760	0.687
320	0.703	780	0.688
340	0.699	800	0.689
360	0.695	820	0.690
380	0.691	840	0.692
400	0.689	860	0.693
420	0.686	880	0.695
440	0.684	900	0.696
460	0.683	920	0.697
480	0.681	940	0.698
500	0.680	960	0.700
520	0.680	980	0.701
540	0.680	1000	0.702

Basic Equations. For two-dimensional steady flow as outlined above, the momentum, continuity, and energy equations are, respectively,

$$\begin{gathered}
\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) - \frac{\partial p}{\partial x} \\
0 = \frac{\partial p}{\partial y} \\
\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0 \\
\rho\left(u\frac{\partial E}{\partial x} + v\frac{\partial E}{\partial y}\right) + p\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \mu\left(\frac{\partial u}{\partial y}\right)^2
\end{gathered} \tag{2u-9}$$

For a perfect gas the equation of state is $p = \rho RT$. In the above equations x may be considered as the distance along the boundary while y is the distance perpendicular to the boundary. The velocity components u and v are interpreted in like manner. The equations then hold also for a slowly curving boundary.

Blasius Flow. For incompressible steady flow past a flat plate with no pressure gradient, the equations of motion are

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}$$
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (2u\text{-}10)$$

with the boundary conditions $u = v = 0$ at $y = 0$ and $u = u_1 = \text{const}$ at $y = \infty$

TABLE 2u-2. PRANDTL NUMBER FOR MOLECULAR HYDROGEN H_2*

T, °K	P_r	T, °K	P_r
60	0.713	440	0.684
80	0.711	460	0.681
100	0.712	480	0.678
120	0.715	500	0.675
140	0.718	520	0.671
160	0.719	540	0.669
180	0.720	560	0.667
200	0.719	580	0.665
220	0.717	600	0.664
240	0.715	620	0.663
260	0.712	640	0.663
280	0.709	660	0.662
300	0.706	680	0.661
320	0.703	700	0.661
340	0.699	720	0.661
360	0.696	740	0.660
380	0.693	760	0.660
400	0.690	780	0.660
420	0.687	800	0.660

* The values in Tables 2u-1 and 2u-2 are taken from the National Bureau of Standards, "NACA Tables of Thermal Properties of Gases" (cf. ref. 6).

and at $x = 0$. u_1 is the free-stream velocity. Blasius solved this problem by means of the change of variable

$$\eta = \frac{1}{2}\left(\frac{u_1}{\nu x}\right)^{\frac{1}{2}} y \qquad u = \frac{1}{2}u_1 f' \qquad v = \frac{1}{2}\left(\frac{u_1\nu}{x}\right)^{\frac{1}{2}}(\eta f' - f) \qquad (2u\text{-}11)$$

This reduces the problem to the ordinary differential equation and boundary conditions

$$\frac{d^3 f}{d\eta^3} + f\frac{d^2 f}{d\eta^2} = 0$$
$$f = f' = 0 \text{ at } \eta = 0 \qquad \text{and} \qquad f' = 2 \text{ at } \eta = \infty \qquad (2u\text{-}12)$$

2u-5. Turbulent Flow. For small values of the Reynolds number most flows are characterized by a certain uniformity of velocity distribution and smoothness of the streamline pattern. This type of flow is called laminar. As the Reynolds number is increased, the flow will remain laminar until a certain critical value of R is reached. At this time swirling or eddying motions begin to appear in the flow. These small-scale eddying motions move with the main flow but also possess an apparent random nature in the way they appear and decay. Such flows are called turbulent.

Turbulent flows also exhibit other striking features. The velocity distribution has a different behavior from that of laminar flow. The viscous drag and heat transfer

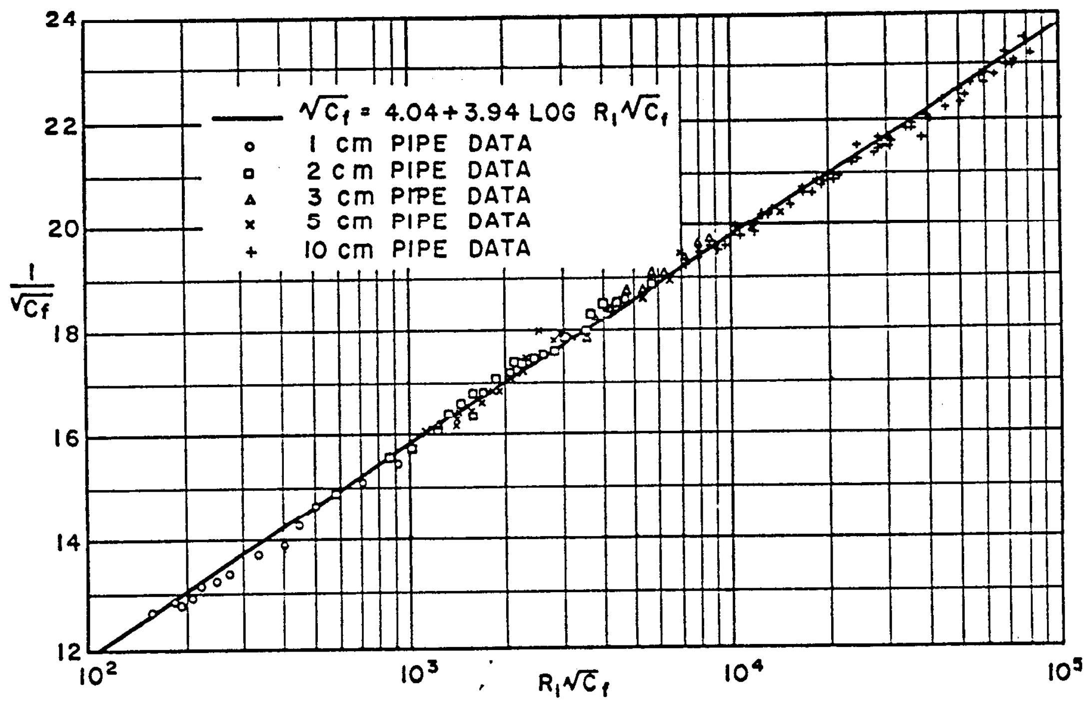

FIG. 2u-1. Universal wall-friction functional relation.

also undergo abrupt changes when turbulent flow begins. The sharp drop in the drag coefficient for the sphere shown in Fig. 2u-4 indicates the onset of turbulent flow.

2u-6. Data on Turbulent Flow through Pipes. The following data show the behavior of the skin friction for incompressible turbulent flow through smooth and rough pipes. These data come from Nikuradse (see refs. 7 and 8).

Smooth Pipes. The skin-friction coefficient c_f is a function of the Reynolds number R_1 for smooth pipes,

$$c_f = \frac{\tau_w}{\frac{1}{2}\rho u_1^2}$$

$$R_1 = \frac{r u_1}{\nu}$$

where τ_w = wall shear stress per unit area
ρ = density
ν = kinematic viscosity
u_1 = velocity in center of pipe
r = pipe radius

The behavior of c_f with R_1 is shown in Fig. 2u-1. An empirical curve which fits the data is also shown.

Rough Pipes. For rough pipes with average projection of the roughness k, the skin-friction data are shown in Fig. 2u-2. The friction factor λ is plotted against

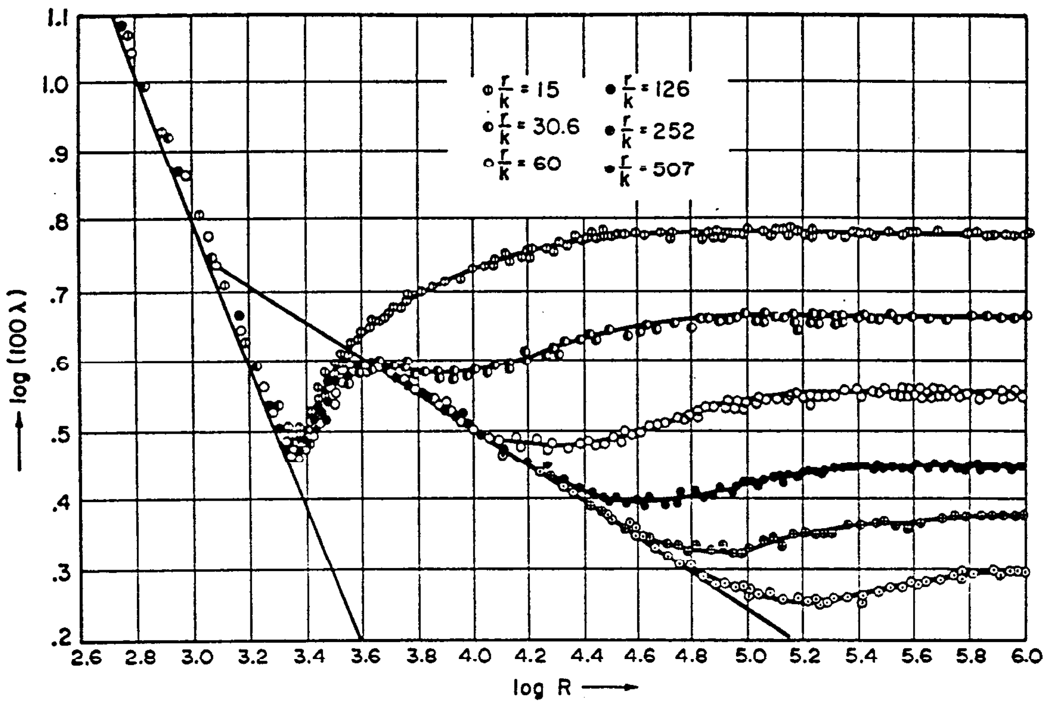

FIG. 2u-2. Relation between log (100λ) and log R (rough pipe).

Reynolds number R for various surface roughnesses r/k,

$$\lambda = 4c_f \left(\frac{u_1}{u}\right)^2$$

$\bar{u}$ = average velocity across pipe
d = pipe diameter
$\frac{r}{k}$ = roughness factor
r = pipe radius

2u-7. Drag Data for Spheres and Cylinders. For incompressible viscous steady flow the drag coefficient is a function of the Reynolds number only. The graphs of Figs. 2u-3 and 2u-4 give curves of the experimental data for C_D, the drag coefficient, for a cylinder in cross flow and for a sphere, respectively.

$$\text{Drag of cylinder } C_D = \frac{\text{drag force}}{\frac{1}{2}\rho u^2 d}$$

where d = diameter of cylinder
u = free-stream velocity
$R = ud/\nu$

$$\text{Drag of sphere } C_D = \frac{\text{drag force}}{\frac{1}{2}\rho u^2(\pi d^2/4)}$$

where d = diameter of sphere
$R = ud/\nu$

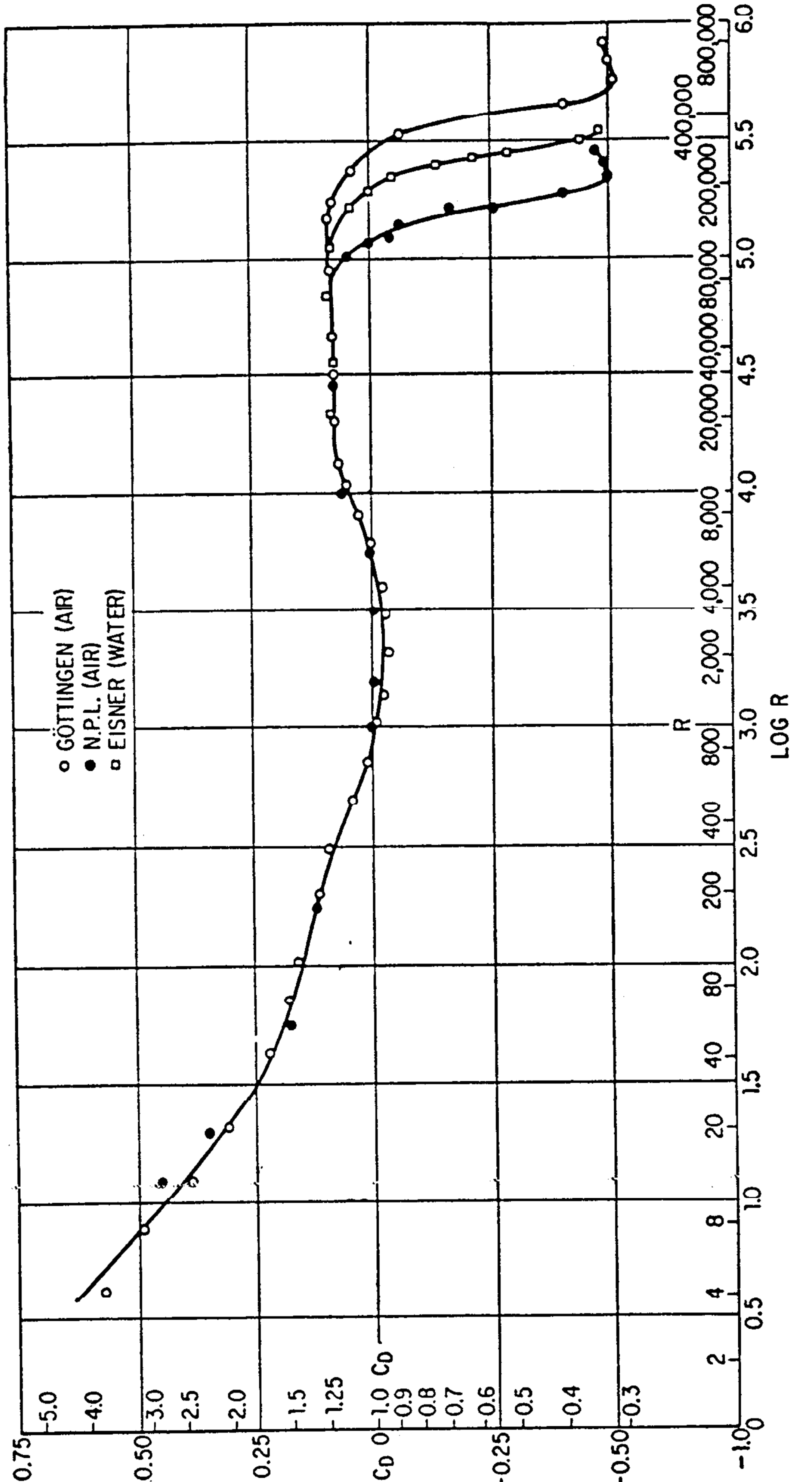

Fig. 2u-3. Relation between log C_D and log R (cylinder).

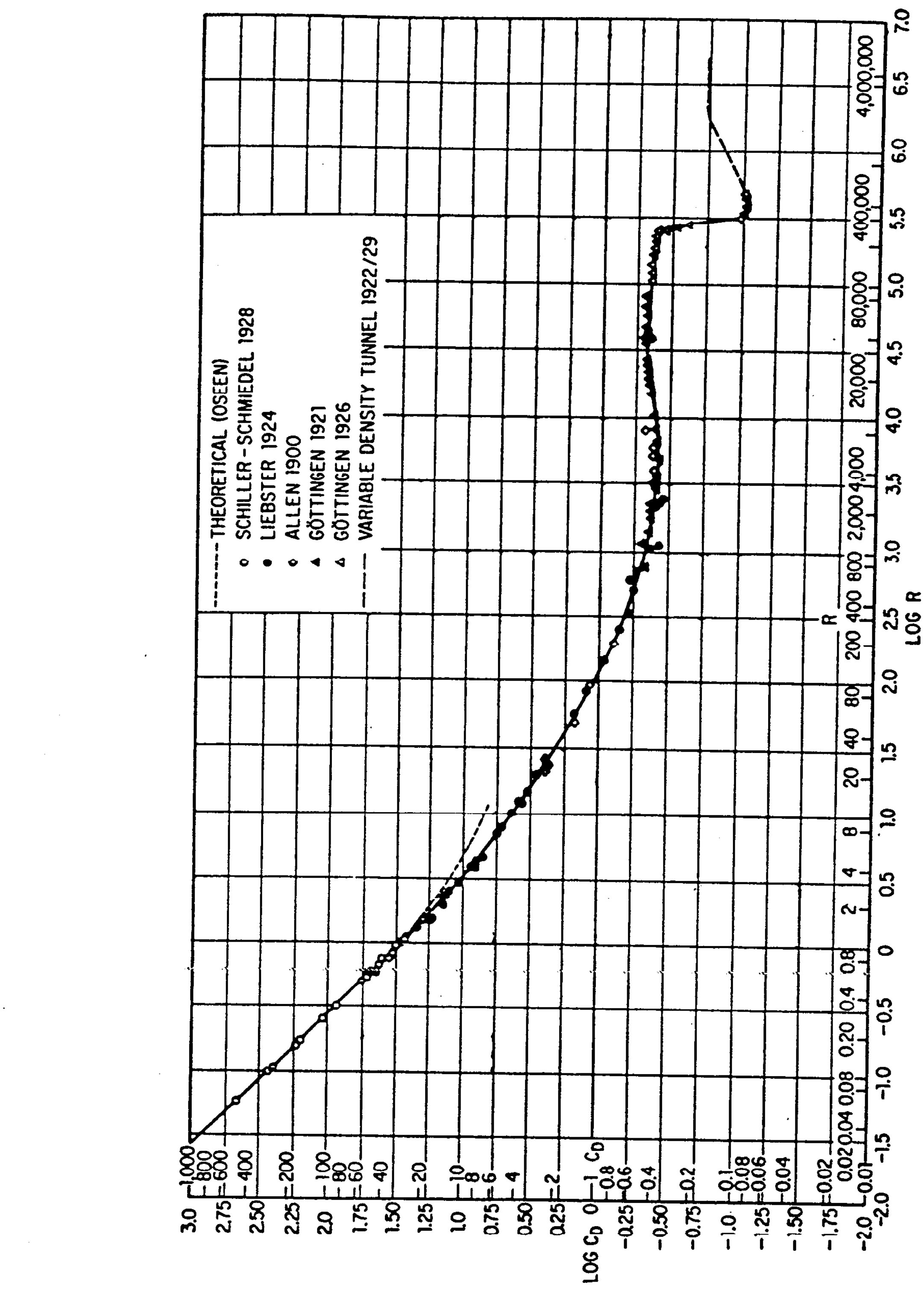

FIG. 2u-4. Relation between log C_D and log R (sphere).

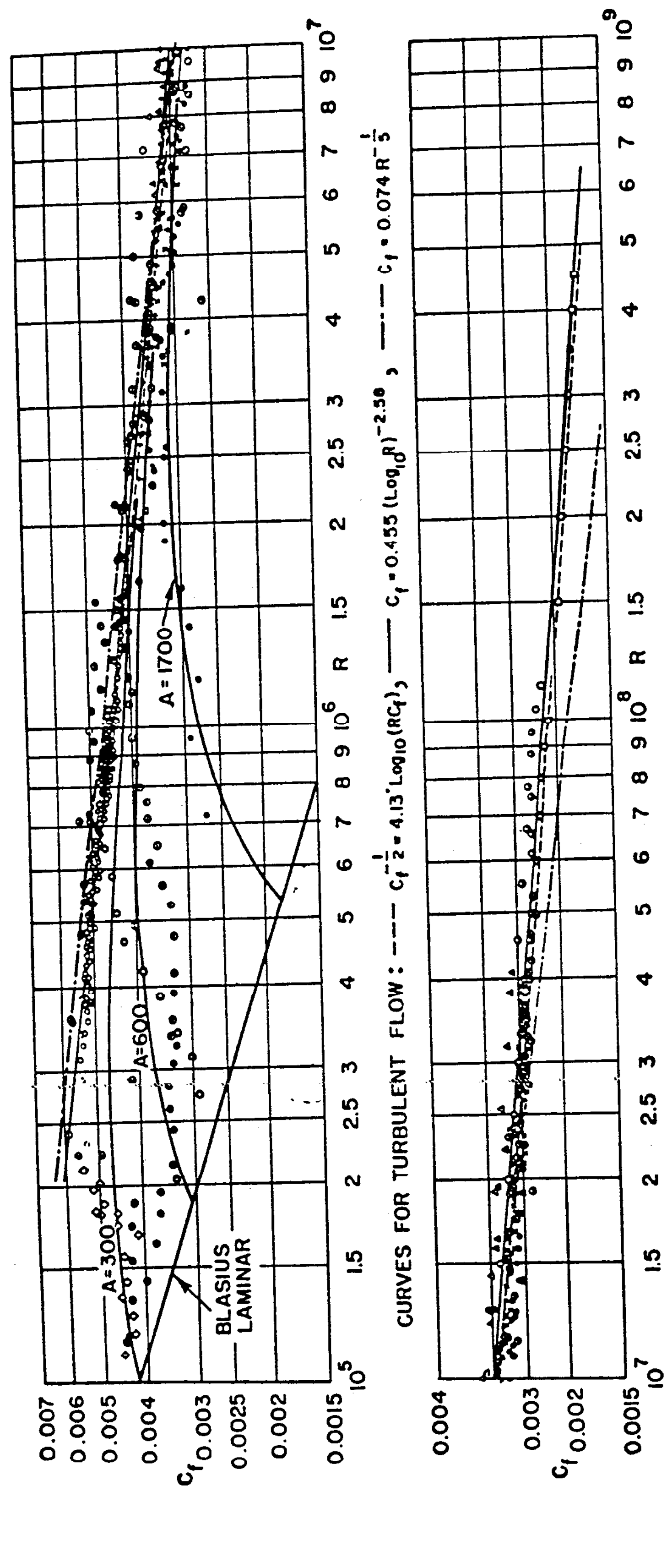

FIG. 2u-5. Curves for turbulent flow along flat plate.

2u-8. Skin-friction Data for a Flat Plate. Figure 2u-5 indicates the behavior of the skin-friction coefficient c_f with Reynolds number for a flat plate in an incompressible fluid. (More details can be found in refs. 1 to 4 and 14.)

$$c_f = \frac{\tau_w}{\frac{1}{2}\rho u^2}$$

where $R = ul/\nu$
l = length of plate

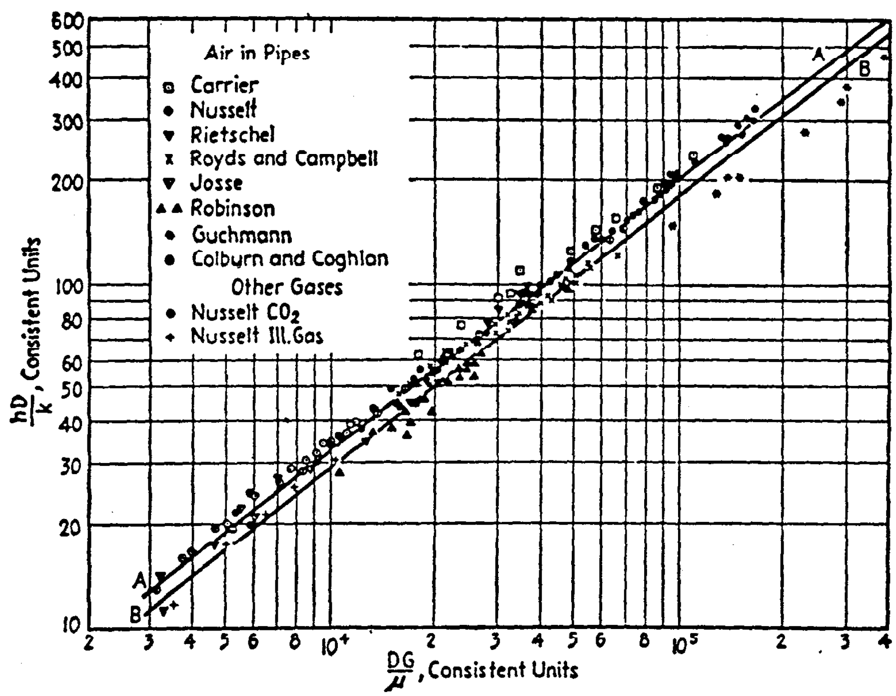

FIG. 2u-6. Data for gases inside tubes compared with recommended line *AA*. Line *BB* is obtained from the Reynolds analogy, taking $f = 0.049(DG/\mu)^{-0.2}$ and $c_p\mu/k = 0.74$. Line *BB* also represents the Prandtl analogy for r_v of 0.3.

2u-9. Heat-transfer Data. The transfer of heat from heated surfaces to gases moving past them is of great importance. This heat transfer is often expressed in dimensionless form in terms of the *Nusselt* number K_N,

$$K_N = \frac{hD}{k}$$

where h = coefficient of heat transfer
D = length
k = thermal conductivity

For incompressible flow K_N is a function of the Reynolds number only. The behavior K_N with R for pipe flow and for a flat plate is given below.

Pipe Flow. The variation of K_N with R for a circular pipe is given in Fig. 2u-6, where D is the pipe diameter.

$$R = \frac{DG}{\mu}$$

where $G = w/s$

w = mass rate of flow

s = cross-sectional area of pipe

Flat Plate. For a flat plate the variation of K_N with R for small R is shown in Fig. 2u-7, where D is the length of the flat plate. For higher values of R recourse must be made to empirical formulas converting the pipe-flow into equivalent flat-plate data (see page 117 of ref. 10) or to *Reynolds analogy.*

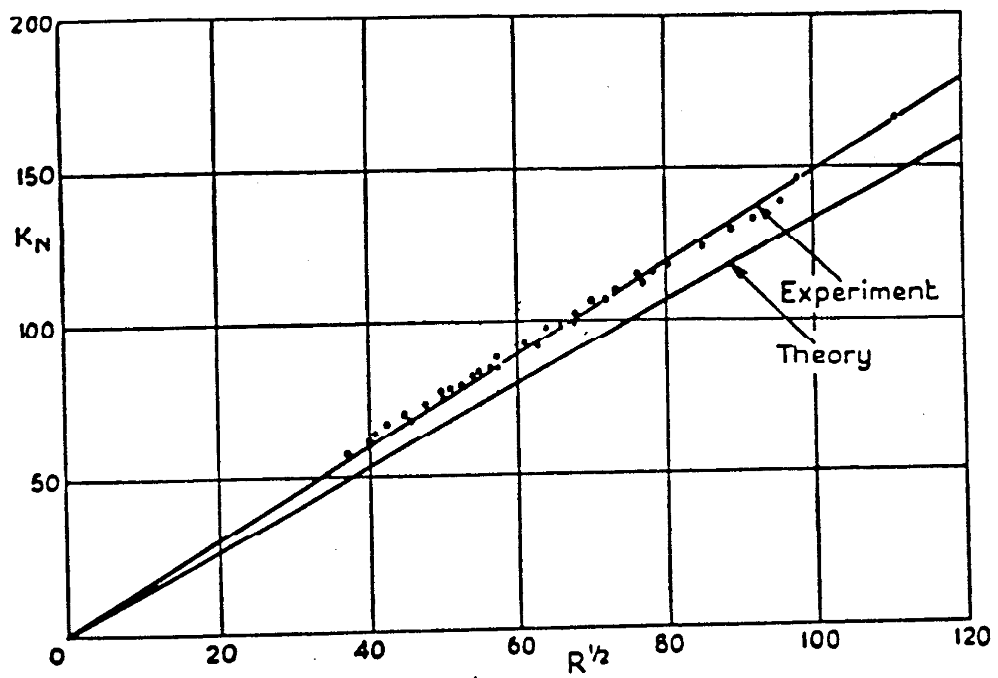

FIG. 2u-7. Comparison between theory and experiment for heat transfer from plate.

Reynolds analogy (see ref. 14) says that heat transfer and skin friction are related in the following way:

$$S_t = \tfrac{1}{2}c_f$$

where S_t is the Stanton number.

$$S_t = \frac{q_w}{\rho_w c_p u_1 (T_w - T_1)}$$

and c_f is the skin-friction coefficient

$$c_f = \frac{\tau_w}{\frac{1}{2}\rho_w u_1^2}$$

The subscripts w and 1 refer to variables at the wall and at the outer edge of the boundary layer, respectively. This analogy holds only approximately and must be modified for compressible flow and high Mach numbers. The extensions of the analogy are given in ref. 14.

2u-10. Effect of Compressibility and Heat Transfer on Skin Friction. For a fixed Reynolds number the ratio of the local skin-friction coefficient c_f to the corresponding incompressible value c_{f_i} is a function of the Mach number and the heat transfer. The graph shown in Fig. 2u-8, taken from ref. 12, represents an excellent theoretical fit to data from refs. 11 and 13. The curves are plotted for zero heat transfer where

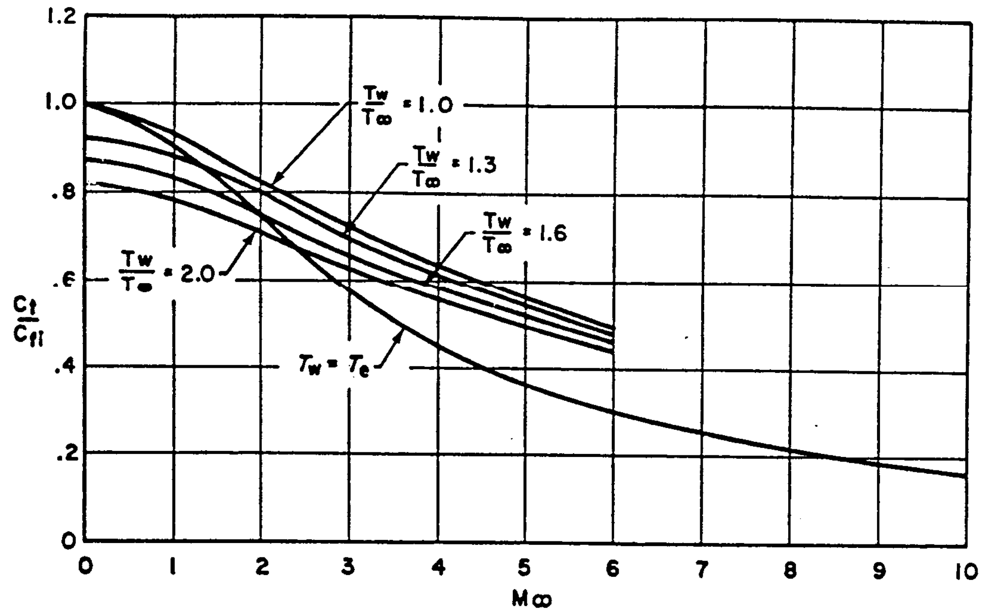

FIG. 2u-8. Variation of skin-friction ratio with Mach number for several constant values of wall-temperature ratio and $Re\theta$ = 13,500.

$T_e = T_w$ and several different constant heat-transfer conditions. The graph is for a single representative Reynolds number R_θ based on momentum thickness.

T_w = wall temperature
T_e = adiabatic wall temperature
T_∞ = free-stream temperature

$$R_\theta = \frac{u_\infty \theta}{\nu}$$

$$\theta = \text{momentum thickness} = \int_0^\delta \frac{\rho u}{\rho_1 u_1}\left(1 - \frac{u}{u_1}\right) dy$$

δ = boundary-layer thickness
ρ_1 = density outside boundary layer
u_1 = velocity outside boundary layer

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