3c. Propagation of Sound in Fluids

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3c-1. Glossary of Symbols[1]

a, a_1; a_i	material coordinate (31); surface element (12)
A; $\mathbf{A}_1$	surface (12), attenuation per wavelength (76), Avogadro's number (95); first order vector potential
B	coefficient relating $\nabla\rho$ and ∇p (58)
c, c_0; c^0, c^∞	speed of sound, reference speed (25); low- and high-frequency limit speeds (84)
c'	speed of thermal wave (78*b*)
C_p, C_v	specific heats at constant pressure, constant volume (14)
d_{ij}	rate of deformation tensor (9)
D	material differential operator (2)
E, F, G, H	algebraic abbreviations (74)
E, E_k, E_I; E_{diss}	energy densities per unit mass (60), (12); degraded component of internal energy (66)
f, $\mathbf{f}_v$, $f(\quad)$, $f(h)$	frequency, sum of viscosity terms (62), "function of" (45), special tabulated function (75)
Δf_c	critical bandwidth (98)
F_i, $\mathbf{F}$	vector body force per unit mass (6)
$g(h)$	tabulated function (75)
h	material mass coordinate (37), argument of tabulated function (75), Planck's constant (89)
i, j, k	coordinate indexes (1)
$\bar{\mathbf{I}}$	average sound-energy-flux density = sound intensity (64)
j	designation of imaginary axis, $[e^{+j\omega t}]$ (69)
$\mathbf{J}$	sound-energy flux vector (54)
k, k_0	phase constant $= \omega/c = 2\pi/\lambda$, Boltzmann's constant (89), $k_0 = \omega/c_0 = 2\pi/\lambda_0$ (47)
K; K_s, K_0, K_T	elastic modulus $= -V(DP/DV)$ (25), material constant $= c^0/c^\infty$ (84); isentropic modulus, reference modulus, isothermal modulus
L	mean free path (86), a sum of linear dimensions (90)
M	peak particle-velocity Mach number $= \omega\xi_0/c_0$ (49), molecular weight (95)
n_V	total number of molecules per unit volume (95)
N	number of modes of vibration (90)
$O(\quad)$	additive terms of indicated order of magnitude (76)

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[1] Numbers indicate equation number in or near which quantity is defined.

$p; p_1, p_2$	incremental, or sound, pressure; first- and second-order sound pressures (25)
$P, P_0; P_m, P_{th}$	total pressure (7), equilibrium or reference pressure (25); mean pressure (7), thermodynamic pressure (14)
$P_1, P_2; \mathcal{P}$	rms fundamental and second-harmonic pressure (49*a*); Prandtl number (72)
$\mathbf{q}, q_i; \mathsf{q}$	heat flux vector (12); Stokes radiation coefficient (21*b*)
$q; q^E, q^L$	exemplar of state or condition variable (30); superscript indicates function of spatial (*E*) variables, or material (*L*) variables (32*b*)
$\mathbf{R}, R; R_1, R_2$	vorticity $= \frac{1}{2}\nabla \times \mathbf{u}$ (11*d*), real part of complex impedance; first- and second-order components of vorticity (57)
s, s_1	specific entropy per unit mass (14), first-order condensation $= \rho_1/\rho_0$ (59)
$S; S'; S_{irr}$	Stokes number $= \omega\eta/\rho_0 c_0^2$ (72), total interior surface (90); frequency number for radiation $= \omega/\mathsf{q}$ (72); entropy generated irreversibly (15*a*)
$t; t_{ij}$	time (2); stress tensor (6)
T	absolute temperature (14)
$\mathbf{u}, u_1; u_1, u_2, u_3$	particle velocity (1); velocity components
$\mathbf{u}_1, \mathbf{u}_2$	first- and second-order components of particle velocity (25)
$v; \bar{v}$	specific volume $= \rho^{-1}$ (1); mean molecular velocity (86)
$\mathcal{V}$	viscosity number $= 2 + \eta'/\eta$ (10)
$V; V_{ij}$	volume (1); residual stress tensor (7)
x_1, x_2, x_3	cartesian coordinates (1)
$X; X'$	frequency number $= \omega\eta\mathcal{V}/\rho_0 c_0^2$ (72), specific acoustic reactance (69); frequency number for relaxation (84)
Y	thermoviscous number $= \kappa/\eta\mathcal{V}C_p$ (72)
z, Z	specific acoustic impedance ratio (87), and impedance (69)
$\alpha; \alpha_K, \alpha_C$	attenuation constant (69); "Kirchhoff" and "classical" attenuation (79*a,b*)
$\beta; \beta_{noise}$	coefficient of thermal expansion $= \rho(\partial v/\partial T)_P$ (22); spectrum level $= 10 \log_{10} [d(p^2/p_0^2)/df]$ (98)
γ	ratio of specific heats $= C_p/C_v$ (14)
$\delta; \delta_{ij}; \Delta$	finite increment (32); Kronecker delta (7); dilatation rate $= \nabla \cdot \mathbf{u}$ (4)
ϵ	specific internal energy per unit mass (13)
η, η', η_B	coefficient of shear viscosity (10), "second" or dilatational viscosity (10), bulk viscosity (10)
θ_1, θ_2	first- and second-order variational components of temperature (25)
κ	thermal conductivity (21*a*)
$\lambda; \lambda_0$	wavelength $= c/f$ (47); $\lambda_0 = c_0/f$
ν, ν', ν_B	kinematic viscosity coefficients (10) $= \eta/\rho$, etc.
$\xi; \xi_t$	displacement of particle from equilibrium (31); partial derivative with respect to subscript variable (41*b*)
$\rho, \rho_0; \rho_1, \rho_2$	densities: total, equilibrium; first- and second-order variational components
τ_r, τ_v, τ_k	relaxation times (83, 85)
$\varphi_s; \phi_\eta, \phi_k$	scalar velocity potential (55); viscous and thermal dissipation functions (16, 18)
χ	complex propagation constant $= \alpha + jk$ (69)
ψ	functional relation (71)
$\omega; \omega_r, \omega_v, \omega_k$	angular frequency $= 2\pi f$; relaxation angular frequencies (84)
$\nabla, \nabla\cdot, \nabla\times$	gradient, divergence, and curl operators
$\langle\ \rangle$	time average

3c-2. The Motion of Viscous Fluids. The motions of a fluid medium that comprise sound waves are governed by equations that include (1) a continuity equation expressing the conservation of mass, (2) a force equation expressing the conservation of momentum, (3) a heat-exchange equation expressing the conservation of energy, and (4) one or more defining equations expressing the constitutive relations that characterize the medium and its response to thermal or mechanical stress. These equations will first be presented in their complete exact form in order to provide a rigorous point of departure for the approximations that must ultimately be made in formulating the linearized, or small-signal, acoustic equations.

The transformation properties of these equations can be indicated by writing them in either vectorial or tensorial form, and both forms will be exhibited in order to facilitate contacts with the rich literature dealing with the motion of fluids.[1]

Cartesian spatial coordinates will be designated x_1, x_2, x_3, and the vector velocity of a material particle will be identified as $\mathbf{u}$ with components u_1, u_2, u_3. These will also be written as x_i and u_i, where it is implied that the subscript i, j, or k takes on successively the values 1, 2, 3. The term "material particle" denotes a finite mass element of the medium small enough for the values assumed by the state variables at every interior point of the particle not to differ significantly from the values they have at the interior reference point whose coordinates "locate" the particle.

Equation of Continuity. The conservation of mass requires that $\rho V = \rho_0 V_0$, where ρ_0 and V_0 are initial and ρ and V are subsequent values assumed by the density and volume of a particular material element of the medium. It follows that

$$\rho DV + V D\rho = 0 \qquad \frac{DV}{V} = -\frac{D\rho}{\rho} \tag{3c-1}$$

If $\rho_0 V_0$ is set equal to 1, V_0 becomes the *specific volume*, $v \equiv 1/\rho$, whence the relation between the total logarithmic time derivatives of v and ρ is

$$\frac{1}{v}\frac{Dv}{Dt} = -\frac{1}{\rho}\frac{D\rho}{Dt} = \frac{D \log v}{Dt} = -\frac{D \log \rho}{Dt} \tag{3c-2}$$

where $D(\ )/Dt$ denotes the "material" derivative, i.e., one that follows the motion of a material "particle" of the medium relative to a fixed spatial coordinate system, and is defined by

$$\frac{D(\ )}{Dt} \equiv \frac{\partial(\ )}{\partial t} + \mathbf{u}\cdot \operatorname{grad}(\ ) \equiv \frac{\partial(\ )}{\partial t} + u_i \frac{\partial(\ )}{\partial x_i} \tag{3c-3}$$

Analysis of the rate of deformation of a volume element yields the kinematical relation

$$\frac{1}{v}\frac{Dv}{Dt} = \operatorname{div}\mathbf{u} \equiv \Delta = \frac{\partial u_i}{\partial x_i} \tag{3c-4}$$

where Δ is the *dilatation rate.* Note that in the last terms of (3c-3) and (3c-4) summation is implied over all the allowable values of the subscript index. Equations (3c-2), (3c-3), and (3c-4) can be combined to yield the following equivalent forms of Euler's *continuity equation:*

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[1] A definitive restatement of the classical-continuum point of view, with critical comments on more than 800 bibliographical references, has been given by C. Truesdell, The Mechanical Foundations of Elasticity and Fluid Dynamics, *J. Rational Mechanics and Analysis* **1**, 125–300 (January and April, 1952), and Corrections and Additions . . . , *J. Rational Mechanics and Analysis* **2**, 593–616 (July, 1953). See also Lamb, "Hydrodynamics," 6th ed., Dover Publications, New York, 1945; Rayleigh, "Theory of Sound," 2d ed., rev., Dover Publications, New York, 1945; and L. Howarth, ed., "Modern Developments in Fluid Dynamics," vol. I, chap. III, Oxford University Press, New York, 1953.

$$\begin{aligned}\frac{D\rho}{Dt} + \rho\frac{\partial u_i}{\partial x_i} &= \frac{\partial\rho}{\partial t} + u_i\frac{\partial\rho}{\partial x_i} + \rho\frac{\partial u_i}{\partial x_i} = \frac{D\rho}{Dt} + \rho\,\mathrm{div}\,\mathbf{u} = 0\\ &= \frac{1}{\rho}\frac{D\rho}{Dt} + \Delta = \frac{\partial\rho}{\partial t} + \mathbf{u}\cdot\mathbf{grad}\,\rho + \rho\,\mathrm{div}\,\mathbf{u}\\ &= \frac{\partial\rho}{\partial t} + \mathbf{u}\cdot\nabla\rho + \rho\nabla\cdot\mathbf{u} = \frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{u})\end{aligned}\tag{3c-5}$$

In the last line of (3c-5), the Gibbs-Hamilton notation has been used for the differential vector operators, $\nabla \equiv$ **grad**; $\nabla\cdot \equiv$ div; $\nabla\times \equiv$ **curl.**

Force Equation. The linear-momentum principle can be stated in terms of Cauchy's first law of motion,

$$\rho\frac{Du_i}{Dt} = \rho F_i + \frac{\partial t_{ij}}{\partial x_j}\tag{3c-6}$$

where the vector F_i is an extraneous body force per unit mass, and where t_{ij} is a second-rank *stress tensor* that represents the net mechanical action of contiguous material on a volume element of the medium due to the actual forces of material continuity. For an isotropic medium in which the stress is a linear function of the rate of deformation, as here assumed, the stress tensor can be resolved arbitrarily as the sum of a scalar, or hydrostatic, pressure function P and a residual stress tensor V_{ij} defined by

$$t_{ij} = -P\delta_{ij} + V_{ij} \qquad t_{ij} = t_{ji}\tag{3c-7}$$

where δ_{ij} is the Kronecker delta which equals unity if $i = j$, but is zero otherwise. Unless V_{ii} vanishes, P is *not* identical with the mean pressure, $P_m = -\frac{1}{3}t_{ii}$. The resolution given by (3c-7) is both unique and useful, however, if P is made equal to the thermodynamic pressure P_{th} defined below. Then the residual stress tensor is given, to a first approximation, by the linear terms of an expansion in powers of the viscosity coefficients,

$$V_{ij} = \eta' d_{kk}\delta_{ij} + 2\eta d_{ij} \qquad V_{ij} = V_{ji}\tag{3c-8}$$

in which d_{ij} is the *rate of deformation* tensor defined by

$$d_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\tag{3c-9}$$

and where η is the "first," or conventional shear, viscosity coefficient. In accordance with current proposals for standardization, η' replaces λ, the symbol used by Stokes, Rayleigh, Lamb, et al., to designate the "second," or dilatational, viscosity coefficient. The term "bulk" viscosity is reserved for $(\lambda + \frac{2}{3}\mu) \rightarrow (\eta' + \frac{2}{3}\eta)$, the linear combination of coefficients that vanishes when the *Stokes relation* holds. Thus, $\eta \equiv$ first, or shear, viscosity; $\eta' \equiv$ second, or dilatational, viscosity; $\eta_B \equiv \eta' + \frac{2}{3}\eta =$ bulk viscosity; $\nu \equiv \eta/\rho$; $\nu' \equiv \eta'/\rho$; $\nu_B \equiv \eta_B/\rho$ (kinematic viscosities);

$$(\lambda + 2\mu) \rightarrow \eta' + 2\eta = \eta_B + \frac{4}{3}\eta = \eta\left(\frac{4}{3} + \frac{\eta_B}{\eta}\right) = \eta\mathcal{V}\tag{3c-10}$$

$$\mathcal{V} \equiv \frac{4}{3} + \frac{\eta_B}{\eta} = 2 + \frac{\eta'}{\eta} \equiv \text{viscosity number}$$

Putting (3c-7), (3c-8), (3c-9) into (3c-6) yields the vector *force equation* in the following equivalent forms:

$$\begin{aligned}\rho\frac{\partial u_i}{\partial t} + \rho u_j\frac{\partial u_i}{\partial x_j} &= \rho F_i - \frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_j}(\eta' d_{kk}\delta_{ij} + 2\eta d_{ij})\\ &= \rho F_i - \frac{\partial P}{\partial x_i} + \eta'\frac{\partial^2 u_k}{\partial x_i\partial x_k} + \eta\frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\\ &\qquad + \frac{\partial u_k}{\partial x_k}\frac{\partial\eta'}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\frac{\partial\eta}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\frac{\partial\eta}{\partial x_j}\end{aligned}\tag{3c-11a}$$

$$\rho \frac{D\mathbf{u}}{Dt} = \rho\mathbf{F} - \mathbf{grad}\, P + (\eta' + \eta)\, \mathbf{grad}\, (\mathrm{div}\, \mathbf{u}) + \eta\nabla^2(\mathbf{u})$$
$$+ (\mathrm{div}\, \mathbf{u})\, \mathbf{grad}\, \eta' + 2\, (\mathbf{grad}\, \eta \cdot \mathbf{grad})\, \mathbf{u} + \mathbf{grad}\, \eta \times \mathbf{curl}\, \mathbf{u} \qquad \text{(3c-11}b\text{)}$$

$$\rho \frac{\partial\mathbf{u}}{\partial t} = \rho\mathbf{F} - \rho(\mathbf{u}\cdot\nabla)\mathbf{u} - \nabla P + (\eta' + 2\eta)\nabla(\nabla\cdot\mathbf{u}) - \eta\nabla \times (\nabla \times \mathbf{u})$$
$$+ (\nabla\cdot\mathbf{u})\nabla\eta' + 2(\nabla\eta\cdot\nabla)\mathbf{u} + \nabla\eta \times (\nabla \times \mathbf{u}) \qquad \text{(3c-11}c\text{)}$$

The vorticity, defined by $\mathbf{R} = \frac{1}{2}\, \mathbf{curl}\, \mathbf{u} = \frac{1}{2}(\nabla \times \mathbf{u})$, and the dilatation rate, $\Delta \equiv \nabla \cdot \mathbf{u}$, can be introduced as useful abbreviations. A somewhat more symmetrical expression in terms of the mass transport velocity $\rho\mathbf{u}$ is obtained if the last form of the continuity equation (3c-5) is multiplied by $\mathbf{u}$ and added to (3c-11*c*), giving

$$\frac{\partial(\rho\mathbf{u})}{\partial t} + \mathbf{u}(\nabla\cdot\rho\mathbf{u}) + (\rho\mathbf{u}\cdot\nabla)\mathbf{u} = \rho\mathbf{F} - \nabla P + \eta\mathcal{V}\nabla\Delta - 2\eta\nabla \times \mathbf{R} + \Delta\nabla\eta'$$
$$+ 2(\nabla\eta\cdot\nabla)\mathbf{u} + 2\nabla\eta \times \mathbf{R} \qquad \text{(3c-11}d\text{)}$$

These equations reduce to the so-called *Navier-Stokes equations* when it is assumed that η and η' are constant ($\nabla\eta = \nabla\eta' = 0$) and that the Stokes relation holds ($\eta_B = 0$, $\mathcal{V} = \frac{4}{3}$); and still further simplification follows if the motion is assumed irrotational so that $\mathbf{R} = 0$. If the viscosity coefficients are to be regarded as functions of one or more of the state variables, however, the gradients of the η's must be retained so that the implicit functional dependence can be introduced by writing, for example, $\nabla\eta = (\partial\eta/\partial T)\nabla T + \cdots$.

Energy Relations and Equations of State. The conservation of energy requires that the following power equation be satisfied:

$$\frac{D(E_k + E_I)}{Dt} = \int_V \rho F_i u_i\, dV + \int_A t_{ij} u_j\, da_i - \int_A q_i\, da_i \qquad \text{(3c-12)}$$

where E_k is the kinetic energy associated with the material velocity, E_I is the total internal energy, V is a volume bounded by the surface A, da_i is the projection of a surface element of A on the plane normal to the $+x_i$ axis, F_i is the extraneous body force (per unit mass), and q_i is the total heat flux vector (mechanical units). After the surface integrals are converted to volume integrals by using the divergence theorem, and with the help of (3c-6), this equation reduces to the Fourier-Kirchhoff-C. Neumann[1] energy equation,

$$\rho \frac{D\epsilon}{Dt} = t_{ij} d_{ij} - \frac{\partial q_i}{\partial x_i} \qquad \text{(3c-13)}$$

where ϵ is the local value of the specific internal energy (per unit mass) defined through $E_I = \int_V \rho\epsilon\, dV$. It is now postulated that the state of the fluid is completely specified by ϵ and two other local state variables, which can be taken as the specific entropy s (per unit mass) and the specific volume $v = \rho^{-1}$, in terms of which the thermodynamic pressure and temperature and the specific heats can be defined by

$$\begin{array}{lll} \epsilon = \epsilon(s,v) & P_{\mathrm{th}} \equiv -\left(\dfrac{\partial\epsilon}{\partial v}\right)_s & T \equiv \left(\dfrac{\partial\epsilon}{\partial s}\right)_v \\ C_p \equiv T\left(\dfrac{\partial s}{\partial T}\right)_p & C_v \equiv T\left(\dfrac{\partial s}{\partial T}\right)_v & \gamma \equiv \dfrac{C_p}{C_v} \end{array} \qquad \text{(3c-14)}$$

The second law of thermodynamics can be introduced in the form of an equality, which replaces the classical Clausius-Duhem inequality, through the expedient of accounting explicitly for the creation of entropy S_{irr} (per unit volume) by irreversible

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[1] See footnote, p. 3-39.

dissipative processes;[1] thus

$$\frac{D}{Dt}\int_V \rho s\,dV = -\int_A \frac{q_i}{T}\,da_i + \int_V \frac{DS_{\text{irr}}}{Dt}\,dV \tag{3c-15a}$$

This relation states that the increase of entropy in a material element is accounted for by the influx of heat and by the irreversible production of entropy within the element. The left-hand side of (3c-15a) can also be written, with the help of the continuity relation, as $\int_V \rho(Ds/Dt)\,dV$. Then, after converting the surface integral to a volume integral, the second law can be given in differential form as

$$\begin{aligned}\rho\frac{Ds}{Dt} &= -\frac{\partial}{\partial x_i}\frac{q_i}{T} + \frac{DS_{\text{irr}}}{Dt}\\ &= -\frac{1}{T}\frac{\partial q_i}{\partial x_i} + \frac{q_i}{T^2}\frac{\partial T}{\partial x_i} + \frac{DS_{\text{irr}}}{Dt}\end{aligned} \tag{3c-15b}$$

A thermal-dissipation function ϕ_κ can be defined by

$$\phi_\kappa = -\frac{q_i}{T}\frac{\partial T}{\partial x_i} \tag{3c-16}$$

whereupon multiplying (3c-15b) by T yields the second-law equality in the form

$$\rho T\frac{Ds}{Dt} = -\frac{\partial q_i}{\partial x_i} - \phi_\kappa + T\frac{DS_{\text{irr}}}{Dt} \tag{3c-15c}$$

Taking the material derivative of the basic equation of state (3c-14_1) (where the subscript added to an equation number indicates the serial number of the equality sign to which reference is made when several relations are grouped under one marginal identification number), introducing the definitions for P_{th} and T, multiplying by ρ, and using (3c-4), gives

$$\rho T\frac{Ds}{Dt} = \rho\frac{D\epsilon}{Dt} + P_{\text{th}}\Delta \tag{3c-17}$$

The energy equation (3c-13) can be recast, using (3c-7) and (3c-9), in the form

$$\rho\frac{D\epsilon}{Dt} + P\Delta + \frac{\partial q_i}{\partial x_i} = V_{ij}d_{ij} = \phi_\eta \tag{3c-18}$$

in which $V_{ij}d_{ij}$, the dissipative component of the stress power $t_{ij}d_{ij}$, is defined as the viscous dissipation function ϕ_η. The usefulness of specifying the arbitrary scalar in (3c-7) as the thermodynamic pressure, so that $P = P_{\text{th}}$, becomes apparent when $\rho\,D\epsilon/Dt$ is eliminated between (3c-18) and (3c-17), giving

$$\begin{aligned}\rho T\frac{Ds}{Dt} &= (P_{\text{th}} - P)\Delta + \phi_\eta - \frac{\partial q_i}{\partial x_i}\\ &= \phi_\eta - \frac{\partial q_i}{\partial x_i}\end{aligned} \tag{3c-19}$$

The viscous dissipation function (dissipated energy per unit volume) is thus seen to account for either an efflux of heat or an increase of entropy. Subtracting (3c-19) from (3c-15c) then allows the rate of irreversible production of entropy to be evaluated directly in terms of the two dissipation functions,

$$T\frac{DS_{\text{irr}}}{Dt} = \phi_\eta + \phi_\kappa \tag{3c-20}$$

The total heat-flux vector q_i, whose divergence is the energy transferred *away* from the volume element, must account for energy transport by either conduction or radi-

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[1] Tolman and Fine, *Revs. Modern Phys.* **20,** 51–77 (1948).

ation. The part due to conduction is given by the Fourier relation, which serves also to define the heat conductivity κ,

$$(q_i)_{\text{cond}} = -\kappa \frac{\partial T}{\partial x_i}$$

$$\frac{\partial (q_i)_{\text{cond}}}{\partial x_i} = -\frac{\partial(\kappa \partial T/\partial x_i)}{\partial x_i} = -\kappa \frac{\partial^2 T}{\partial x_i{}^2} - \frac{\partial T}{\partial x_i}\frac{\partial \kappa}{\partial x_i} \tag{3c-21a}$$

The last term, containing the gradient of κ, must be retained if implicit dependence of κ on the state variables is to be represented. On the other hand, if κ is assumed to be constant, (3c-21a) reduces to the more familiar form

$$\nabla \cdot \mathbf{q}_{\text{cond}} = -\kappa \nabla^2 T$$

The component of heat flux due to radiation can be approximated, for small temperature differences, by Newton's law of cooling,

$$\frac{\partial (q_i)_{\text{rad}}}{\partial x_i} = \rho C_v \mathfrak{q}(T - T_0) = \nabla \cdot \mathbf{q}_{\text{rad}} \tag{3c-21b}$$

where $(T - T_0)$ is the local temperature excess and $\mathfrak{q}$ is a radiation coefficient introduced by Stokes.[1] The foregoing thermal relations can be combined with the equations of continuity and momentum more readily if the term $T(Ds/Dt)$ appearing in (3c-19) is expressed in terms of the variables $\mathbf{u}$, v, and T. The defining equations (3c-14) establish that $P = P(v,s)$ and $T = T(v,s)$, from which it follows that one may also write $s = s(T,v)$ or $s = s(T,P)$. Using both of the latter leads, after some manipulation,[2] to the identity

$$\rho T \frac{Ds}{Dt} = \rho C_v \left[(\gamma - 1)\frac{\Delta}{\beta} + \frac{DT}{Dt} \right] \tag{3c-22}$$

in which β is the coefficient of thermal expansion, $\beta \equiv \rho(\partial v/\partial T)_P$. After (3c-22) and (3c-21) are combined with (3c-19), the energy equation can be written in the alternate forms

$$\frac{\rho C_v DT}{Dt} + \rho C_v \frac{\gamma - 1}{\beta}\frac{\partial u_i}{\partial x_i} + \frac{\partial q_i}{\partial x_i} - \phi_\eta = 0$$

$$\rho C_v \left(\frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T \right) + \frac{\rho (C_p - C_v)}{\beta} \Delta - \nabla \cdot (\kappa \nabla T) + \rho C_v \mathfrak{q}(T - T_0) - \phi_\eta = 0 \tag{3c-23}$$

$$\frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T + \frac{(\gamma - 1)}{\beta}\Delta - \frac{\kappa}{\rho C_v}\nabla^2 T - \frac{\nabla T \cdot \nabla \kappa}{\rho C_v} + \mathfrak{q}(T - T_0) - \frac{\phi_\eta}{\rho C_v} = 0$$

The viscous dissipation function ϕ_η can be evaluated, with the aid of (3c-8) and (3c-9) in the explicit form

$$\begin{aligned} \phi_\eta &= V_{ij} d_{ji} = \eta' d_{kk} d_{ii} + 2\eta d_{ij} d_{ji} \\ &= \eta_B \Delta^2 + \frac{4}{3}\eta \left[\left(\frac{\partial u_1}{\partial x_1}\right)^2 + \left(\frac{\partial u_2}{\partial x_2}\right)^2 + \left(\frac{\partial u_3}{\partial x_3}\right)^2 - \frac{\partial u_1}{\partial x_1}\frac{\partial u_2}{\partial x_2} - \frac{\partial u_2}{\partial x_2}\frac{\partial u_3}{\partial x_3} - \frac{\partial u_3}{\partial x_3}\frac{\partial u_1}{\partial x_1} \right] \\ &\quad + \eta \left[\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right)^2 + \left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2} \right)^2 + \left(\frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3} \right)^2 \right] \end{aligned} \tag{3c-24a}$$

The thermal dissipation function ϕ_κ due to heat conduction can be evaluated, with the aid of (3c-16) and (3c-21a), in the form

$$\phi_\kappa = -\frac{q_i}{T}\frac{\partial T}{\partial x_i} = +\frac{\kappa}{T}\left(\frac{\partial T}{\partial x_i}\right)^2 = \frac{\kappa}{T}(\nabla T)^2 \tag{3c-24b}$$

It does not appear explicitly in (3c-23), but it is there implicitly as a consequence of the heat-transfer processes described by (3c-23).

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[1] *Phil. Mag.* (4) **1**, 305–317 (1851).

[2] See, for example, Zemansky, "Heat and Thermodynamics," 3d ed., pp. 246–255, McGraw-Hill Book Company, New York, 1951.

Summary of Assumptions. The fluid considered is assumed to be continuous except at boundaries or interfaces, locally homogeneous and isotropic when at rest, viscous, thermally conducting, and chemically inert, and its local thermodynamic condition is assumed to be completely determined by specifying three "state" variables, any two of which determine the third uniquely through an equation of state. No structural or thermal "relaxation" mechanism has been presumed up to this point in the analysis, except to the extent that ordinary heat conduction and viscous losses may be described in such terms. Local thermodynamic reversibility has been assumed in using conventional thermodynamic identities based on the second law, but the irreversible production of entropy by dissipative processes has been accounted for explicitly. It is also assumed that the stress tensor is a linear function of the rate of deformation, and that the tractions due to viscosity can be represented by the linear terms of an expansion in powers of the viscosity coefficients. The viscosity and heat-exchange parameters of the fluid η, η', κ, and $\mathfrak{q}$ may depend in any continuous way on the state variables and hence may be implicit functions of time and the spatial coordinates. Within the scope thus defined the equations given are exact.

The functional dependence on time and the spatial coordinates of the condition and motion variables P, T, ρ, and $\mathbf{u}$ can be evaluated, in a formal sense at least, by solving the set of four simultaneous equations connecting these variables [Eqs. (3c-5), (3c-11), (3c-23), and (3c-15) or one of its alternates]. No general solution of these complete equations has been given, however, and one or another of the least important terms is usually omitted in order to render the equations tractable for dealing with specific problems.

3c-3. The Small-signal Acoustic Equations. The physical theory of sound waves deals with systematic motions of a material medium relative to an equilibrium state and thus comprises the variational aspects of elasticity and fluid dynamics. Such perturbations of state can be described by incremental, or acoustic, variables and approximate equations governing them can be obtained by arbitrarily "linearizing" the general equations of motion. These results, as well as higher-order approximations, can be derived in an orderly way by invoking a modified perturbation analysis.[1] This consists of replacing the dependent variables appearing in (3c-5), (3c-11), and (3c-23) by the sum of their equilibrium or zero-order values and their first- and second-order variational components, and then forming the separate equations that must be satisfied by the variables of each order. Two of the composite state variables, for example ρ and T, can be defined arbitrarily, whereupon the third, P, is determined by the functional equation of state. These definitions, some self-evident manipulations, and the subscript notation identifying the orders can be exhibited as follows:

$$\rho \equiv \rho_0 + \rho_1 + \rho_2 \qquad T \equiv T_0 + \theta_1 + \theta_2$$

$$\nabla\rho = \nabla\rho_1 + \nabla\rho_2 \qquad \nabla T = \nabla\theta_1 + \nabla\theta_2$$

$$P(\rho,T) \equiv P_0(\rho_0,T_0) + p_1 + p_2$$

$$p_1 + p_2 = \left[\left(\frac{\partial P}{\partial \rho}\right)_T\right]_0 (\rho - \rho_0) + \left[\left(\frac{\partial P}{\partial T}\right)_\rho\right]_0 (T - T_0) + \cdots \tag{3c-25}$$

$$K - K_T \equiv \rho\left(\frac{\partial P}{\partial \rho}\right)_T \qquad \beta \equiv -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_P \qquad c_0^2 \equiv \left[\left(\frac{\partial P}{\partial \rho}\right)_s\right]_0 \equiv \frac{(K_s)_0}{\rho_0}$$

$$\gamma = \frac{K_s}{K_T} = \frac{C_p}{C_v}$$

$$p_1 = \frac{c_0^2}{\gamma}(\rho_1 + \beta_0\rho_0\theta_1) \qquad p_2 = \frac{c_0^2}{\gamma}(\rho_2 + \beta_0\rho_0\theta_2)$$

$$\mathbf{u} \equiv 0 + \mathbf{u}_1 + \mathbf{u}_2 \qquad \nabla\cdot\mathbf{u} \equiv \Delta \equiv \Delta_1 + \Delta_2 = \nabla\cdot\mathbf{u}_1 + \nabla\cdot\mathbf{u}_2$$

$$\rho\mathbf{u} = [\rho_0\mathbf{u}_1]_1 + [\rho_1\mathbf{u}_1 + \rho_0\mathbf{u}_2]_2 + \cdots$$

$$\nabla\cdot(\rho\mathbf{u}) = [\rho_0\nabla\cdot\mathbf{u}_1]_1 + [\rho_1\nabla\cdot\mathbf{u}_1 + \mathbf{u}_1\cdot\nabla\rho_1 + \rho_0\nabla\cdot\mathbf{u}_2]_2 + \cdots$$

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[1] Eckart, *Phys. Rev.* **73**, 68–76 (1948).

Terms containing $\nabla\rho_0$ have been omitted in writing out $\nabla \cdot (\rho\mathbf{u})$, on the assumption that ρ_0, T_0, and P_0 are constant and $\mathbf{u}_0 = 0$. The reference state need not be so restricted to one of static equilibrium provided its time and space rates of change are presumed small in comparison with the corresponding change rates of the acoustic variables. The extraneous body force $\mathbf{F}$ will also be omitted hereafter; it would become important in cases involving electromagnetic interaction, but it usually derives from a gravitation potential and affects primarily the equilibrium configuration.[1] Little generality is sacrificed by omitting $\mathbf{F}$ and assuming a static reference, moreover, since the basic equations characterize directly the equilibrium condition and since the "cross-modulation" effects brought in by nonlinearity are dealt with adequately through second- or higher-order approximations.

Notice that the foregoing represents a mathematical-approximation procedure that is concerned only with the *precision* achieved in interpreting the content of the basic equations. The *accuracy* with which the basic equations themselves delineate the behavior of a real fluid is an entirely different question that must be considered independently on its own merits. It follows that, while good judgment may restrain the effort, there is no impropriety involved in pursuing higher-order solutions of the acoustic equations, even though the equations themselves may embody first-order approximations to reality such as that represented by assuming linear dependence on the viscosity coefficients and the deformation rate.

When the appropriate relations from (3c-25) are substituted in (3c-5), (3c-11), and (3c-23), the *first-order acoustic equations* can be separated out in the form

$$\frac{\partial\rho_1}{\partial t} + \rho_0(\nabla \cdot \mathbf{u}_1) = 0 \tag{3c-26a}$$

$$\rho_0 \frac{\partial \mathbf{u}_1}{\partial t} + \frac{c_0^2}{\gamma}\left(1 + \beta_0\rho_0 \frac{\nabla\theta_1}{\nabla\rho_1}\right)\nabla\rho_1 - (\eta_0\mathcal{V})\nabla(\nabla \cdot \mathbf{u}_1) + \eta_0\nabla \times (\nabla \times \mathbf{u}_1) = 0 \tag{3c-26b}$$

$$\rho_0 C_v \frac{\partial\theta_1}{\partial t} + \frac{\rho_0 C_v(\gamma - 1)}{\beta_0}(\nabla \cdot \mathbf{u}_1) - \kappa_0\nabla^2\theta_1 + \rho_0 C_v \mathfrak{q}\theta_1 = 0 \tag{3c-26c}$$

Inasmuch as the first-order effects of both shear and dilatational viscosity and of heat conduction and radiation have been included, these equations comprehend a *viscothermal theory* of small-signal sound waves. The sound absorption and velocity dispersion predicted by this theory are discussed below. Note especially that taking heat exchange into account explicitly by including (3c-26*c*) has precluded the conventional adiabatic assumption and denied the simplifying assumption that $P = P(\rho)$.

Adiabatic behavior would be assured, on the other hand, if it were assumed at the outset that $\kappa = \mathfrak{q} = 0$, but the behavior would *not* at the same time be strictly isentropic so long as irreversible viscous losses are still present and accounted for. The difference between adiabatic and isentropic behavior in this case is of second order, however, as indicated by the fact that the second-order dissipation functions ϕ do not appear in the first-order energy equation (3c-26*c*), which is thereby reduced to yielding just the isentropic relation between dilatation and excess temperature. It is allowable, therefore, in this first-order approximation, to replace the quotient $(\nabla\theta_1/\nabla\rho_1)$ appearing in (3c-26*b*) with the isentropic derivative $(\partial T/\partial\rho)_s = (\gamma - 1)/\rho\beta$, whereupon the first-order equation of motion for an *adiabatic viscous* fluid can be written as

$$\rho_0 \frac{\partial \mathbf{u}_1}{\partial t} + c_0^2\nabla\rho_1 - \eta_0\mathcal{V}\nabla(\nabla \cdot \mathbf{u}_1) + 2\eta_0(\nabla \times \mathbf{R}_1) = 0 \tag{3c-27}$$

If the effects of viscosity, as well as of heat exchange, are to be neglected, the divergence of what is left of (3c-27) can be subtracted from the time derivative of (3c-26*a*)

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[1] But, for a case in which $\mathbf{F}$ and $\nabla\rho_0$ cannot be neglected, see Haskell, *J. Appl. Phys.* **22**, 157–168 (February, 1951).

to yield the typical *small-signal scalar wave equation* of classical acoustics,

$$\frac{\partial^2 \rho_1}{\partial t^2} = \left(\frac{\partial P}{\partial \rho}\right)_s \nabla^2 \rho_1 \tag{3c-28a}$$

and, with the help of the first-order isentropic relation $p_1 = c_0^2(\rho_1)_s$, this wave equation becomes, in terms of the sound pressure,

$$\frac{\partial^2 p_1}{\partial t^2} = c_0^2 \nabla^2 p_1 \tag{3c-28b}$$

3c-4. The Second-order Acoustic Equations. The same substitution of composite variables that delivered (3c-26*a*), (3c-26*b*), and (3c-26*c*) will also yield directly the second-order equations of acoustics, which can now be marshaled as follows:

$$\frac{\partial \rho_2}{\partial t} + \rho_0(\nabla \cdot \mathbf{u}_2) + \nabla \cdot (\rho_1 \mathbf{u}_1) = 0 \tag{3c-29a}$$

$$\begin{aligned} \rho_0 \frac{\partial \mathbf{u}_2}{\partial t} + \frac{\partial(\rho_1 \mathbf{u}_1)}{\partial t} &+ \rho_0 \mathbf{u}_1(\nabla \cdot \mathbf{u}_1) + \rho_0(\mathbf{u}_1 \cdot \nabla)\mathbf{u}_1 \\ &+ \frac{c_0^2}{\gamma}\left(1 + \beta_0 \rho_0 \frac{\nabla \theta_2}{\nabla \rho_2}\right) \nabla \rho_2 - \eta_0 \mathcal{V} \nabla(\nabla \cdot \mathbf{u}_2) + 2\eta_0(\nabla \times \mathbf{R}_2) \\ &- (\nabla \eta_1')(\nabla \cdot \mathbf{u}_1) - 2(\nabla \eta_1 \cdot \nabla)\mathbf{u}_1 - 2(\nabla \eta_1) \times \mathbf{R}_1 = 0 \end{aligned} \tag{3c-29b}$$

$$\begin{aligned} \frac{\partial \theta_2}{\partial t} + \mathbf{u}_1 \cdot (\nabla \theta_1) &+ \frac{\gamma - 1}{\beta_0}(\nabla \cdot \mathbf{u}_2) - \frac{\kappa_0}{\rho_0 C_v} \nabla^2 \theta_2 \\ &+ \frac{\kappa_0}{\rho_0^2 C_v} \rho_1 \nabla^2 \theta_1 - \frac{\nabla \theta_1 \cdot \nabla \kappa_1}{\rho_0 C_v} + q\theta_2 - \frac{\phi_\eta}{\rho_0 C_v} = 0 \end{aligned} \tag{3c-29c}$$

The subscripts appended to κ and the η's imply that each may be expressed in the generic form

$$\eta(T, \rho, \cdots) = \eta_0(T_0, \rho_0, \cdots) + \eta_1 \qquad \eta_1 = \frac{\partial \eta}{\partial T}\theta_1 + \frac{\partial \eta}{\partial \rho}\rho_1 + \cdots \tag{3c-30}$$

No general solution of these complete second-order equations has been given, but they provide a useful point of departure for making approximations and for investigating some second-order phenomena that cannot be predicted by the first-order equations alone.

3c-5. Spatial and Material Coordinates. Equations (3c-26) and (3c-29) are couched in terms of the local values assumed by the dependent variables ρ, P, T, and $\mathbf{u}$ at *places* identified by their coordinates x_i in a fixed *spatial* reference frame, commonly called *Eulerian* coordinates (in spite of their first use by d'Alembert). As an alternate method of representation, the behavior of the medium can be described in terms of the sequence of values assumed by the dependent condition and state variables pertaining to identified *material* particles of the medium no matter how these particles may move with respect to the spatial coordinate system. The independent variables in this case are the identification coordinates a_i, rather than the position coordinates; the latter then become dependent variables that describe, as time progresses, the travel history of each particle of the medium. Such a representation in terms of *material* coordinates is commonly called *Lagrangian* (in spite of its first introduction and use by Euler).

The Wave Equation in Material Coordinates. The use of material coordinates can be demonstrated by deriving the exact equations governing one-dimensional (plane-wave) propagation in a *nonviscous adiabatic* fluid. Consider a cylindrical segment of the medium of unit cross section with its axis along $+x$, the direction of propagation, and let x and $x + \delta x$ define the boundaries of a thin laminar "particle" whose undisturbed equilibrium position is given by a and $a + \delta a$. The difference $x - a = \xi$ defines the displacement of the a particle from its equilibrium position and provides a convenient incremental, or acoustic, dependent variable in terms of which to describe the position, velocity, and acceleration of the particle; thus

$$x(a,t) = a + \xi(a,t) \qquad \frac{\partial x}{\partial t} = u^L(a,t) = \frac{\partial \xi}{\partial t} \qquad \frac{\partial u^L}{\partial t} = \frac{\partial^2 \xi}{\partial t^2} \tag{3c-31}$$

Continuity requires that the mass of the particle remain constant during any displacement, which means that

$$\rho_0 \delta a = \rho^L \delta x = \rho^L \left(\delta a + \frac{\partial \xi}{\partial a} \delta a \right) \qquad \frac{\rho_0}{\rho^L} = \frac{\partial x}{\partial a} = 1 + \frac{\partial \xi}{\partial a} \tag{3c-32a}$$

or, for three-dimensional disturbances and in general,

$$\frac{\rho_0}{\rho^L} = \frac{\partial(x_1,x_2,x_3)}{\partial(a_1,a_2,a_3)} \tag{3c-32b}$$

in which the symbolic derivative stands for the Jacobian functional determinant. The superscript L is used here and below as a reminder that the dependent variable so tagged adheres to, or "follows" in the Lagrangian sense, a specific particle, and that it is a function of the independent identification coordinates. When not so tagged, or with superscript E added for emphasis, the state variables ρ, P, T and the condition variable u are each assumed to be functions of time and the spatial coordinate x.

The net force per unit mass acting on the particle at time t is $-(\rho^L)^{-1} \partial P^L / \partial x$, where ρ^L and P^L are the density and pressure at x, the "now" position of the moving particle. However, inasmuch as x is not an independent variable in this case, the pressure gradient must be rewritten as $(\partial P^L / \partial a)(\partial a / \partial x)$, from which the second factor can be eliminated by recourse to (3c-32a). The momentum equation then becomes just

$$\frac{\rho_0 \partial^2 \xi}{\partial t^2} = \frac{-\partial P^L}{\partial a} \tag{3c-33}$$

The adiabatic assumption makes available the simplified equation of state, $P = P(\rho)$, and this relation, in turn, allows the material gradient, $\partial P^L / \partial a$, to be written as

$$\frac{-\partial P^L}{\partial a} = -\left(\frac{\partial P^L}{\partial \rho^L} \right)_s \frac{\partial \rho^L}{\partial a} = -c^2 \frac{\partial \rho^L}{\partial a} \tag{3c-34}$$

from which the last factor can be eliminated by using (3c-32a) again. This leads at once to the exact wave equation[1]

$$\frac{\partial^2 \xi}{\partial t^2} = \left(\frac{c \rho^L}{\rho_0} \right)^2 \frac{\partial^2 \xi}{\partial a^2} = c^2 \left(1 + \frac{\partial \xi}{\partial a} \right)^{-2} \frac{\partial^2 \xi}{\partial a^2} \tag{3c-35}$$

The pressure-density relation for a perfect adiabatic gas is $P = P_0(\rho/\rho_0)^\gamma$, from which it can be deduced that

$$c^2 = \left(\frac{\partial P}{\partial \rho} \right)_s = \frac{\gamma P_0}{\rho_0} \left(\frac{\rho}{\rho_0} \right)^{\gamma - 1} = c_0^2 \left(\frac{\rho}{\rho_0} \right)^{\gamma - 1} \tag{3c-36}$$

No generalization of comparable simplicity is available for liquids.[2] When (3c-36) is introduced in (3c-35), the exact "Lagrangian" wave equation for an adiabatic perfect gas becomes

$$\frac{\partial^2 \xi}{\partial t^2} = c_0^2 \left(\frac{\rho^L}{\rho_0} \right)^{\gamma + 1} \frac{\partial^2 \xi}{\partial a^2} = c_0^2 \left(1 + \frac{\partial \xi}{\partial a} \right)^{-(\gamma + 1)} \frac{\partial^2 \xi}{\partial a^2} \tag{3c-37}$$

In the Lagrangian formulation illustrated above, the choice of a, the initial-position coordinate, as the independent variable is useful but any other coordinate that

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[1] Rayleigh, "Theory of Sound," vol. II, §249; Lamb, "Hydrodynamics," §§13–15, 279–284.

[2] But see Courant and Friedrichs, "Supersonic Flow and Shock Waves," p. 8, Interscience Publishers, Inc., New York, 1948.

identifies the particles would serve the same purpose. For example, the particle located momentarily at x can be uniquely identified by the material coordinate $h \equiv \int_0^x \rho\, dx$, where h represents the mass of fluid contained between the origin and the particle. Inasmuch as this included mass will not change as the particle moves, the use of h as an independent "mass" variable automatically satisfies the requirements of continuity, with some attendant simplification in the analysis of transient disturbances. In the undisturbed condition, $\rho = \rho_0$ and $x = a$, whence the relation $a = h/\rho_0$ allows the independent variables to be interchanged by direct substitution in (3c-37).

Material and Spatial Coordinate Transforms. It is useful to have available a systematic procedure for converting a functional expression for one of the state variables from the form involving material coordinates to the corresponding form in spatial coordinates, or the inverse. One should avoid, however, the trap of referring to the state variables themselves as Lagrangian or Eulerian *quantities;* density and pressure, for example, are scalar point functions that can have only one value at a given place and time. On the other hand, it is of prime importance to distinguish carefully (and to specify!) the independent variables when computing the derivatives of these quantities.

The E and L functions are tied together by the displacement variable ξ, which provides a single-valued connection between the a particle and its instantaneous position coordinate x and which may therefore be regarded as a function of either of its terminal coordinates a or x. This can be indicated [cf. (3c-31)] by writing $x(a,t) = a + \xi(a,t)$, or the inverse relation $a(x,t) = x - \xi(x,t)$, from which follow the alternate expressions

$$a = x - \xi(a,t) \qquad x = a + \xi(x,t) \tag{3c-38}$$

The desired coordinate transforms can then be established by means of Taylor series expansions, the two forms following according to whether the expansion is centered on the instantaneous particle position or spatial coordinate x, or on the particle's equilibrium position or material coordinate a. Thus, if q is used to represent any one of the variables ρ, P, T, or u, one of the expansions can be based on the obvious identity

$$\begin{aligned} q^L(a,t) &= q^E(x,t)_{x=a+\xi(x,t)} \\ &= q^E(x,t)_{x=a} + \left[\xi(x,t)\,\frac{\partial q^E(x,t)}{\partial x}\right]_{x=a} + \frac{1}{2}\left[\xi^2(x,t)\,\frac{\partial^2 q^E(x,t)}{\partial x^2}\right]_{x=a} + \cdots \end{aligned} \tag{3c-39}$$

Note that all terms on the right of (3c-39) are functions of the spatial coordinates and that each is to be evaluated at the equilibrium position coordinate a. This transform yields, therefore, the instantaneous value in material coordinates of the variable represented by q, in terms of the local value of q modified by correction terms (comprising the succeeding terms of the series) based on the spatial rate of change of q and the instantaneous displacement.

The inverse transform is derived in a similar way from the identity

$$\begin{aligned} q^E(x,t) &= [q^L(a,t)]_{a=x-\xi(a,t)} \\ q^E(x,t) &= [q^L(a,t)]_{a=x} - \left[\xi(a,t)\,\frac{\partial q^L(a,t)}{\partial a}\right]_{a=x} + \frac{1}{2}\left[\xi^2(a,t)\,\frac{\partial^2 q^L(a,t)}{\partial a^2}\right]_{a=x} - \cdots \end{aligned} \tag{3c-40}$$

In symmetrical contrast with (3c-39), all terms on the right in (3c-40) are functions of the material coordinates and are to be evaluated for $a = x$. This transform, therefore, yields the instantaneous local value of the variable q at the place x, in terms of the instantaneous value of q for the now-displaced particle whose equilibrium position or material coordinate is $a = x$, modified by the succeeding terms of the series in accordance with the material-coordinate rate of change of q and the instantaneous displacement.

The transforms (3c-39) and (3c-40) indicate that the differences between q^L and q^E are of second order, which explains why the troublesome distinction between spatial and material coordinates does not intrude when only first-order effects are being considered. It also follows that the first two terms of these transforms are sufficient to deliver all terms of q^L or q^E through the second order. The use of these transforms can be illustrated by writing them out explicitly for u and ρ, including all second-order terms,

$$u^L \equiv \xi_t \qquad u^E = u^L - \xi u_a{}^L = \xi_t - \xi\xi_{ta} \tag{3c-41a}$$

$$\rho^L = \rho_0(1 + \xi_a)^{-1} = \rho_0(1 - \xi_a + \xi_a{}^2 - \cdots)$$

$$\rho^E = \rho_0(1 - \xi_a + \xi_a{}^2 + \xi\xi_{aa}) = \rho_0[1 - \xi_a + (\xi\xi_a)_a] \tag{3c-41b}$$

in which the subscripts indicate partial differentiation with respect to a or t. The product of (3c-41a) and (3c-41b) gives at once the relation between the material and spatial coordinate expressions for the mass transport ρu; thus, through second order,

$$\rho^E u^E = \rho^L u \quad - \xi(\rho^L u^L)_a + \xi^2(\rho_a{}^L u_a{}^L) = \rho_0[\xi_t - (\xi\xi_t)_a] = \rho_0[\xi - \xi\xi_a]_t \tag{3c-42}$$

It is then straightforward to show that, if the particle velocity w is simple harmonic, the time average of the local mass transport $\rho^E u^E$ will vanish through the second order, even though the average value of u^E is not zero. Note, however, that the displacement velocity ξ_t is measured from an equilibrium position that is here assumed to be static; the average mass transport may indeed take on nonvanishing values if the wave motion as a whole leads to gross streaming (see Sec. 3c-7).

3c-6. Waves of Finite Amplitude.[1] A distinguished tradition adheres to the study of the propagation of unrestricted compressional waves. That the particle velocity is forwarded more rapidly in the condensed portion of the wave was known early (Poisson, 1808; Earnshaw, 1858; Riemann, 1859); and that this should lead eventually to the formation of a discontinuity or shock wave was recognized by Stokes (1848), interpreted by Rayleigh,[2] discussed more recently by Fubini,[3] and has been reviewed still more recently with heightened interest by modern students of blast-wave transmission.[4]

By virtue of the adiabatic assumption underlying $P = P(\rho)$, the speed of sound is also a function of density alone and may be approximated by the leading terms of its expansion about the equilibrium density:

$$c^2 \doteq c_0{}^2\left[1 - 2\xi_a \frac{\rho_0}{c_0}\left(\frac{Dc}{D\rho}\right)_0 + \cdots\right] \tag{3c-43}$$

When (3c-43) is introduced in the exact wave equation in material coordinates, (3c-35), the latter can be recast in the following form, using the subscript convention for partial differentiation and retaining only, but all, terms through second order:

$$\xi_{tt} - c_0{}^2\xi_{aa} = -c_0{}^2\left[1 + \frac{\rho_0}{c_0}\left(\frac{Dc}{D\rho}\right)_0\right](\xi_a{}^2)_a \tag{3c-44}$$

If it is then assumed that an arbitrary plane displacement $\xi(0,t) = f(t)$ is impressed at the origin, it can be verified by direct substitution that a solution of (3c-44) is

$$\xi(a,t) = f\left(t - \frac{a}{c_0}\right) + \frac{a}{2c_0{}^2}\left[1 + \frac{\rho_0}{c_0}\left(\frac{Dc}{D\rho}\right)_0\right]\left[f'\left(t - \frac{a}{c_0}\right)\right]^2 \tag{3c-45}$$

The density variations associated with these displacements are to be found by entering (3c-45) in (3c-32), and the variational pressure can then be evaluated in terms of the adiabatic compressibility of the medium.

Relatively more attention has been devoted to the analysis of solutions of (3c-37) for the case of an adiabatic perfect gas. For an arbitrary initial displacement, as

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[1] For more recent developments see Sec. 3n, Nonlinear Acoustics (Theoretical), pp. 3-183 to 3-205.

[2] "Theory of Sound," vol. II, §§249–253. *Proc. Roy. Soc.* (*London*) **84**, 247–284 (1910).

[3] *Alta Frequenza* **4**, 530–581 (1935).

[4] See also Sec. 2y of this book, Shock Waves, pp. 2-273 to 2-278.

above, the solution of the corresponding wave equation (3c-37), again including all terms through second order, is

$$\xi(a,t) = f\left(t - \frac{a}{c_0}\right) + \frac{a}{2c_0^2}\frac{\gamma + 1}{2}\left[f'\left(t - \frac{a}{c_0}\right)\right]^2 \qquad (3c\text{-}46)$$

Technological interest in this problem centers on the generation of spurious harmonics, which can be studied by assuming the initial displacement to be simple harmonic, viz., $f(t) = \xi_0(1 - \cos \omega t)$ at the origin. The solution then takes the explicit form

$$\xi(a,t) = \xi_0[1 - \cos(\omega t - k_0 a)] + \frac{\gamma + 1}{8} k_0^2 \xi_0^2 a[1 - \cos 2(\omega t - k_0 a)] \qquad (3c\text{-}47)$$

in which k_0 is written for the phase constant, $k_0 = \omega/c_0 = 2\pi/\lambda_0$.

The most striking feature of the solutions (3c-45) and (3c-47) is the appearance of the material coordinate a in the coefficient of the second-harmonic term. As a consequence, the condensation wave front becomes progressively steeper as the wave propagates, the energy supplied at fundamental frequency being gradually diverted toward the higher harmonic components. The compensating diminution of the fundamental-frequency component would be exhibited explicitly if third-order terms had been retained in (3c-46) and (3c-47) inasmuch as all odd-order terms include a "contribution" to the fundamental. When such higher terms are retained it is predicted that propagation will always culminate in the formation of a shock wave at a distance from the source given approximately by $a \doteq 2\xi_0/(\gamma + 1)M^2$, where M is the peak value of the particle-velocity Mach number.[1] On the other hand, when dissipative mechanisms are taken into account, the fact that attenuation increases with frequency for either liquids or gases leads to the result that, except for very large initial disturbances, the wavefront will achieve a maximum steepness when the propagation distance is such that the rate of energy conversion to higher frequencies by nonlinearity is just compensated by the increase of absorption at higher frequencies. Note, however, that this steepest wave front does not qualify as a "disturbance propagated without change of form." When attention is centered on the fundamental component, the diversion of energy to higher frequencies appears as an attenuation and accounts for the relatively more rapid absorption sometimes observed near a sound source.[2]

The variational or acoustic pressure, in material coordinates, can be expressed generally as a function of the displacement gradients by using the adiabatic pressure-density relation $P^L = P_0(\rho^L/\rho_0)^\gamma$ in conjuction with the continuity relation (3c-32); thus,

$$P^L - P_0 = p^L = \gamma P_0[-\xi_a + \tfrac{1}{2}(\gamma + 1)\xi_a^2] = \langle p^L \rangle + p_1{}^L + p_2{}^L \qquad (3c\text{-}48)$$

in which the last member identifies the steady-state alteration of the average pressure and the fundamental and second-harmonic components of sound pressure. When the harmonic solution (3c-47) is introduced in (3c-48), the two alternating components of pressure for $a^2 \gg (\lambda/4\pi)^2$ can be shown, after some algebraic manipulation, to be

$$p_1{}^L = +\gamma P_0 M \sin(\omega t - k_0 a) = +\sqrt{2}\, P_1 \sin(\omega t - k_0 a) \qquad (3c\text{-}49a)$$

$$p_2{}^L = \gamma P_0 M^2 k_0 a \tfrac{1}{4}(\gamma + 1) \sin 2(\omega t - k_0 a) = \sqrt{2}\, P_2 \sin 2(\omega t - k_0 a) \qquad (3c\text{-}49b)$$

in which P_1 and P_2 are the rms values of the fundamental and second-harmonic sound pressures, and $M = k_0\xi_0 = \omega\xi_0/c_0$ is again the peak value of the particle-velocity Mach number at the origin. The relative magnitude of P_2 increases linearly with distance from the origin and is directly proportional to the peak Mach number, as may be deduced from (3c-49a) and (3c-49b); thus

$$\frac{P_2}{P_1} = \frac{1}{4}(\gamma + 1)Mk_0 a \qquad P_2 = \frac{P_1{}^2 k_0 a(\gamma + 1)}{2\sqrt{2}\,\gamma P_0} \qquad (3c\text{-}50)$$

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[1] Fubini, *Alta Frequenza* **4**, 530–581 (1935).

[2] Fox and Wallace, *J. Acoust. Soc. Am.* **26**, 994–1006 (1954). Blackstock, *J. Acoust. Soc. Am.* **36**, 534–542 (1964).

Various experimental studies of second-harmonic generation have given results in reasonably good agreement with the predictions of (3c-50).[1]

The sound-induced alteration of mean total pressure, or "average" acoustic pressure, is given by the time-independent terms yielded by the substitution of (3c-47) in (3c-48), viz.,

$$\langle p^L \rangle = + \frac{\gamma P_0 M^2(\gamma + 1)}{8} \tag{3c-51}$$

Note that this pressure increment is given as a function of the material coordinates, which means that it pertains to a *moving* element of the fluid. The *local* value of the pressure change can be found by means of the transform (3c-40), which gives, through second-order terms, the following replacement for (3c-48),

$$p^E = p^L - \xi \frac{\partial p^L}{\partial a} = \gamma P_0 \left[-\xi_a + \frac{1}{2} (\gamma + 1)\xi_a{}^2 + \xi\xi_{aa} \right] \tag{3c-52}$$

When (3c-47) is introduced in (3c-52), the time-independent terms give the local change in mean pressure as

$$\langle p^E \rangle = + \frac{\gamma P_0 M^2(\gamma - 3)}{8} \tag{3c-53}$$

and since γ is usually less than 2, it follows that the local value of mean pressure will be *reduced* by the presence of the sound wave, in striking contrast to the *increase* of mean pressure that would be observed when following the motion of a particle of the medium. Negative pressure increments as large as 10 newtons m^{-2} (100 dynes cm^{-2}) have been reported experimentally, in reasonably good agreement with (3c-53).

The mean value of the material particle velocity, $u^L \equiv \xi_t$, vanishes, as may be seen by differentiating (3c-47). The local particle velocity that would be observed at a fixed spatial position does not similarly vanish, however, and may be shown, by using the transform (3c-40) again, to be

$$u^E = \xi_t - \xi\xi_{ta} \qquad \langle u^E \rangle = -\frac{1}{2} c_0 M^2 = -\frac{\rho_0 c_0 \omega^2 \xi_0{}^2}{2\rho_0 c_0{}^2} = -(\rho_0 c_0{}^2)^{-1}\langle J \rangle \tag{3c-54}$$

where $\langle J \rangle$ is the average sound energy flux, or sound intensity.[2]

3c-7. Vorticity and Streaming. As suggested above, and with scant respect for the traditional symmetry of simple-harmonic motion, sound waves are found experimentally to exert net time-independent forces on the surfaces on which they impinge, and there is often aroused in the medium a pattern of steady-state flow that includes the formation of streams and eddies. The exact wave equation considered in the preceding section has been solved only for one-parameter waves (i.e., plane or spherical), and these solutions do not embrace some of the gross rotational flow patterns that are observed to occur. It is necessary, therefore, to revert for the study of these phenomena to the perturbation procedures introduced by the first- and second-order equations (3c-26) and (3c-29).

It is plausible that vortices and eddies should arise, if there is any net transport at all, inasmuch as material continuity would require that any net flow in the direction of sound propagation must be made good in the steady state by recirculation toward the source. Streaming effects can be studied most usefully, therefore, in terms of the generation and diffusion of circulation, or vorticity. More specifically, the time average of the second-order velocity $\mathbf{u}_2$ will be a first-order measure of the streaming

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[1] Thuras, Jenkins, and O'Neil, *J. Acoust. Soc. Am.* **6**, 173–180 (1935); Fay, *J. Acoust. Soc. Am.* **3**, 222–241 (October, 1931); O. N. Geertsen, unpublished (ONR) Tech. Report no. **III**, May, 1951, U.C.L.A.; D. T. Blackstock, Report of the Fourth International Congress on Acoustics, Part I, 1962.

[2] Westervelt, *J. Acoust. Soc. Am.* **22**, 319–327 (1950).

velocity. The vector function describing $\mathbf{u}_2$ can always be resolved into solenoidal and lamellar components defined by

$$\mathbf{u}_2 \equiv -\nabla\varphi_2 + \nabla \times \mathbf{A}_2 \qquad \nabla^2\varphi_2 \equiv -\nabla \cdot \mathbf{u}_2 \qquad \nabla^2\mathbf{A}_2 = -(\nabla \times \mathbf{u}_2) \tag{3c-55}$$

The irrotational component that represents the compressible, or acoustic, part of the fluid motion is derived from the scalar potential φ_2. The vector potential $\mathbf{A}_2$ is associated with the rotational component comprising the incompressible circulatory flow that is of primary interest in streaming phenomena.

The failure of the first-order equations to predict streaming can be demonstrated by writing directly the curl of the first-order force equation (3c-26*b*). The gradient terms are eliminated by this operation, since $\nabla \times \nabla(\quad) \equiv 0$, leaving just

$$\frac{\partial \mathbf{R}_1}{\partial t} - \nu_0\nabla^2\mathbf{R}_1 = 0 \tag{3c-56}$$

Thus the first-order vorticity, $\mathbf{R}_1 \equiv \frac{1}{2}(\nabla \times \mathbf{u}_1)$, if it has any value other than zero, obeys a typical homogeneous diffusion equation. On the other hand, it would appear to follow that, if $\mathbf{R}_1$ were ever zero everywhere, its time derivative would also vanish everywhere and $\mathbf{R}_1$ would be constrained always thereafter to remain zero. This is *not* a valid proof of the famous Lagrange-Cauchy proposition on the permanence of the irrotational state, but the absence of any source terms on the right-hand side of (3c-56) does indicate correctly[1] that first-order vorticity cannot be generated in the interior of a fluid even when viscosity and heat conduction are taken into account. Instead, first-order vorticity, if it exists at all, must diffuse inward from the boundaries under control of (3c-56).

A notably different result is obtained when the second-order equations are dealt with in the same way. It is useful, before taking the curl of (3c-29*b*), to eliminate the second and third terms of this equation by subtracting from it the product of (ρ_1/ρ_0) and (3c-26*b*), and the product of $\mathbf{u}_1$ and (3c-26*a*). In effect this raises the first-order equations to second order and then combines the information in both sets. The augmented second-order force equation can then be arranged in the form

$$\begin{aligned}\rho_0 \frac{\partial \mathbf{u}_2}{\partial t} &+ 2\eta_0(\nabla \times \mathbf{R}_2) + \nu_0\mathcal{V}\rho_1\nabla(\nabla \cdot \mathbf{u}_1) - 2\nu_0\rho_1(\nabla \times \mathbf{R}_1) - 2\rho_0(\mathbf{u}_1 \times \mathbf{R}_1)\\ &-2[(\nabla\eta_1 \cdot \nabla)\mathbf{u}_1 + \nabla\eta_1 \times (\nabla \times \mathbf{u}_1)] + 2(\nabla\eta_1 \times \mathbf{R}_1) + \rho_0\nabla\left(\frac{1}{2}\mathbf{u}_1 \cdot \mathbf{u}_1\right) + B_2\nabla\rho_2\\ &- B_1\nabla\left(\frac{1}{2}\rho_1^2\right) - \eta_0\mathcal{V}\nabla(\nabla \cdot \mathbf{u}_2) - \nabla\eta_1'(\nabla \cdot \mathbf{u}_1) = 0\end{aligned} \tag{3c-57}$$

The following abbreviations have been used for the coefficients of $\nabla\rho_1$ in (3c-26*b*) and of $\nabla\rho_2$ in (3c-29*b*):

$$B_1 \equiv \frac{c_0^2}{\gamma}\left[1 + \beta_0\rho_0\left(\frac{D\theta_1}{D\rho_1}\right)_0\right] \qquad B_2 \equiv \frac{c_0^2}{\gamma}\left[1 + \beta_0\rho_0\left(\frac{D\theta_2}{D\rho_2}\right)_0\right] \tag{3c-58}$$

in which the quotients $(\nabla\theta_1/\nabla\rho_1)$ and $(\nabla\theta_2/\nabla\rho_2)$ have been replaced by the corresponding material derivatives $D\theta/D\rho$, which must be evaluated, of course, for the particular conditions of heat exchange satisfying the energy equations (3c-26*c*) and (3c-29*c*). This evaluation can be evaded temporarily (at the cost of neglecting ∇B_1 and ∇B_2) by observing that each of the last five terms of (3c-57) contains a gradient. These disappear on taking the curl of (3c-57), whereupon the vorticity equation emerges as

$$\begin{aligned}\frac{\partial \mathbf{R}_2}{\partial t} - \nu_0\nabla^2\mathbf{R}_2 = \frac{1}{2}\nu_0\mathcal{V}\left(\nabla s_1 \times \nabla\frac{\partial s_1}{\partial t}\right) &+ \rho_0^{-1}\nabla \times (\mathbf{u}_1 \cdot \nabla)\nabla\eta_1 + \nu_0 s_1\nabla^2\mathbf{R}_1\\ - \nu_0\nabla s_1 \times (\nabla \times \mathbf{R}_1) - \nabla \times (\mathbf{u}_1 \times \mathbf{R}_1) &+ \rho_0^{-1}\nabla \times (\nabla\eta_1 \times \mathbf{R}_1)\end{aligned} \tag{3c-59}$$

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[1] St. Venant. *Compt. rend.* **68**, 221–237 (1869).

in which s_1 has been introduced as an abbreviation for the first-order condensation, $s_1 = \rho_1/\rho_0$. This inhomogeneous diffusion equation puts in evidence various second-order sources of vorticity: four vanish if the first-order motion is irrotational ($\mathbf{R}_1 = 0$), and two drop out when the shear viscosity is constant ($\nabla\eta_1 = 0$). It is notable that the dilatational viscosity η' does not appear in any of these source terms except through the ratio η'/η that forms part of the dimensionless viscosity number $\mathcal{V} \equiv 2 + (\eta'/\eta)$.

Except for the third source term, which (3c-56) shows to be one order smaller than the change rate of $\mathbf{R}_1$, *all* the vorticity sources would vanish—and the streaming would "stall"—if the wave front were strictly plane with $\mathbf{u}_1$, s_1, and η functions of only one space coordinate. Wave fronts cannot remain strictly plane at grazing incidence, however,[1] and rapid changes in the direction and magnitude of $\mathbf{u}_1$ will occur near reflecting surfaces, in the neighborhood of sound-scattering obstacles, and in thin viscous boundary layers. As a consequence, the "surface" source terms containing $\mathbf{R}_1$ become relatively more important in these cases.[2] In other circumstances, when the sound field is spatially restricted by source directionality, the first source term in (3c-59) dominates and leads to a steady-state streaming velocity proportional to the *ratio* of the dilatational and shear viscosity coefficients—and hence to a unique independent method of measuring this moot ratio.[3] Both the force that drives the fluid circulation and the viscous drag that opposes it are proportional to the kinematic viscosity, which does not therefore control the final value of streaming velocity but only the time constant of the motion, i.e., the time required to establish the steady state.[4]

Evaluating the second-order vorticity source terms in any specific case requires that the first-order velocity field be known, and this calls in the usual way for solutions that satisfy the experimental boundary conditions and the wave equation. Unusual requirements of exactness are imposed on such solutions, moreover, by the fact that even the second-order acoustic equations yield only a first approximation to the mean particle velocity.

The analysis of vorticity can be recast, by skillful abbreviation and judicious regrouping of the elements of (3c-57), in such a way as to yield a general law of rotational motion, according to which the average rate of increase of the moment of momentum of a fluid element responds to the difference between the sound-induced torque and a viscous torque arising from the induced flow.[5] A close relation has also been shown to exist in some cases between the streaming potential and the attenuation of sound by the medium without regard for whether the attenuation is caused by viscosity, heat conduction, or by some relaxation process; in effect the average momentum of the stream "conserves" the momentum diverted from the sound wave by absorption.[6] This principle has so far been established rigorously only for the adiabatic assumption under which $P = P(\rho)$, and under restrictive assumptions on the variability of η and $\mathcal{V}$, but its prospective importance would appear to justify efforts to extend the generalization.

3c-8. Acoustical Energetics and Radiation Pressure. If the kinetic energy density that appeared briefly in (3c-12) is restored to (3c-18), the change rate of the specific

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[1] Morse, "Vibration and Sound," 2d ed., pp. 368–371, McGraw-Hill Book Company, New York, 1948.

[2] Medwin and Rudnick, *J. Acoust. Soc. Am.* **25**, 538–540 (1953).

[3] Liebermann, *Phys. Rev.* **75**, 1415–1422 (1949); Medwin, *J. Acoust. Soc. Am.* **26**, 332–341 (1954).

[4] Eckart, *Phys. Rev.* **73**, 68–76 (1948).

[5] Nyborg, *J. Acoust. Soc. Am.* **25**, 938–944 (1953); Westervelt, *J. Acoust. Soc. Am.* **25**, 60–67 and *errata*, 799 (1953).

[6] Nyborg, *J. Acoust. Soc. Am.* **25**, 68–75 (1953); Doak, *Proc. Roy. Soc.* (*London*), ser. A, **226**, 7–16 (1954); Piercy and Lamb, *Proc. Roy. Soc.* (*London*), ser. A, **226**, 43–50 (1954).

total energy density (per unit mass), E/ρ, can be formulated in terms of

$$\begin{aligned}\rho\frac{D(E/\rho)}{Dt} &= \rho\frac{D(\frac{1}{2}\mathbf{u}\cdot\mathbf{u})}{Dt} + \rho\frac{D\epsilon}{Dt}\\ &= \rho\frac{D(\frac{1}{2}\mathbf{u}\cdot\mathbf{u})}{Dt} - \rho P\frac{Dv}{Dt} - \nabla\cdot\mathbf{q} + \phi_\eta \qquad (3c\text{-}60)\end{aligned}$$

Material derivatives are used here so that the energy balance reckoned for a particular volume element will continue to hold as the derivatives "follow" the motion of the material particles. The mechanical work term on the right in (3c-60) can be resolved into two components by writing $P = P_0 + p$, where the excess, or sound, pressure p now represents the sum of the variational components of all orders

$$(p = p_1 + p_2 + \cdots)$$

Thus

$$\rho\frac{D(E/\rho)}{Dt} = \rho\frac{D(\frac{1}{2}\mathbf{u}\cdot\mathbf{u})}{Dt} - \rho p\frac{Dv}{Dt} + \rho P_0\frac{Dv}{Dt} - \nabla\cdot\mathbf{q} + \phi_\eta \qquad (3c\text{-}61)$$

A second equation involving the first two terms on the right of (3c-61) can be formed by multiplying the continuity equation (3c-5) by p and adding it to the scalar product of the vector **u** and the vector force equation (3c-11*b*); thus

$$\begin{aligned}\rho\mathbf{u}\cdot\frac{D\mathbf{u}}{Dt} + \mathbf{u}\cdot\nabla p + p\left(\frac{1}{\rho}\frac{D\rho}{Dt} + \nabla\cdot\mathbf{u}\right) &= \mathbf{u}\cdot\mathbf{f}_v(\eta, \eta', \mathbf{u})\\ &= \rho\mathbf{u}\cdot\frac{D\mathbf{u}}{Dt} - p\rho\frac{Dv}{Dt} + \mathbf{u}\cdot\nabla p + p\nabla\cdot\mathbf{u} \qquad (3c\text{-}62)\end{aligned}$$

where $\mathbf{f}_v$ stands for the sum of the five viscosity terms that appear on the right-hand side of (3c-11*b*). Combining this result with (3c-61) gives

$$\begin{aligned}\rho\frac{D(\frac{1}{2}\mathbf{u}\cdot\mathbf{u})}{Dt} - \rho p\frac{Dv}{Dt} + \nabla\cdot(p\mathbf{u}) &= +\mathbf{u}\cdot\mathbf{f}_v\\ \rho\frac{D(E/\rho)}{Dt} + \nabla\cdot(p\mathbf{u}) &= -\rho P_0\frac{Dv}{Dt} - \nabla\cdot\mathbf{q} + \phi_\eta + \mathbf{u}\cdot\mathbf{f}_v\end{aligned} \qquad (3c\text{-}63)$$

The significance of this result can be made more apparent by using the continuity equation again, this time in the form $(E/\rho)[\partial\rho/\partial t + \nabla\cdot(\rho\mathbf{u})] = 0$. Adding this "zero" to the left-hand side of (3c-63), after first using (3c-3) to express the material derivative in terms of fixed spatial coordinates, allows the continuity of acoustic energy to be expressed by

$$\begin{aligned}\rho\frac{D(E/\rho)}{Dt} + \nabla\cdot(\rho\mathbf{u}) &= \rho\frac{\partial(E/\rho)}{\partial t} + \rho\mathbf{u}\cdot\nabla\frac{E}{\rho} + \nabla\cdot(p\mathbf{u})\\ &\qquad + \left[\frac{E}{\rho}\frac{\partial\rho}{\partial t} + \frac{E}{\rho}\nabla\cdot(\rho\mathbf{u})\right]\\ \frac{\partial E}{\partial t} &= -\nabla\cdot(p\mathbf{u} + E\mathbf{u}) - P_0\Delta - \nabla\cdot\mathbf{q} + \mathbf{u}\cdot\mathbf{f}_v + \phi_\eta \qquad (3c\text{-}64)\end{aligned}$$

The acoustic energy-flux vector can be identified as $p\mathbf{u} = \mathbf{J}$, inasmuch as this term represents the instantaneous rate at which one portion of the medium does mechanical work on a contiguous portion in the process of forwarding the sound energy. The time average of the sound-energy flux through unit area normal to **u** is defined as the *sound intensity*, $\langle\mathbf{J}\rangle \equiv \mathbf{I}$. Ordinarily it is only the time average of each term of (3c-64) that is of interest, but the equation itself holds at every instant and asserts that growth of the total energy density of a volume element is accounted for by the influx of acoustic and thermal energy across the boundaries of the element, by the energy dissipated in viscous losses, and by the work done by the equilibrium pressure on the volume element during condensation. The latter component is represented by $(-P_0\Delta)$ and by a corresponding linear term contained implicitly in E [cf. (3c-19)]. It is omitted in most textbook descriptions of acoustic energy density, the neglect being justified if at all on the grounds that the stored energy varies linearly with the dilatation and hence will have a vanishing net value when averaged over an integral number of periods or wavelengths, or over the entire region occupied by the sound field. Care must be taken to ensure that it does indeed vanish rigorously on the average inasmuch as the peak values of this component of energy storage are larger than the acoustic energy in the ratio P_0/p.

Acoustic Radiation Pressure. The appearance of the product $E\mathbf{u}$ as an additive term in the first right-hand member of (3c-64) is notable and represents the net energy density carried across the boundary of a volume element by convection, the net flow being measured by the divergence of the particle velocity.[1] No approximations have been made in deducing (3c-64), which holds, therefore, within the scope of validity of the basic assumptions.

It is significant to remark the fact that E is directly additive to p when the divergence term is written as $\nabla \cdot (p + E)\mathbf{u}$, thereby identifying the additive term as a *radiation pressure* whose magnitude at every instant is just equal to the total energy density, $E = \frac{1}{2}\rho\mathbf{u}\cdot\mathbf{u} + \rho\epsilon$. This interpretation can be fortified by revising (3c-64) by expanding $\nabla \cdot (E\mathbf{u}) = E(\nabla \cdot \mathbf{u}) + \mathbf{u} \cdot \nabla E$. The last term can be used to restore the material time derivative of E and the other can be merged with the linear term in P_0, yielding a revised power equation in the form

$$\frac{DE}{Dt} = -\nabla \cdot (p\mathbf{u}) - (P_0 + E)\Delta - \nabla \cdot \mathbf{q} + \phi_\eta + \mathbf{u} \cdot \mathbf{f}_v \qquad \text{(3c-65)}$$

The role of E as an additive or radiation pressure is thus retained in (3c-65) where its time-independent part is now exhibited appropriately as a slight change in the equilibrium pressure.

When seeking to evaluate the net mechanical force due to radiation pressure on a material obstacle or screen exposed to a sound field, care must be taken to specify the boundary conditions and to account for *all* the reaction forces involved, including the steady-state interaction of the obstacle with the medium as well as the dynamic interaction of the obstacle with the sound field itself. Thus, for example, if a long tube is "filled" with a progressive plane wave, the walls of the tube, which interact only with the medium, would experience only the mean increment of the equilibrium pressure [cf. (3c-53)], and this would disappear if the walls were permeable to the medium, but not to the sound wave (e.g., with capillary holes). On the other hand, if a sound-absorbing screen were freely suspended athwart the wavefronts, it would experience just the pressure E shown by (3c-64) to be additive to p; but if the screen were to form an impermeable termination of the tube it would experience both components of pressure, including changes due to the enhancement of $\langle E \rangle$ by the reflected wave.[2]

3c-9. Sound Absorption and Dispersion. The basic manifestation of the absorption or attenuation of sound is the conversion of organized systematic motions of the particles of the medium into the uncoordinated random motions of thermal agitation.

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[1] Schock, *Acustica* **3,** 181–184 (1953).

[2] Suggested references: On fundamentals, see L. Brillouin, "Les Tenseurs en mécanique et en élasticité," Dover Publications, New York, 1946. On influence of oblique incidence and of obstacle's reflection coefficient, see F. E. Borgnis, On the Forces upon Plane Obstacles Produced by Acoustic Radiation, *J. Madras Inst. Technol.* **1** (2), 171–210 (November, 1953), and (3), 1–33 (September, 1954); also condensed in *Revs. Modern Phys.*, **25,** 653–664 (1953). For review, critical bibliography, and sophisticated analysis of general topic, see E. J. Post, *J. Acoust. Soc. Am.* **25,** 55–60 (1953); *Phys. Rev.* **118,** 1113–1118 (1960).

Various agencies of conversion can be identified as viscosity, heat conduction, or as some other mechanism that gives rise to a delay in the establishment of thermodynamic equilibrium; but all are mechanisms of interaction that lead to the same result, viz. that the energy of mass motion imparted intermittently to the medium by the sound source becomes increasingly disordered and "unavailable." Describing this in terms of the irreversible production of entropy leads to the definition of dissipation functions and paves the way for formulating an acoustic energy balance.

Equation of Continuity for Acoustic Energy. This may take the form of a statement that the mean net influx of sound energy across the boundaries of a volume element situated in a sound field must just balance the average time rate at which this energy is degraded, or made unavailable, throughout the volume element by irreversible increase of entropy; thus, by extension of (3c-20),

$$-\int_A J_i\,da_i = \int_V \frac{DE_{\text{diss}}}{Dt}\,dV = \int_V T\,\frac{DS_{\text{irr}}}{Dt}\,dV = \int_V (\phi_\kappa + \phi_\eta)\,dV \qquad \text{(3c-66)}$$

where the sound energy flux vector is $J_i = pu_i$, and E_{diss} is the degraded component of internal energy associated with the irreversible entropy S_{irr}.

The differential form of (3c-66) can be obtained in the usual way by using the divergence theorem to convert the surface integral to a volume integral. Then, after introducing the explicit forms of the dissipation functions, (3c-24*a*) and (3c-24*b*), the acoustic energy continuity relation becomes

$$\begin{aligned}-\nabla\cdot\mathbf{J} = -\frac{\partial(pu_i)}{\partial x_i} = \phi_\kappa + \phi_\eta = \frac{\kappa}{T}\left(\frac{\partial T}{\partial x_i}\right)^2 + \eta'\,\frac{\partial u_k}{\partial x_k}\,\frac{\partial u_i}{\partial x_i}\\ + \frac{1}{2}\eta\left[\left(\frac{\partial u_i}{\partial x_j}\right)^2 + \left(\frac{\partial u_j}{\partial x_i}\right)^2 + 2\,\frac{\partial u_i}{\partial x_j}\,\frac{\partial u_j}{\partial x_i}\right]\end{aligned} \qquad \text{(3c-67a)}$$

where it is understood that only the time-independent parts of each side of (3c-67*a*) are to be retained. The algebraic complexity of dealing with (3c-67*a*) is considerably abated by considering only plane waves, for which case the running subscripts each reduce to unity and can be dropped. The plane-wave form of the acoustic-energy relation then becomes, after introducing P as an implicit variable in ∇T,

$$-\frac{\partial(pu)}{\partial x} = \frac{\kappa}{T}\left(\frac{DT}{DP}\right)^2\left(\frac{\partial p}{\partial x}\right)^2 + \eta\mathcal{V}\left(\frac{\partial u}{\partial x}\right)^2 \qquad \text{(3c-67b)}$$

in which $\eta\mathcal{V}$ has been written for $\eta' + 2\eta$ [cf. (3c-10)]. The thermal dissipation term can then be maneuvered into more suggestive form by further manipulation involving the equation of state $T = T(P,\rho)$ and various thermodynamic identities including the useful relation that holds for all fluids, $T\beta^2c^2 = C_p(\gamma - 1)$. This leads, still without approximation, and with the time average explicitly indicated, to

$$\left\langle -\frac{\partial(pu)}{\partial x}\right\rangle = \left\langle \eta\mathcal{V}\left(\frac{\partial u}{\partial x}\right)^2\right\rangle + \left\langle \frac{\kappa}{\rho C_p}\,\frac{[(\rho c^2/K_T) - 1]^2}{(\gamma - 1)\rho c^2}\left(\frac{\partial p}{\partial x}\right)^2\right\rangle \qquad \text{(3c-68)}$$

It can now be observed that p, u, and their derivatives must be known throughout the sound field in order to evaluate the sound energy flux and the dissipation functions that make up (3c-67*a*) or its reduced form (3c-68). On the other hand, if these field variables are known explicitly, the effects of dissipation will already be in evidence without recourse to (3c-68). Such a continuity equation for *acoustic* energy is therefore redundant, as might have been expected inasmuch as the conservation of energy has already been incorporated in the basic equations (3c-5), (3c-15), and (3c-23). Nevertheless, (3c-68) retains some logical utility as an auxiliary relation, even though it no longer needs to be relied on for the pursuit of absorption measures, at least for plane waves.

Exact Solution of the First-order Equations. An exact solution of the complete first-order equations (3c-26*a*), (3c-26*b*), (3c-26*c*) for the plane-wave case and a definitive discussion of its implications have been given recently by Truesdell.[1] The specific problem considered is that of forced plane damped waves in a viscous, conducting fluid medium. It is assumed that each of the first-order incremental state and field variables can be described by the real parts of

$$u_1 = u_{10} e^{j\omega t} e^{-(\alpha + jk)x} \tag{3c-69}$$

and of similar equations for ρ_1, p_1, θ_1. It is assumed that $(u_1)_{x=0} = u_{10}e^{j\omega t}$ is the simple-harmonic velocity imparted to the medium by the vibrating surface of a source located at $x = 0$, but the other amplitude coefficients may be complex in order to embody the phase angles by which these variables lead or lag u_1. The exponent expressing time dependence is written $+j\omega t$, as required in order to preserve both the conventional form $R + jX$ for complex impedances *and* the positive sign for inductive or mass reactance. The attenuation constant α and the phase constant $k \equiv \omega/c$, or $k_0 \equiv \omega/c_0$, are the real and imaginary parts of the complex propagation constant $\chi \equiv \alpha + jk$; and $c_0 \equiv (\partial P/\partial \rho)_s^{1/2}$ is the *reference* value of sound speed.

When the assumed solutions (3c-69) are systematically introduced in (3c-26*a*), (3c-26*b*), and (3c-26*c*), three algebraic equations in ρ_1, u_1, θ_1 are obtained, as follows:

$$\begin{aligned}
&\rho_0(\alpha + jk)u_1 && -j\omega\rho_1 && = 0 \\
&[j\omega\rho_0 - \eta\mathcal{V}(\alpha + jk)^2]u_1 && -(\alpha + jk)\left[\frac{c_0^2}{\gamma}(\rho_1 + \beta_0\rho_0\theta_1)\right] && = 0 \\
&-\frac{\gamma - 1}{\beta_0}(\alpha + jk)u_1 && + \left[j\omega - \frac{\kappa}{\rho_0 C_v}(\alpha + jk)^2 + \mathfrak{q}\right]\theta_1 && = 0
\end{aligned} \tag{3c-70}$$

If these equations are indeed to admit solutions of the assumed form (3c-69), the determinant of the coefficients of u_1, ρ_1, and θ_1 must vanish. The characteristic or *secular equation* formed in this way (Kirchhoff, for perfect gases, 1868; extended to any fluid with arbitrary equation of state by P. Langevin[2]) turns out to be a biquadratic in the dimensionless complex propagation variable $(\alpha + jk)/k_0$. Writing this out in full, however, will be facilitated by first considering the question of how best to specify the properties of the medium.

Dimensional Analysis and Absorption Measure. Examination of (3c-70) reveals that, in addition to $(\alpha + jk)/k_0$ and the three independent variables, there are 10 parameters that pertain to the behavior of the medium at the angular frequency ω. One of these could be eliminated, in principle at least, by using the relation $T\beta^2c^2 = (\gamma - 1)C_p$, leaving 9 that are independent: C_p, C_v, η, η', κ, ρ_0, c_0, $\mathfrak{q}$, and ω. Then, since each of these can be expressed in terms of 4 basic dimensional units (e.g., mass, length, time, and temperature), it follows from the pi theorem of dimensional analysis[3] that just 5 independent dimensionless ratios can be formed out of combinations of these 9 parameters. This leads to a functional expression of the absorption measure in the symbolic form

$$\frac{\alpha + jk}{k_0} = \psi\left(\frac{C_p}{C_v}, \frac{\eta'}{\eta}, \frac{\eta C_p}{\kappa}, \frac{\omega\eta}{\rho_0 c_0^2}, \frac{\mathfrak{q}}{\omega}\right) \tag{3c-71}$$

The first two ratios have already been incorporated in γ and the viscosity number $\mathcal{V} \equiv 2 + \eta'/\eta$; the third is the Prandtl number $\mathcal{P} \equiv \eta C_p/\kappa$, and the fourth and fifth can be identified as Stokes numbers $S \equiv \omega\eta/\rho_0 c_0^2$ and $S' \equiv \omega/\mathfrak{q}$. The present purpose

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[1] C. A. Truesdell, Precise Theory of the Absorption and Dispersion of Forced Plane Infinitesimal Waves According to the Navier-Stokes Equations, *J. Rational Mechanics and Analysis* **2**, 643–741 (October, 1953).

[2] Reported by Biquard, *Ann. phys.* (11) **6**, 195–304 (1936).

[3] E. Buckingham, *Phys. Rev.* **4**, 345 (1914); *Phil. Mag.* (6) **42**, 696 (1921).

is served somewhat better by substituting for the third and fourth ratios their products with the dimensionless viscosity number, thus defining a frequency number X and thermoviscous number Y through

$$X \equiv \mathcal{V}S = \frac{\omega\eta\mathcal{V}}{\rho_0 c_0^2} \qquad Y \equiv (\mathcal{P}\mathcal{V})^{-1} = \frac{\kappa}{\eta\mathcal{V}C_p} \qquad XY = \frac{\omega\kappa}{\rho_0 c_0^2 C_p} \tag{3c-72}$$

The frequency parameter X also provides a natural criterion for designating frequencies as "low," "medium," or "high" according to whether X is much less than, comparable with, or much greater than unity. It may also be noted that, for nearly perfect gases, $\rho_0 c_0^2 \doteq \gamma P_0$, from which it follows that $X_{\text{gas}} \doteq (\omega/P_0)(\eta\mathcal{V}/\gamma)$. Hence variation of pressure may be used to extend in effect the accessible range of frequency in measurements on gases, and the ratio ω/P_0 is a proper parameter in terms of which to report such results.

Solutions of the Characteristic Equation. If the dimensionless ratios discussed above are now introduced in the expanded determinant of the coefficients of (3c-70), the resulting Kirchhoff-Langevin secular equation can be written as

$$\left(1 - \frac{j}{S'}\right) + \left(\frac{\alpha + jk}{k_0}\right)^2 \left[1 + jX(1 + \gamma Y) + \frac{\gamma X - j}{\gamma S'}\right] + \left(\frac{\alpha + jk}{k_0}\right)^4 XY(j - \gamma X) = 0 \tag{3c-73}$$

The standard "quadratic formula" can be used at once to solve (3c-73) for the reciprocal square of the propagation constant,

$$\begin{aligned} -2\left(1 - \frac{j}{S'}\right)\left(\frac{k_0}{\alpha + jk}\right)^2 = 1 + \frac{X}{S'} + j\left[X(1 + \gamma Y) - \frac{1}{\gamma S'}\right] \\ \pm \left[\left(1 + \frac{X}{S'}\right)^2 - \left[X(1 - \gamma Y) - \frac{1}{\gamma S'}\right]^2 \right. \\ \left. + 2j\left\{X[1 - (2 - \gamma)Y] + X^2\frac{1 - \gamma Y}{S'} - \frac{[1 + (X/S')]}{\gamma S'}\right\}\right]^{\frac{1}{2}} \end{aligned} \tag{3c-74a}$$

Skillful abbreviation might allow this complete solution to be carried somewhat further but no algebraic magic can lighten very much the burden of depicting the behavior of α and k as a function of *four* independent parameters—and it might have been five but for the welcome fact that $\mathcal{V}$ does not appear except as embodied in X and Y. Moreover, each parameter that does appear in (3c-74*a*) occurs in one or more product combinations, and hence it can *not* be assumed in general that the effects of viscosity and heat exchange will be linearly additive. The common practice of assessing these one at a time and then superimposing the results must therefore be considered unreliable unless justified explicitly and quantitatively. Nevertheless, something must give, and it is customary to abandon first the radiant-heat exchange, at least temporarily, by letting S' become infinite in (3c-74*a*). With this simplification, and with some new abbreviations, (3c-74*a*) becomes

$$\begin{aligned} -2\left(\frac{k_0}{\alpha + jk}\right)^2 &= 1 + jX(1 + \gamma Y) \pm \{1 - X^2(1 - \gamma Y)^2 + j2X[1 - (2 - \gamma)Y]\}^{\frac{1}{2}} \\ &\equiv G + jH = 1 + jX(1 + \gamma Y) \pm (E + jF)^{\frac{1}{2}} \\ E &\equiv 1 - X^2(1 - \gamma Y)^2 \qquad F \equiv 2X[1 - (2 - \gamma)Y] \end{aligned} \tag{3c-74b}$$

This equation has two pairs of noncoincident complex roots, but only the one of each pair that has a nonnegative real part corresponding to real attenuation is to be retained. These two physical solutions comprise the two branches of a complex square root; one branch pertains to typical compressional sound waves identified as type I, the other to so-called thermal waves identified as type II. It is an unwarranted oversimplification, however, to describe these simply as "pressure" waves and "thermal" waves inasmuch as *all* the state and condition variables—pressure, density, velocity, temperature, heat flux, etc.—are simultaneously entrained and propagated by *each* wave type, and waves of *both* types are always excited simultaneously by any source. On the other hand, the absorption and dispersion measures for waves of type I and type II will, in general, be quite different and will vary differently with the frequency parameter X and with the thermoviscous parameters γ and Y that characterize the fluid. For example, type II waves are so rapidly attenuated in ordinary fluids at accessible frequencies that they cannot be observed, whereas in strongly conducting liquids such as mercury (and perhaps in liquid helium II) the absorption for type II waves becomes less than for type I waves when the frequency is high enough for X to exceed $\frac{1}{3}$.

It should be noticed, parenthetically, that if the basic first-order equations (3c-70) had not been restricted to plane waves, the last term of (3c-26*b*) would not have dropped out. Instead, there would have turned up eventually in (3c-70) a pair of terms in the first-order vector velocity potential $\mathbf{A}_1$ [see (3c-55)] on the basis of which it would have been predicted that still another type of allowed wave motion can exist in viscous fluids—a transverse *viscous wave* that is propagated by virtue of the transverse shear reactions due to viscosity.[1]

Viscothermal Absorption and Dispersion Measures. The problem of branch determination arising in the solution of (3c-74*b*) has been discussed thoroughly by Truesdell.[2] One view of it can be expressed by writing the formal solution in the explicit form

$$\frac{\alpha}{k} \equiv \frac{A}{2\pi} = \frac{H}{+(G^2 + H^2)^{\frac{1}{2}} + G} \qquad \left(\frac{c}{c_0}\right)^2 = \frac{2(G^2 + H^2)}{+\,(G^2 + H^2)^{\frac{1}{2}} + G}$$
$$2G = 1 \pm f(h)(+E^{\frac{1}{2}}) \qquad 2H = X(1 + \gamma Y) \pm (\operatorname{sgn} F)g(h)(+E^{\frac{1}{2}}) \qquad \text{(3c-75}a\text{)}$$

(upper signs yield type I waves, lower signs type II waves)

$$h \equiv \frac{F}{E} \qquad \begin{aligned} f(h) &\equiv +\sqrt{2}\,[+(1 + h^2)^{\frac{1}{2}} + 1] = +\cosh \tfrac{1}{2}(\sinh^{-1} h) \\ g(h) &\equiv +\sqrt{2}\,[+(1 + h^2)^{\frac{1}{2}} - 1] = +\sinh \tfrac{1}{2}(\sinh^{-1} h) \end{aligned} \qquad \text{(3c-75}b\text{)}$$

where the plus signs associated with roots denoted by fractional exponents indicate that the principal or positive root is to be used. The solution (3c-75*a*) can now be attacked frontally, either by means of power-series expansions for large or small values of X or by resorting to brute-force numerical computation for intermediate frequencies. The several square-root operations on complex quantities required by the latter procedure are often facilitated by using the f and g functions defined by (3c-75*b*), for which the principal values have been tabulated.[3]

The clue to a basis for classifying fluids according to their viscothermal behavior is afforded by noting that the algebraic sign of F appears in (3c-75*a*) in such a way as to interchange the wave types when F changes sign, and that this occurs when $(2 - \gamma)Y$ passes through unity. On this basis, one may categorize fluids as *strong* conductors if Y is greater than $(2 - \gamma)^{-1}$. The contrary alternative can be further subdivided usefully[2] into *weak* conductors for which Y is less than γ^{-1}, and *moderate* conductors for which Y has intermediate values. Most liquids (including the liquefied noble gases) qualify as weak conductors, most gases as moderate conductors. On the other hand, the fact that mercury, the molten metals, and liquid helium II rank as strong

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[1] Rayleigh, "Theory of Sound," vol. II, §§347; Mason, *Trans. ASME* **69**, 359–367 (1947); Epstein and Carhart, *J. Acoust. Soc. Am.* **25**, 553–565 [557] (1953).

[2] C. A. Truesdell, Precise Theory of the Absorption and Dispersion of Forced Plane Infinitesimal Waves According to the Navier-Stokes Equations, *J. Rational Mechanics and Analysis* **2**, 643–741 (October, 1953).

[3] G. W. Pierce, *Proc. Am. Acad. Arts Sci.* **57**, 175–191 (1922).

conductors emphasizes the value of including a wide range of parameter values in any general survey of thermoviscous behavior.

For weak or moderate conductors, the absorption and dispersion measures for type I waves at moderately low frequencies can be expressed with any desired precision by means of power-series expansions in the frequency number X:

$$\left(\frac{c}{c_0}\right)^2 = 1 + \frac{1}{4}X^2[3 + 10(\gamma - 1)Y - (\gamma - 1)(7 - 3\gamma)Y^2] + 0(X^4)$$

$$\frac{\alpha}{k_0} \equiv \frac{A_0}{2\pi} = \frac{1}{2}X\left\{1 + (\gamma - 1)Y - \frac{1}{8}X^2[5 + 35(\gamma - 1)Y + (\gamma - 1)(35\gamma - 63)Y^2 \right.$$
$$\left. + (\gamma - 1)(5\gamma^2 - 30\gamma + 33)Y^3]\right\} + O(X^5) \qquad (3c\text{-}76)$$

$$\frac{\alpha}{k} \equiv \frac{A}{2\pi} = \frac{1}{2}X\left\{1 + (\gamma - 1)Y - \frac{1}{4}X^2[1 + 11(\gamma - 1)Y - (\gamma - 1)(23 - 11\gamma)Y^2 \right.$$
$$\left. + (\gamma - 1)(\gamma^2 - 10\gamma + 13)Y^3]\right\} + O(X^5)$$

Note that $\alpha/k \equiv \alpha\lambda/2\pi \equiv A/2\pi$, where A is the amplitude attenuation per wavelength, and that α/k_0 is similarly related to the attenuation per reference wavelength λ_0. The series (3c-76) can be used with confidence for almost any values of γ and Y so long as the frequency is low enough to keep $X < 0.1$, and for a somewhat wider range of X when certain restrictions on γ and Y are satisfied.[1]

On the other hand, for frequencies high enough to make $X^{-2} \ll 1$, the absorption and dispersion are given, within $O(X^{-2})$, by

$$\frac{(c/c_0)^2}{2X} = \frac{\alpha}{k} \equiv \frac{A}{2\pi} = \frac{A_0{}^2X}{2\pi^2} = \left(\frac{\alpha}{k_0}\right)^2 2X$$
$$= 1 - \frac{1 - Y}{(1 - \gamma Y)X} \qquad (3c\text{-}77)$$

It can be inferred at once from (3c-77) that, for sufficiently high frequencies, dispersion is always anomalous (i.e., speed *increases* with frequency) regardless of γ and Y; that $\alpha/k = A/2\pi$ approaches the limit 1, and that α/k_0 and A_0 recede to zero as the actual wavelength decreases with respect to the reference wavelength λ_0. It also follows, from comparison of this result with (3c-76$_3$), that as frequency increases, $\alpha = A/\lambda = A_0/\lambda_0$ *will always have at least one maximum that is characteristic of viscothermal resonance.* The frequency at which this resonance occurs lies in the range $X = 1$ to 1.7, but the peak is relatively broad and flat and often cannot be located experimentally with high precision.

It can also be deduced from (3c-77) that the asymptotic speed of sound at very high frequencies will always be determined by viscosity alone, without regard for the form of the equation of state; thus,

$$(c^2)_{X\to\infty} = \frac{2\omega\eta\mathcal{V}}{\rho} \qquad (3c\text{-}78a)$$

Under the same limiting conditions, the asymptotic speed of type II, or "thermal," waves is similarly determined by thermal conductivity alone, according to

$$(c'^2)_{X\to\infty} = \frac{2\omega\kappa}{\rho_0 C_p} \qquad (3c\text{-}78b)$$

The steady increase of c' with $\omega^{\frac{1}{2}}$ predicted by (3c-78b) has sometimes been cited as a basis for denying that second sound in helium II, which displays small dispersion and low attenuation,[2] can be a type II thermal wave of the sort predicted by viscothermal

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[1] Truesdell, *J. Rational Mechanics and Analysis* **2**, 643–741 (October, 1953).

[2] Peshkof, *J. Phys.* (*U.S.S.R.*) **8**, 381 (1944); **10**, 389–398 (1946); Lane, Fairbank, and Fairbank, *Phys. Rev.* **71**, 600–605 (1947).

theory. This conclusion is probably correct but the argument is faulty inasmuch as the vanishing viscosity of the superfluid would make it more appropriate to use as a type criterion the behavior predicted for the limiting condition $X \to 0$. Thus, if the Kirchhoff-Langevin secular equation (3c-73) is reduced by letting $X \to 0$ while XY is held fixed, and if XY is then allowed to increase indefinitely as required by the superconductivity of helium II, what is left of (3c-73) *does* have a pair of roots for which the attenuation vanishes and the speed is nondispersive, viz., $\alpha = A_0 = 0$ and $c = c_0/\gamma^{\frac{1}{2}}$. This result looks, at first sight, like just an isothermal velocity for type I waves, as might be expected to prevail if uniform temperature were enforced by infinite conductivity. On the other hand, the wave types would be expected to interchange, according to (3c-75*a*), as Y becomes very large; and one has also to deal with the standing conclusion that any viscosity however small will eventually take over control of dispersion when X departs sufficiently from zero. These remarks are intended to emphasize primarily the fact that the problem of branch determination, or type identification, under such extreme circumstances needs probably to be attacked by considering the relative *rates* at which the various limiting conditions are approached. Other considerations need also to be taken into account, of course, in dealing with the two-fluid-mixture theory of liquid helium; but it seems clear that further inquiry is warranted concerning the relevance of classical viscothermal concepts now that a more exact theory of these effects is available.

The Kirchhoff approximation for weak or moderate conductors at low frequencies can be obtained directly from (3c-76) by neglecting terms in X^2 or higher. The dispersion is thereby predicted to be negligible, so that $c \doteq c_0$; and the "Kirchhoff" attenuation α_K is given by

$$\begin{aligned}\alpha_K &= \frac{1}{2}k_0[X + (\gamma - 1)XY] = \frac{1}{2}k_0 S\left(\mathcal{V} + \frac{\gamma - 1}{\mathcal{P}}\right)\\ &= \frac{\omega^2}{2\rho_0 c_0^3}\left[\eta\mathcal{V} + \frac{(\gamma - 1)\kappa}{C_p}\right]\end{aligned} \qquad (3c\text{-}79a)$$

If the Stokes relation is then presumed, by setting $\mathcal{V} = \frac{4}{3}$ (which neither Kirchhoff nor Stokes himself did in this connection), (3c-79*a*) becomes

$$\alpha_C = \frac{1}{2}k_0 S\left(\frac{4}{3} + \frac{\gamma - 1}{\mathcal{P}}\right) = \frac{\omega^2}{2\rho_0 c_0^3}\left[\frac{4}{3}\eta + \frac{(\gamma - 1)\kappa}{C_p}\right] \qquad (3c\text{-}79b)$$

The absorption predicted by (3c-79*b*) is commonly, but not very appropriately, referred to as "classical"; but such an emasculated theoretical prediction neither accounts adequately for the attenuation observed experimentally, except in the case of a few monatomic gases, nor does justice to the essential content of the classical theory of viscous conducting fluids.

Even when terms through X^2 are included, no change occurs in the odd function α/k_0, but dispersion is then predicted according to (3c-76$_1$) which accounts for the second-order effects of both compressional and shear viscosity, heat conduction, and their interaction. This dispersion is anomalous for weak or moderate conductors (small Y) but becomes normal if the speed-reducing influence of thermal conductivity becomes large enough to make $(7 - 3\gamma)Y > 10$. On the other hand, if heat exchange were to be ignored altogether, the first two terms of (3c-76$_1$) would give, for the dispersion due to viscosity alone,

$$\begin{aligned}\left(\frac{c}{c_0}\right)^2 &\doteq 1 + \frac{3}{4}X^2 = 1 + \frac{3}{4}\left(\frac{\omega\eta\mathcal{V}}{\rho_0 c_0^2}\right)^2\\ c &\doteq c_0\left[1 + \frac{3}{8}\left(\frac{\omega\eta\mathcal{V}}{\rho_0 c_0^2}\right)^2\right]\end{aligned} \qquad (3c\text{-}80)$$

Absorption and Dispersion Due to Heat Radiation. The effects of heat exchange by radiation, which were abandoned above in order to make (3c-74) more manageable, can now be assessed by reverting to (3c-73). The nonlinear interaction between radiation and viscosity will be neglected, for the sake of expediency, even though (3c-74) suggests that it may be as large as second order. The primary effects of viscosity and heat conduction can be eliminated from (3c-73) by letting both X and XY go to zero while holding the frequency variable $S' = \omega/\mathfrak{q}$ finite. This reduces the characteristic secular equation to the simple quadratic form

$$\gamma(S' - j) + \left(\frac{\alpha + jk}{k_0}\right)^2 (\gamma S' - j) = 0 \qquad \text{(3c-81)}$$

which can be solved directly to yield the following exact expressions for the attenuation and dispersion due to radiation alone:

$$\begin{aligned} \frac{A}{2\pi}\left(\frac{c_0}{c}\right)^2 &= \frac{\alpha}{k}\left(\frac{c_0}{c}\right)^2 = \gamma S' \frac{\gamma - 1}{2[1 + (\gamma S')^2]} \\ \left(\frac{A_0}{2\pi}\right)^2 &= \left(\frac{\alpha}{k_0}\right)^2 = \frac{1}{2}\gamma \frac{(1 + S')^{\frac{1}{2}}(1 + \gamma^2 S'^2)^{\frac{1}{2}} - (1 + \gamma S'^2)}{1 + (\gamma S')^2} \\ \left(\frac{c}{c_0}\right)^2 &= \frac{2}{\gamma} \frac{1 + (\gamma S')^2}{(1 + \gamma S'^2) + (1 + S'^2)^{\frac{1}{2}}(1 + \gamma^2 S'^2)^{\frac{1}{2}}} \end{aligned} \qquad \text{(3c-82)}$$

These equations indicate that both attenuation and dispersion become vanishingly small for either very large or very small values of S', and that a maximum of attenuation occurs in mid-range, near the single point of inflection of the dispersion curve. This absorption peak is characterized by

$$\begin{aligned} &\left(\frac{\alpha}{k}\right)_{\max} = \frac{\gamma^{\frac{1}{2}} - 1}{\gamma^{\frac{1}{2}} + 1} \qquad S'_{\max A} = \gamma^{-\frac{1}{2}} \qquad \tau_{\text{rad}} = \frac{2\pi\gamma^{\frac{1}{2}}}{\mathfrak{q}} \\ &\left(\frac{\alpha}{k_0}\right)_{\max} = \left(\frac{A_0}{2\pi}\right)_{\max} = \frac{\gamma - 1}{[8(\gamma + 1)]^{\frac{1}{2}}} \qquad S'_{\max A_0} = \gamma^{-1}\frac{(3\gamma + 1)^{\frac{1}{2}}}{(\gamma + 3)^{\frac{1}{2}}} \end{aligned} \qquad \text{(3c-83)}$$

There is a curious dearth of quantitative information concerning the radiation coefficient $\mathfrak{q}$, and little is added to this by noticing the low attenuation and negligible dispersion observed for a wide range of audible sounds in air since these might correspond to values of S' either far above or far below the resonance peak described by (3c-83). The choice $S' \gg 1$ is unambiguously dictated, however, by the fact that the observed speed of sound is very close to the isentropic value c_0, whereas (3c-82_5) indicates that the isothermal speed $c_0/\gamma^{\frac{1}{2}}$ would prevail if $\mathfrak{q}$ were large enough to make S' small for all audio frequencies. Truesdell[1] has pointed out that these conclusions leave still in effect a prediction that at some lower subaudible frequency a peak of attenuation should appear with a magnitude $A_0 = 0.185\pi$ ($\approx$5 dB per reference wavelength). This absorption peak has not been observed yet, at least deliberately, although its possible bearing on the acoustical character of thunder might be worth investigating.

Relaxation Processes and Sound Absorption. The foregoing analysis of heat exchange by radiation puts in evidence the first example of what would now be called a typical relaxation process. The characteristic feature of such a process, in so far as the gross hydrodynamical response of the medium is concerned, is the existence of two relations among the state variables, one of which prevails asymptotically for slow variations, the other for rapid changes. Such bivalent behavior is typical of fluid mixtures containing two interacting components, such as a partly dissociated gas[2] or an ionic solution.[3] In these cases the relative concentrations of the two components either follow faithfully, in quasi-static equilibrium, the dictates of slowly changing external variables, or else, at the other asymptotic limit, they do not change at all

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[1] C. A. Truesdell, *J. Rational Mechanics and Analysis* **2,** 643–741 [666] (October, 1953).

[2] Einstein, *Sitzber. deut. Akad. Wiss. Berlin Math.-Phys. Kl.* **1920,** 380–385.

[3] Liebermann, *Phys Rev.* **76,** 1520–1524 (1949).

when the finite reaction rate is such that the external variables can complete cyclic changes too rapidly for the concentrations to "follow." A different but comparable kind of mixture is exemplified by an ensemble of atoms or molecules capable of being excited to different energy levels, of which the most common example is a diatomic gas in which the rotational degrees of freedom may or may not share the cyclic work of compression depending on whether an appropriately normalized frequency variable is "low" or "high."

The *physical* problem of characterizing the rate-dependent properties of mixtures can be studied without regard for its acoustical consequences, and various approaches to this problem have turned on the assignment of two or more different internal or "partial" temperatures, different compressibilities, specific heats, etc. All the physical theories of pure relaxation appear to converge, however, in predicting the same *acoustical* behavior; viz., at low frequencies an asymptotic speed of sound c^0, a transition region of anomalous dispersion ($dc/d\omega > 0$) within which a maximum of attenuation occurs, and at high frequencies an asymptotic sound speed c^∞ which can be related to c^0 by writing $K \equiv c^0/c^\infty \leq 1$, where K is a material constant of the two-component medium. It follows then that, when the constant K and a dimensionless frequency variable X' can be properly identified and interpreted in terms of the *physical* mechanism involved, the *acoustical* behavior for any pure relaxation process will be described exactly by the following expressions derived from (3c-82) and (3c-83) by substitution:

$$\begin{aligned}
\left(\frac{c}{c^0}\right)^2 &= \frac{2(1+X'^2)}{1+K^2X'^2+[(1+K^4X'^2)(1+X'^2)]^{\frac{1}{2}}} \\
&\doteq \frac{1+X'^2}{1+K^2X'^2} \\
\frac{\alpha}{k}\left(\frac{c^0}{c}\right)^2 &= \frac{1}{2}\,\frac{(1-K^2)X'}{1+X'^2} \\
\left(\frac{\alpha}{k}\right)_{\max} &= \frac{1-K}{1+K} \qquad \left(\frac{\alpha}{k_0}\right)_{\max} = \frac{1-K^2}{[8(1+K^2)]^{\frac{1}{2}}} \\
X'_{\max A} &= K^{-1} = \frac{c^\infty}{c^0} \qquad X'_{\max A_0} = \left(\frac{3+K^2}{1+3K^2}\right)^{\frac{1}{2}}
\end{aligned} \tag{3c-84}$$

These equations revert exactly to (3c-82) and (3c-83) when the substitutions $K^2 = \gamma^{-1}$ and $X' = \gamma S'$ are made, and when a factor γ^{-1} is introduced to convert the low-frequency reference speed c^0 to the usual isentropic reference c_0. .

The "resonance" frequency characterizing a relaxation process is usually defined as the angular frequency at which the maximum attenuation per wavelength, $A = \alpha\lambda$, occurs; thus, $\omega_r \equiv 2\pi/\tau_r = (\omega/X')X'_{\max A}$, where τ_r is the related "relaxation period." It has been pointed out that *any* mechanism of sound absorption can be interpreted as a relaxation phenomenon by suitably defining its relaxation time. For example, viscosity and heat-conduction "relaxation times" and their associated "resonance frequencies" can be defined by writing

$$\tau_v = \frac{2\pi}{\omega_v} = \frac{X}{\omega}\,\frac{4}{3\mathcal{V}} = \frac{\frac{4}{3}\eta}{\rho_0 c_0^2} \qquad \tau_\kappa = \frac{2\pi}{\omega_\kappa} = \frac{XY}{\omega} = \frac{\kappa}{\rho_0 c_0^2 C_p} \tag{3c-85}$$

Note that ω_v is specified in such a way that it reduces to ω/X when $\mathcal{V}$ has the Stokes-relation value $\frac{4}{3}$. When these relaxation frequencies are introduced in (3c-79) and (3c-80), the second-order dispersion and the Kirchhoff linear approximation for attenuation become

$$\begin{aligned}
c &\doteq c_0\left[1+\frac{3}{8}(2\pi)^2\left(\frac{3\mathcal{V}}{4}\right)^2\frac{\omega^2}{\omega_v^2}\right] \\
\alpha_K &= \pi k_0\left[\frac{3\mathcal{V}}{4}\,\frac{\omega}{\omega_v}+(\gamma-1)\,\frac{\omega}{\omega_\kappa}\right]
\end{aligned} \tag{3c-86}$$

When the fluid medium consists of an ideal monatomic gas, the physical significance of the relaxation times τ_v and τ_κ can readily be interpreted as the time required for subsidence of a momentary departure from the equilibrium distribution of energy among the translational degrees of freedom. In the classical kinetic theory of gases, this recovery time is shown to be approximately $L/\bar{v}$, the mean free path divided by the mean molecular velocity.[1] The conformity of the definitions (3c-85) with this concept can then be verified by recalling the kinetic-theory evaluations of viscosity $[\eta \doteq \frac{1}{2}\rho\bar{v}L]$, thermal conductivity $[(\kappa/C_p) \doteq (5/4\gamma)\rho\bar{v}L]$, and the speed of sound $[c \doteq 0.74\bar{v}]$. These considerations show, incidentally, that for such a gas the attenuation per reference wavelength is contributed almost equally by viscosity and heat conduction, and is proportional to the ratio of mean free path to wavelength.

The precise physical significance of τ_v and τ_κ is less obvious for polyatomic gases and liquids; but if this is glossed over, the frequency ratios $\frac{3}{2}\pi\mathcal{V}\omega/\omega_v$, $\frac{4}{3}\omega_v/\omega_\kappa\mathcal{V}$, and $2\pi\omega/\omega_\kappa$ can be substituted directly for X, Y, and XY in any of the viscothermal relations deduced above. Merely introducing these "relaxation" frequencies, however, does not invest heat conduction or viscosity with any new or different relaxation-like properties, and the exact viscothermal theory, in whatever symbols expressed, continues to predict that sound speed will increase indefinitely with frequency, that A_0 will display a typical broad maximum for some X in the range 1 to 1.7 (depending on the thermoviscous parameters γ and Y), that $(A_0)_{\max}$ will always have about the same magnitude $(\alpha/k_0 \approx \frac{1}{3})$, and that the peak in A_0 can be made to occur at any chosen actual frequency by suitable assignment of the viscosity number $\mathcal{V}$ [cf. (3c-72), (3c-85)]. In contrast with this behavior, a pure relaxation phenomenon would call for the sound speed to level off at the high-frequency limit given by K^{-1}, and would display a maximum in A_0 that increases in height and retreats toward higher frequencies as the speed increment $c^\infty - c^0$ increases and K varies from 1 toward zero.

Allusion has already been made to the established fact that measured values of attenuation usually exceed the "classical" prediction (3c-79*b*) and often exhibit one or more maxima at finite frequencies. As a matter of fact, even when the complete consequences of the classical theory are taken into account, and when the viscosity number is adjusted to make the predicted attenuation at low frequencies correspond with experiment, the classical viscothermal theory still fails to account for all the experimental facts, but *for a reason that is just the opposite of that usually advanced*, namely, because it then predicts *too much* attenuation at the resonance peak and at higher frequencies! In spite of this latent contradiction, the alleged failure of "classical" theory as represented by (3c-79*b*) (which is, after all, only *part* of an *approximate* solution of the *linearized first-order* equations) has stimulated widespread efforts to repair its deficiency by invoking a wide variety of relaxation and other theories,[2] many of which have been marred by an *ad hoc* flavor that renders them little more than examples of ingenuity in curve fitting.

Measurements of absorption and dispersion in rarefied helium gas over a wide range of the frequency variable S have confirmed in all essential details the pattern of behavior predicted by the *exact* viscothermal theory.[3] Unless the classical concepts of viscosity and heat conduction are to be abandoned altogether, therefore, logic demands that the *exact* viscothermal theory be accepted as the foundation on which to erect any more complete analysis of sound absorption in media less idealized than rarefied

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[1] Jeans, "Dynamical Theory of Gases," 2d ed., pp. 260–262, Cambridge University Press, Cambridge, England, 1916.

[2] For reviews of what has been called the "exuberant literature" dealing with relaxation and other theories of sound absorption, see Kneser, *Ergeb. exakt. Naturwiss.* **22,** 121–185 (1949); Markham, Beyer, and Lindsay, *Revs. Modern Phys.* **23,** 353–411 (1951); Kittel, *Phys. Soc. (London), Repts. Progr. in Phys.* **11,** 205–247 (1948); see also, for background, W. T. Richards, *Revs. Modern Phys.* **11,** 36–64 (1939).

[3] Greenspan, *Phys. Rev.* **75,** 197–198 (1949); *J. Acoust. Soc. Am.* **22,** 568–571 (1950).

helium. A good many "honest" relaxation mechanisms do exist and must be accounted for, but in the accounting these effects should presumably be regarded as factors perturbing the fundamental thermoviscous behavior rather than the converse. The two-fluid-mixture theory of relaxation effects seems best adapted for inclusion in such a compound analysis, and a start in this direction has already been made.[1] Much remains to be done, however, before this basic acoustical problem can be said to be understood.

3c-10. Characteristic Acoustic Impedance of a Thermoviscous Medium. When the first-order sound pressure p_1 is put back into (3c-70$_2$) [by tracing its last term back through (3c-25$_{11}$)], this equation of motion can be rewritten at once in terms of the specific acoustic impedance, as follows:

$$[j\omega\rho_0 - (\alpha + jk)^2\eta\mathcal{V}]u_1 - (\alpha + jk)p_1 = 0$$

$$\begin{aligned} \frac{p_1}{u_1} \equiv Z &= jk\rho_0c(\alpha + jk)^{-1} - \eta\mathcal{V}(\alpha + jk) \\ &= \rho_0c\left(1 - j\frac{\alpha}{k}\right)^{-1} - j\rho_0c\frac{\omega\eta\mathcal{V}}{\rho_0{c_0}^2}\left(\frac{c_0}{c}\right)^2\left(1 - j\frac{\alpha}{k}\right) \qquad \text{(3c-87)} \\ \frac{p_1}{\rho_0cu_1} \equiv z &= \left(1 - j\frac{\alpha}{k}\right)^{-1} - jX\left(\frac{c_0}{c}\right)^2\left(1 - j\frac{\alpha}{k}\right) \end{aligned}$$

The normalized specific impedance, or *specific impedance ratio*, $(p_1/\rho_0cu_1) \equiv z$, which would be unity in the nondissipative case, is now in a form to be evaluated by direct substitution of the series expansions (3c-76). After some manipulation, and retaining only terms through X^2 and Y^2, the impedance ratio can be put in the form

$$\begin{aligned} \frac{p_1}{\rho_0cu_1} &= 1 - \frac{\alpha}{k}\left[\frac{\alpha}{k} + X\left(\frac{c_0}{c}\right)^2\right] + j\left[\frac{\alpha}{k} - X\left(\frac{c_0}{c}\right)^2\right] \\ &= 1 - \frac{1}{4}X^2[3 + 4(\gamma - 1)Y + (\gamma - 1)^2Y^2] + O(X^4) \\ &\qquad - j\left\{\frac{1}{2}X[1 - (\gamma - 1)Y] + O(X^3)\right\} \qquad \text{(3c-88)} \end{aligned}$$

It follows that sound pressure *lags* the particle velocity when $(\gamma - 1)\kappa/\eta\mathcal{V}C_p$ is less than unity, as it is for the common fluids under ordinary conditions; but pressure *leads* the particle velocity when the ratio of heat conductivity to viscosity is high enough to make $(\gamma - 1)\kappa > \eta\mathcal{V}C_p$.

3c-11. Thermal Noise in the Acoustic Medium. The mode of motion that is heat furnishes a restless background of noise that underlies all acoustical phenomena. The magnitude and nature of this thermal noise can be assessed by appealing to concepts drawn from such apparently unrelated sources as architectural acoustics, elementary quantum theory, and the classical kinetic theory of gases.

The scheme of analysis can be described simply: the thermoacoustic noise energy density, as measured by the mean-square sound pressure, is set equal to the density of the internal energy of thermal agitation associated with the translational degrees of freedom of the molecules composing the medium. It is then postulated that these molecular motions of thermal agitation can be regarded as a vector summation of the motions associated with a three-dimensional manifold of compressional standing waves, each behaving as it would in an ideal continuous medium having the same gross mechanical and elastic properties that characterize the actual medium. Each of these standing-wave systems thus constitutes an allowed, thermally excited, normal mode of vibration, or degree of freedom, to which can be assigned, in accordance with elementary quantum theory, the average energy

[1] Z. Sakadi, *Proc. Phys.-Math. Soc. Japan* (3) **23**, 208–213 (1941); Meixner, *Acustica* **2**, 101–109 (1952).

$$\frac{\text{Energy}}{\text{Mode}} = \frac{hf}{\exp(hf/kT) - 1} \tag{3c-89}$$

where h is Planck's constant, k is Boltzmann's constant, T is the absolute temperature, and f is the frequency in hertz.

The incremental number of such energy-bearing modes of vibration is given by the count of normal frequencies lying between f and $f + df$; and this is given, as in the theory of room acoustics,[1] by

$$dN = \left(\frac{4\pi V f^2}{c^3} + \frac{\pi S f}{2c^2} + \frac{L}{2c}\right) df \tag{3c-90}$$

where V is the volume, S the total surface, and L the sum of the three dimensions of the region under consideration, and where the three terms represent, respectively, the normal-frequency "points" distributed throughout the volume, over the coordinate planes, and along the coordinate axes of an octant of frequency space. If the three dimensions of the region are not too disparate, S can be approximated by $6V^{2/3}$, and L by $3V^{1/3}$, giving

$$dN = \frac{4\pi V f^2 df}{c^3}\left[1 + \frac{3\lambda}{4V^{1/3}} + \frac{3\lambda^2}{8\pi V^{2/3}}\right] \tag{3c-91}$$

For sufficiently high frequencies, this reduces to the classical expression (Rayleigh, 1900; Jeans, 1905) for the distribution of normal frequencies,

$$dN = \frac{4\pi V f^2 df}{c^3} \tag{3c-92}$$

an aymptotic form that can be shown (Weyl, 1911) to be independent of the shape of V and rigorously valid in the limit when $\lambda = c/f$ becomes small in comparison with $V^{1/3}$.

If attention is confined for the moment to finite frequency bands that do not include the lower frequencies, the incremental translational energy density of thermal agitation will be given by the product of (3c-89) and (3c-92). Then, by hypothesis, this can be set equal to the incremental energy density of the diffuse sound field, which is given by $d(\langle p^2\rangle/\rho c^2)$, where p is the rms sound pressure; thus

$$d\frac{\langle p^2\rangle}{\rho c^2} = \frac{(4\pi f^2 df/c^3)hf}{\exp(hf/kT) - 1} \tag{3c-93}$$

$$= \frac{(4\pi kT/c^3)f^2 df(hf/kT)}{\exp(hf/kT) - 1}$$

$$= \frac{4\pi kT}{c^3} f^2 df\left[1 - \frac{1}{2}\frac{hf}{kT} + \frac{1}{12}\left(\frac{hf}{kT}\right)^2 - \cdots\right], \quad \left(\frac{hf}{kT}\right)^2 < 4\pi^2 \tag{3c-94}$$

The total energy density associated with all the allowed modes of vibration is then to be found by extending the integral of (3c-94) over all frequencies less than the upper limiting frequency for which the mode count [by (3c-92)] is just equal to three times n_V, the total number of molecules in unit volume. This upper frequency limit, f_{lim}, is given, for either liquids or gases, by the integral of (3c-92),

$$\frac{N_{\text{lim}}}{V} = \frac{4\pi f_{\text{lim}}^3}{3c^3} = 3n_V = 3A\frac{\rho}{M} \qquad f_{\text{lim}}^3 = \frac{9c^3 A\rho}{4\pi M} \tag{3c-95}$$

where A is Avogadro's number (6.025×10^{26} molecules/kg mole), ρ is in kg/m^3, and M is the molecular weight (numeric, $O_2 = 32$). At ordinary room temperature, $f_{\text{lim}} \approx 2 \times 10^{10}$ Hz for air, $\approx 4 \times 10^{12}$ Hz for water. These frequencies are well outside the range so far accessible for acoustical experimentation and need not be

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[1] Maa, *J. Acoust. Soc. Am.* **10**, 235–238 (1939); Bolt, *J. Acoust. Soc. Am.* **10**, 228–234 (1939).

considered further except when the foregoing notions are used as the basis for a theory of specific heats, in which case it is necessary also to take into account vibrational and rotational degrees of freedom, and to reexamine the equilibrium statistics that underlie (3c-89). Note in passing that the *phonon* of specific-heat theory merely identifies the burden of internal energy carried by each of the normal modes of vibration postulated above.

Within the ranges of frequency and temperature ordinarily of interest in the assessment of thermal noise, the exponent hf/kT is so small that even the linear term in the series expansion of (3c-94) can be omitted. This amounts to a reversion to the classical analysis of energy partition in continuous media[1] and to the assignment of an energy kT to each allowed mode of vibration. With this simplification, (3c-94) can be integrated at once to yield the mean-square sound pressure, in the frequency band $f_2 - f_1$, as

$$\langle p^2 \rangle = \frac{4}{3} \pi k T \frac{\rho}{c} (f_2^3 - f_1^3) \qquad (\text{newtons/m}^2)^2 \qquad (3c\text{-}96)$$

in which Boltzmann's constant $k = 1.380 \times 10^{-23}$ joule/K, T is in kelvins, ρ in kg/m³, and c in m/sec. To facilitate computation, it is useful to rearrange (3c-96) in the following forms:

$$p_{\text{rms}} = 1.3 \times 10^{-12} \left(\frac{\rho}{c}\right)^{\frac{1}{2}} \left[\frac{T}{293} (f_2^3 - f_1^3)\right]^{\frac{1}{2}} \qquad \text{newtons/m}^2 \qquad (3c\text{-}97a)$$

$$(p_{\text{rms}})_{\text{air}} = 0.76 \times 10^{-10} \left[\frac{T}{293} (f_2^3 - f_1^3)\right]^{\frac{1}{2}} \qquad \text{dynes/cm}^2 = \mu\text{b} \qquad (3c\text{-}97b)$$

$$(p_{\text{rms}})_{\text{sea water}} = 10.6 \times 10^{-10} \left[\frac{T}{293} (f_2^3 - f_1^3)\right]^{\frac{1}{2}} \qquad \mu\text{b} \qquad (3c\text{-}97c)$$

in which the constants have been adjusted to make the temperature factor reduce to unity at 20°C, and where ρ/c has been taken as 0.00345 for air and 0.67 for sea water. It follows, for example, that the rms thermal noise pressure, for the wide-range audio-frequency band extending to 19 kHz in air, is just equal to the reference sound pressure, $p_0 = 0.0002\ \mu\text{b}$.

The power spectrum of thermal noise can be deduced from either (3c-94) or (3c-97*b*) and may be expressed as a *sound spectrum level* by writing

$$\begin{aligned} \beta_{\text{noise}} &= 10 \log_{10} \left(\frac{d(\langle p^2 \rangle / p_0^2)}{df}\right) = 10 \log_{10} \frac{4\pi k T f^2 \rho}{c p_0^2} \\ &= 10 \log_{10} \left[4.33 \times 10^{-7} (f_{\text{kc/s}})^2 \frac{T}{293}\right] \\ &= -63.6 + 20 \log_{10} f_{\text{kc/s}} + 10 \log_{10} \frac{T}{293} \qquad \text{db} \qquad (3c\text{-}98) \end{aligned}$$

Note that this noise spectrum is *not* "white" but has instead a uniform positive slope of 6 dB/octave, corresponding to an rms thermal-noise sound pressure that is directly proportional to frequency. On the other hand, for frequencies low enough to make the additive "correction" terms of (3c-91) significant, the noise spectrum level tends increasingly to lie *above* the +6 dB/octave line as the frequency approaches the low-frequency cutoff at which only the gravest mode of vibration can be excited. The noise spectrum level can also be expected to vary erratically as the low-frequency limit is approached and the population of normal frequencies becomes sparse, in much the same way that the steady-state pressure response of small rooms varies irregularly with frequency when only a few normal modes of vibration are available for excitation. It does not follow, however, that thermal noise in such a small enclosure could be

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[1] Jeans, "Dynamical Theory of Gases," 2d ed., pp. 381–391.

"quieted" by the application of sound absorbents. The boundary surfaces, without regard for their acoustical character, will always reach the same radiative equilibrium with the interior medium if both are at the same temperature; otherwise there would be a net flow of thermal "noise" energy across the boundaries in the guise of ordinary heat transfer.

The possibility that thermal noise might be the factor that limits human hearing acuity can be assessed with the help of (3c-98). If the critical-band theory of masking by wide-band noise continues to hold for subliminal stimuli, the effective masking level of thermal noise can be found by adding, at any frequency, the critical bandwidth (expressed as $10 \log_{10} \Delta f_c$) and the spectrum level given by (3c-98). Comparing this result with the binaural threshold for random incidence then leads to the conclusion that thermal noise remains about 11 to 13 dB below threshold at the frequency of greatest vulnerability (ca. 3 to 5 kHz), even for young people with exceptionally acute hearing. On this basis human hearing might be assigned a "noise figure" of approximately 12 dB. It is probable that some at least of this failure to achieve ideal function can be ascribed to internal noise of physiological origin. The near miss on thermal noise limiting gives comforting reassurance, however, that not more than a few decibels of additional hearing acuity could be utilized effectively by humans even if biological adaptation were to make it available.