3h. Frequencies of Simple Vibrators. Musical Scales

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3h-1. Strings. The fundamental frequency of vibration of an ideal string is

$$f_0 = \frac{1}{2l}\sqrt{\frac{F}{m}} \tag{3h-1}$$

where f_0 is the frequency, l is the free length, F is the force (tension) stretching the string, and m is the mass per unit length. Values of m for steel and gut strings are given in Table 3h-1.

In addition to the vibration in a single loop which gives rise to the fundamental frequency, the ideal string may vibrate in harmonics whose frequencies are

$$f_n = nf_0 \tag{3h-2}$$

where n is the integer denoting the particular mode of vibration. The length of each vibration loop is l/n. These successive lengths and the corresponding periods of vibration (i.e., the reciprocals of the frequencies) constitute a harmonic series according to the strict mathematical definition; nowadays, however, the frequencies themselves are usually said to make up a harmonic series.

The frequencies of actual strings depart somewhat from the frequencies computed from the simple formula because actual strings are stiff, they may be partially clamped at the ends, they are not infinitely thin, the tension increases with amplitude of vibration, the mass per unit length is not exactly uniform, there is internal damping and damping due to the surrounding air and supports, and the supports are not infinitely rigid. In the formulas which follow damping has been neglected.

For an actual string set

$$f = nf_0(1 + G) \tag{3h-3}$$

where the factor $(1 + G)$ is a measure of the departure (i.e., the inharmonicity) from the ideal harmonic values. Table 3h-2 lists values of G for various small perturbations. The approximations are valid only when G is small.

For musical purposes it is often convenient to give the inharmonicity in cents (hundredths of an equally tempered semitone) by setting

$$1 + G = 2^{\delta/1,200} = e^{\delta/1,731} \tag{3h-4}$$

where δ is the inharmonicity. To a usually acceptable approximation, $\delta = 1{,}731G$.

If the stiff string listed in Table 3h-2 is of steel music wire, $Y/\rho = 25.5 \times 10^6$ m^2/sec^2, Y being Young's modulus and ρ the density. The tension is very nearly $F = l^2\rho f_0^2\pi d^2$. Thus for steel wire, and by virtue of the stiffness formula, the inharmonicity in cents is $\delta = 3.4 \times 10^{13} d^2 n^2/f_0^2 l^4$, provided that the diameter and length are in centimeters.

TABLE 3h-1. MASS PER UNIT LENGTH OF STEEL AND GUT STRINGS*

Diam		Steel, g/m	Gut, g/m	Diam		Steel, g/m	Gut, g/m	Diam		Steel, g/m	Gut, g/m
mm	in.			mm	in.			mm	in.		
0.20	0.0079	0.25	0.04	1.00	0.0394	6.15	1.10	1.80	0.0709	19.9	3.56
0.22	0.0087	0.30	0.05	1.02	0.0402	6.40	1.14	1.82	0.0717	20.4	3.64
0.24	0.0094	0.35	0.06	1.04	0.0409	6.65	1.19	1.84	0.0724	20.8	3.72
0.26	0.0102	0.42	0.07	1.06	0.0417	6.91	1.24	1.86	0.0732	21.3	3.80
0.28	0.0110	0.48	0.09	1.08	0.0425	7.17	1.28	1.88	0.0740	21.7	3.88
0.30	0.0118	0.55	0.10	1.10	0.0433	7.44	1.33	1.90	0.0748	22.2	3.97
0.32	0.0126	0.63	0.11	1.12	0.0441	7.71	1.38	1.92	0.0756	22.7	4.05
0.34	0.0134	0.71	0.13	1.14	0.0449	7.99	1.43	1.94	0.0764	23.1	4.14
0.36	0.0142	0.80	0.14	1.16	0.0457	8.27	1.48	1.96	0.0772	23.6	4.22
0.38	0.0150	0.89	0.16	1.18	0.0465	8.56	1.53	1.98	0.0780	24.1	4.31
0.40	0.0157	0.98	0.18	1.20	0.0472	8.86	1.58	2.00	0.0787	24.6	4.40
0.42	0.0165	1.08	0.19	1.22	0.0480	9.15	1.64	2.02	0.0795	25.1	4.49
0.44	0.0173	1.19	0.21	1.24	0.0488	9.46	1.69	2.04	0.0803	25.6	4.58
0.46	0.0181	1.30	0.23	1.26	0.0496	9.76	1.75	2.06	0.0811	26.1	4.67
0.48	0.0189	1.42	0.25	1.28	0.0504	10.1	1.80	2.08	0.0819	26.6	4.76
0.50	0.0197	1.54	0.27	1.30	0.0512	10.4	1.86	2.10	0.0827	27.1	4.85
0.52	0.0205	1.66	0.30	1.32	0.0520	10.7	1.92	2.12	0.0835	27.6	4.94
0.54	0.0213	1.79	0.32	1.34	0.0528	11.1	1.97	2.14	0.0843	28.2	5.04
0.56	0.0220	1.93	0.34	1.36	0.0535	11.4	2.03	2.16	0.0850	28.7	5.13
0.58	0.0228	2.07	0.37	1.38	0.0543	11.7	2.09	2.18	0.0858	29.2	5.23
0.60	0.0236	2.21	0.40	1.40	0.0551	12.1	2.16	2.20	0.0866	29.8	5.32
0.62	0.0244	2.36	0.42	1.42	0.0559	12.4	2.22	2.22	0.0874	30.3	5.42
0.64	0.0252	2.52	0.45	1.44	0.0567	12.8	2.28	2.24	0.0882	30.9	5.52
0.66	0.0260	2.68	0.48	1.46	0.0575	13.1	2.34	2.26	0.0890	31.4	5.62
0.68	0.0268	2.84	0.51	1.48	0.0583	13.5	2.41	2.28	0.0898	32.0	5.72
0.70	0.0276	3.01	0.54	1.50	0.0591	13.8	2.47	2.30	0.0906	32.5	5.82
0.72	0.0283	3.19	0.57	1.52	0.0598	14.2	2.54	2.32	0.0913	33.1	5.92
0.74	0.0291	3.37	0.60	1.54	0.0606	14.6	2.61	2.34	0.0921	33.7	6.02
0.76	0.0299	3.55	0.64	1.56	0.0614	15.0	2.68	2.36	0.0929	34.3	6.12
0.78	0.0307	3.74	0.67	1.58	0.0622	15.4	2.74	2.38	0.0937	34.8	6.23
0.80	0.0315	3.94	0.70	1.60	0.0630	15.7	2.81	2.40	0.0945	35.4	6.33
0.82	0.0323	4.14	0.74	1.62	0.0638	16.1	2.89	2.42	0.0953	36.0	6.44
0.84	0.0331	4.34	0.78	1.64	0.0646	16.5	2.96	2.44	0.0961	36.6	6.55
0.86	0.0339	4.55	0.81	1.66	0.0654	16.9	3.03	2.46	0.0968	37.2	6.65
0.88	0.0346	4.76	0.85	1.68	0.0661	17.4	3.10	2.48	0.0976	37.8	6.76
0.90	0.0354	4.98	0.89	1.70	0.0669	17.8	3.18	2.50	0.0984	38.4	6.87
0.92	0.0362	5.20	0.93	1.72	0.0677	18.2	3.25	2.52	0.0992	39.1	6.98
0.94	0.0370	5.43	0.97	1.74	0.0685	18.6	3.33	2.54	0.1000	39.7	7.09
0.96	0.0378	5.67	1.01	1.76	0.0693	19.0	3.41	2.56	0.1008	40.3	7.21
0.98	0.0386	5.91	1.06	1.78	0.0701	19.5	3.48	2.58	0.1016	40.9	7.32

* This table is based on a density of steel of 7.83 g/cm³. Density of gut is assumed to be 1.4 g/cm³, about one-sixth that of steel. This is only approximate, since the density of gut varies from sample to sample, and increases markedly with humidity. Brass wire has a density of 8.7 g/cm³, about 1.1 times that of steel.

3h-2. Air Columns and Rods. The air within a simple tube of constant cross section, open at both ends or closed at both ends, vibrates freely at a frequency near

$$f = \frac{nc}{2l} \tag{3h-5}$$

where n is an integer (mode of vibration number), c is the speed of sound in the contained air, and l is the length of the tube. (See Sec. 3d for speed of sound in air and its dependence on temperature.) The diameter of the tube must be relatively small;

TABLE 3h-2. PERTURBATION IN FREQUENCY OF A STRING

Cause	G	Explanation
Stiffness	$\frac{n^2\pi^3 d^4 Y}{128 l^2 F}$	Y is Young's modulus, d is the diameter of the string
Yielding support	$\frac{4ml}{4\pi^2 n^2 M - K/f_0^2}$	The support consists of a mass M on a spring of transverse force constant K. Multiply by 2 if there are two such supports
Variable density	$-\frac{1}{l}\int_0^l g(x) \sin^2 \frac{\pi n x}{l}\, dx$	The mass per unit length is $m = m_0[1 + g(x)]$ where m_0 is the mean value over the string and x is the distance from one end of the string; the function $g(x)$ must be small in comparison with unity

plane sound waves propagated longitudinally are assumed. The same formula applies to thin rods vibrating longitudinally and suitably supported (say, at distances $l/2n$ from the ends) so that the vibration is not inhibited. (See Sec. 3f for speed of sound in solids.)

An open organ pipe is an example of a doubly open tube of constant cross section. To calculate its frequency adequately it must be recognized, however, that the air beyond the physical ends of the tube partakes of the vibration and adds inertia to the vibrating system. (This does *not* mean, however, that there is a velocity antinode beyond the end of the tube.) The necessary corrections to the simple formula are usually introduced as empirical "end corrections" to be added to the geometrical length; thus

$$f = \frac{nc}{2(l + x_1 + x_2)} \tag{3h-6}$$

where $x_1 = 0.3d$ is the correction for the unimpeded end (d being the inside diameter of the pipe) and $x_2 = 1.4d$ is the correction for the mouth of the pipe. These are rough approximations; the literature on the end correction is extensive.[1]

The air inside a cylindrical tube that is closed at one end and open at the other vibrates at frequency

$$f = \frac{nc}{4(l + x)} \tag{3h-7}$$

where $x = 0.3d$ if the open end is unimpeded. In the case of the "closed" organ pipe (meaning closed at one end only), for the mouth $x = 1.4d$.

[1] E. G. Richardson, ed., "The Technical Aspects of Sound," vol. I, pp. 493–496, 578, Elsevier Publishing Company, Amsterdam, 1953; Harold Levine, *J. Acoust. Soc. Am.* **26**, 200–211 (1954).

The speed of sound c (and thus the frequency of vibration) in a gas contained within a tube is reduced somewhat from its value c_0 in free space, as a consequence of friction and loss of heat to the wall of the tube. If the frequency of vibration f and the tube diameter d are such that $df^{\frac{1}{2}} > 2\nu^{\frac{1}{2}}$, ν being the kinematic viscosity of the gas, the speed of sound (longitudinal phase velocity) within the tube is[1]

$$c = \frac{c_0}{[1 + 2(\nu/\pi f)^{\frac{1}{2}}/d]^{\frac{1}{2}}[1 + 2(\gamma - 1)(\nu/\pi f P_r)^{\frac{1}{2}}/d]^{\frac{1}{2}}}$$

where γ is the ratio of specific heats, and P_r the Prandtl number for the gas. For air at 20°C, and when $df^{\frac{1}{2}} > 0.8$ with d in cm and f in hertz, with slight approximation the Helmholtz-Kirchhoff correction for the speed of sound is

$$c = c_0\left(1 - \frac{0.33}{df^{\frac{1}{2}}}\right)$$

Correspondingly the interval by which the frequency of vibration is lowered owing to friction and heat conduction is $572/df^{\frac{1}{2}}$ cents. As $df^{\frac{1}{2}}$ becomes less than $2\nu^{\frac{1}{2}}$ a transition[1] occurs to an even more marked reduction in the speed of sound propagation in the tube.

The air in a conical tube is resonant in some cases at the same frequencies as a doubly open cylindrical tube of the same length, but there is the important difference that the contained sound waves are spherical rather than plane. Table 3h-3 gives equations[2] to be solved for each combination of end conditions; $k = 2\pi f/c$. "Closed-open," for example, means that the smaller end of the truncated cone is closed while the larger end is open; r_1 is the slant distance from the extrapolated apex of the cone to the smaller end and r_2 is the slant distance to the larger end. The slant length of the resonator is thus $r_2 - r_1$. When $r_1 = 0$, the length is r_2 and the cone is complete to the apex. Formulas for computing frequency when the cone is complete are shown at the right of Table 3h-3. As in the case of cylindrical tubes, the length should be

TABLE 3h-3. FREQUENCIES OF CONICAL RESONATORS

Ends	Equation	For $r_1 = 0$
Closed-closed	$kr_2 - \tan^{-1} kr_2 = kr_1 - \tan^{-1} kr_1$	$\tan kr_2 = kr_2$
Closed-open	$\tan k(r_2 - r_1) = -kr_1$	$f_1 = \frac{nc}{2r_2}$
Open-closed	$\tan k(r_2 - r_1) = kr_2$	$\tan kr_2 = kr_2$
Open-open	$f = \frac{nc}{2(r_2 - r_1)}$	$f = \frac{nc}{2r_2}$

slightly modified by end corrections. As the angle of the cone increases the correction decreases and may even become negative.[3]

3h-3. Volume Resonators. The Helmholtz resonator consists of a nearly closed cavity of volume V with an opening of acoustical conductance C. If the opening is

[1] A. H. Benade, *J. Acoust. Soc. Am.* **44**, 616–623 (1968). Multiplication by the correction term is erroneously shown there in eq. (13c), instead of division.

[2] Eric J. Irons, *Phil. Mag.* **9**, 346–360 (1930).

[3] A. E. Bate and E. T. Wilson, *Phil. Mag.* **26**, 752–757 (1938).

in a thin wall the conductance is simply d, the diameter of the hole. If the opening is through a short neck of length l, approximately

$$C = \frac{\pi d^2}{4(l + 0.8d)} \tag{3h-8}$$

The natural frequency of the resonator is

$$f = \frac{c}{2\pi}\sqrt{\frac{C}{V}} \tag{3h-9}$$

the speed of sound in the opening being c. The equation is valid for wavelengths large in comparison with the dimensions of the resonator.

The ocarina may be recognized as an instrument of the resonator type because the *position* of an open hole of given size is immaterial; when the holes are all equal, they can be opened in any order to give the same scale. The total conductance for use in the formula given above is the sum of the conductance of individual holes, provided that they are separated far enough that there is no interaction.

TABLE 3h-4. FREQUENCIES OF TRANSVERSE VIBRATION OF BARS

Ends	Frequency	Ratio			Cents		
	Mode → 1	2	3	4	2	3	4
Clamped-free	$f_1 = \frac{0.5597\kappa}{l^2}\sqrt{\frac{Y}{\rho}}$	6.267	17.548	34.387	3,177	4,960	6,124
Free-free, or clamped-clamped	$f_1 = \frac{3.561\kappa}{l^2}\sqrt{\frac{Y}{\rho}}$	2.756	5.404	8.933	1,755	2,921	3,791

3h-4. Bars. A long thin bar clamped and/or free at the end(s) can vibrate transversely at the fundamental frequencies listed in Table 3h-4 under mode 1. The length of the bar is l, Y is Young's modulus, ρ is the density, and κ is the radius of gyration about the neutral axis of the cross section. For a round bar $\kappa = d/4$, where d is the diameter. For a flat bar of thickness t (in the plane of vibration) $\kappa = t/\sqrt{12}$; the width is immaterial. The frequency of a bar clamped at both ends is the same as that of a bar free at both ends. The frequency of a higher mode of vibration can be found by multiplying the fundamental frequency by the ratio indicated in Table 3h-4; the intervals in cents corresponding to these ratios are given at the extreme right of the table. These are the classic[1] values for thin bars; the frequencies of actual bars are lowered slightly as a consequence of rotatory inertia, lateral inertia, and shear.[2] For example, for a steel bar whose length is 40 times the thickness, the frequencies of the first four modes of vibration are expected to be 0.997, 0.992, 0.984, and 0.974 times the corresponding "thin" values (i.e., lowered 5, 14, 28, and 46 cents, respectively).

[1] Lord Rayleigh, "Theory of Sound," vol. I, p. 280, Macmillan & Co., Ltd., London, 1894. The interval erroneously given as 2.4359 octaves has been corrected here to 2.4340 octaves = 2,921 cents.

[2] William T. Thomson, *J. Acoust. Soc. Am.* **11**, 199–204 **(1939)**. There is an error: $m = \beta/[1 + \beta^2(k/L)^2]^{\frac{1}{2}}$, not $m = \beta/[1 + \beta^2(k/L)^2]^{\frac{1}{4}}$.

TABLE 3h-5. FREQUENCIES OF THE EQUALLY TEMPERED SCALE, BASED ON THE INTERNATIONAL STANDARD A = 440 HERTZ

Note	S	f	$2\pi f$	Note	S	f	$2\pi f$	Note	S	f	$2\pi f$
C_0	0	16.352	102.74	C_3	36	130.81	821.92	C_6	72	1,046.5	6,575.4
	1	17.324	102.74		37	138.59	870.79		73	1,108.7	6,966.4
D_0	2	18.354	115.32	D_3	38	146.83	922.58	D_6	74	1,174.7	7,380.6
	3	19.445	122.18		39	155.56	977.43		75	1,244.5	7,819.5
E_0	4	20.602	129.44	E_3	40	164.81	1,035.6	E_6	76	1,318.5	8,284.4
F_0	5	21.827	137.14	F_3	41	174.61	1,097.1	F_6	77	1,396.9	8,777.1
	6	23.125	145.30		42	185.00	1,162.4		78	1,480.0	9,299.0
G_0	7	24.500	153.93	G_3	43	196.00	1,231.5	G_6	79	1,568.0	9,851.9
	8	25.957	163.09		44	207.65	1,304.7		80	1,661.2	10,438
A_0	9	27.500	172.59	A_3	45	220.00	1,382.3	A_6	81	1,760.0	11,058
	10	29.135	183.06		46	233.08	1,464.5		82	1,864.7	11,716
B_0	11	30.868	193.95	B_3	47	246.94	1,551.6	B_6	83	1,975.5	12,413
C_1	12	32.703	205.48	C_4	48	261.63	1,643.8	C_7	84	2,093.0	13,151
	13	34.648	217.70		49	277.18	1,741.6		85	2,217.5	13,933
D_1	14	36.708	230.64	D_4	50	293.66	1,845.2	D_7	86	2,349.3	14,761
	15	38.891	244.36		51	311.13	1,954.9		87	2,489.0	15,639
E_1	16	41.203	258.89	E_4	52	329.63	2,071.1	E_7	88	2,637.0	16,569
F_1	17	43.654	274.28	F_4	53	349.23	2,194.3	F_7	89	2,793.8	17,554
	18	46.249	290.59		54	369.99	2,324.7		90	2,960.0	18,598
G_1	19	48.999	307.87	G_4	55	392.00	2,463.0	G_7	91	3,136.0	19,704
	20	51.913	326.18		56	415.30	2,609.4		92	3,322.4	20.875
A_1	21	55.000	345.58	A_4	57	440.00	2,764.6	A_7	93	3,520.0	22,117
	22	58.270	366.12		58	466.16	2,929.0		94	3,729.3	23,432
B_1	23	61.735	387.90	B_4	59	493.88	3,103.2	B_7	95	3,951.1	24,825
C_2	24	65.406	410.96	C_5	60	523.25	3,287.7	C_8	96	4,186.0	26,301
	25	69.296	435.40		61	554.37	3,483.2		97	4,434.9	27,865
D_2	26	73.416	461.29	D_5	62	587.33	3,690.3	D_8	98	4,698.6	29,522
	27	77.782	488.72		63	622.25	3,909.7		99	4,978.0	31,278
E_2	28	82.407	517.78	E_5	64	659.26	4,142.2	E_8	100	5,274.0	33,138
F_2	29	87.307	548.57	F_5	65	698.46	4,388.5	F_8	101	5,587.7	35,108
	30	92.499	581.19		66	739.99	4,649.5		102	5,919.9	37,196
G_2	31	97.999	615.74	G_5	67	783.99	4,926.0	G_8	103	6,271.9	39,408
	32	103.83	652.36		68	830.61	5,218.9		104	6,644.9	41,751
A_2	33	110.00	691.15	A_5	69	880.00	5,529.2	A_8	105	7,040.0	44,234
	34	116.54	732.25		70	932.33	5,858.0		106	7,458.6	46,864
B_2	35	123.47	775.79	B_5	71	987.77	6,206.3	B_8	107	7,902.1	49,651

Numerous subscript notations have been employed to distinguish the notes of one octave from those of another. The particular scheme used here assigns to C_0 a frequency which corresponds roughly to the lowest audible pitch. S is the number of semitones counted from this C_0.

The simple tuning fork may be recognized as an example of dual clamped-free bars. The frequency of a tuning fork made of ordinary steel can be computed approximately from

$$f = \frac{80{,}000t}{l^2}\ \text{Hz} \tag{3h-10}$$

provided that the thickness t and length l of the prongs are given in centimeters.

It is evident from Table 3h-4 that the different modes of vibration of a uniform bar are inharmonic. However, the cross section of the bar in the modern xylophone or marimba is often given an empirical lengthwise "undulation" such that the second

Table 3h-6. Intervals in Cents Corresponding to Certain Frequency Ratios

Name of interval	Frequency ratio	Cents
Unison	1:1	0
Minor second or semitone	1.059463:1	100
Semitone	16:15	111.731
Minor tone or lesser whole tone	10:9	182.404
Major second or whole tone	1.122462:1	200
Major tone or greater whole tone	9:8	203.910
Minor third	1.189207:1	300
Minor third	6:5	315.641
Major third	5:4	386.314
Major third	1.259921:1	400
Perfect fourth	4:3	498.045
Perfect fourth	1.334840:1	500
Augmented fourth	45:32	590.224
Augmented fourth	1.414214:1	600
Diminished fifth	1.414214:1	600
Diminished fifth	64:45	609.777
Perfect fifth	1.498307:1	700
Perfect fifth	3:2	701.955
Minor sixth	1.587401:1	800
Minor sixth	8:5	813.687
Major sixth	5:3	884.359
Major sixth	1.681793:1	900
Harmonic minor seventh	7:4	968.826
Grave minor seventh	16:9	996.091
Minor seventh	1.781797:1	1,000
Minor seventh	9:5	1,017.597
Major seventh	15:8	1,088.269
Major seventh	1.887749:1	1,100
Octave	2:1	1,200.000

mode of vibration of the free-free bar is changed in frequency to 3 or 4 times the fundamental frequency.[1] The frequencies of the higher modes of vibration are also modified by variation in cross section for special purposes such as the simulation of the sound of a bell.[2]

3h-5. Membranes. The membrane often assumed for vibration calculations is flexible, thin, and of uniform mass per unit area σ. The membrane is stretched by a tension T, this being the force per unit length anywhere in the membrane. The

[1] See U.S. Pats. 1,838,502 (1931) and 1,632,751 (1927).

[2] See U.S. Pats. 2,273,333 (1942), 2,516,725 (1950), 2,536,800 (1951), and 2,606,474 (1952).

characteristic frequencies of transverse vibration for such a rectangular membrane clamped at its edges are given by

$$f = \frac{c}{2}\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]^{\frac{1}{2}} \qquad \text{(3h-11)}$$

where

$$c = \sqrt{\frac{T}{\sigma}} \qquad \text{(3h-12)}$$

is the speed of propagation of transverse wave motion, a and b are the lengths of the sides, and m and n are integers. Note the similarity of Eq. (3h-11) to Eq. (3h-1).

TABLE 3h-7. RATIOS FOR INTERVALS TO 100 CENTS

Cents	Ratio	Cents	Ratio	Cents	Ratio	Cents	Ratio
0	1.000000	25	1.014545	50	1.029302	75	1.044274
1	1.000578	26	1.015132	51	1.029896	76	1.044877
2	1.001158	27	1.015718	52	1.030492	77	1.045481
3	1.001734	28	1.016305	53	1.031087	78	1.046085
4	1.002313	29	1.016892	54	1.031683	79	1.046689
5	1.002892	30	1.017480	55	1.032079	80	1.047294
6	1.003472	31	1.018068	56	1.032876	81	1.047899
7	1.004052	32	1.018656	57	1.033473	82	1.048505
8	1.004632	33	1.019244	58	1.034070	83	1.049111
9	1.005212	34	1.019833	59	1.034667	84	1.049717
10	1.005793	35	1.020423	60	1.035265	85	1.050323
11	1.006374	36	1.021012	61	1.035863	86	1.050930
12	1.006956	37	1.021602	62	1.036462	87	1.051537
13	1.007537	38	1.022192	63	1.037060	88	1.052145
14	1.008120	39	1.022783	64	1.037660	89	1.052753
15	1.008702	40	1.023374	65	1.038259	90	1.053361
16	1.009285	41	1.023965	66	1.038859	91	1.053970
17	1.009868	42	1.024557	67	1.039459	92	1.054579
18	1.010451	43	1.025149	68	1.040060	93	1.055188
19	1.011035	44	1.025741	69	1.040661	94	1.055798
20	1.011619	45	1.026334	70	1.041262	95	1.056408
21	1.012204	46	1.026927	71	1.041864	96	1.057018
22	1.012789	47	1.027520	72	1.042466	97	1.057629
23	1.013374	48	1.028114	73	1.043068	98	1.058240
24	1.013959	49	1.028708	74	1.043671	99	1.058851

The characteristic frequencies of a circular membrane clamped at its boundary are given by

$$f = \frac{c}{2a}\beta_{mn}$$

where a is the radius of the membrane. For $n = 1$, 2, and 3, $\beta_{0n} = 0.766$, 1.757, and 2.755, these numbers being the first three roots divided by π of the Bessel function of zero order set equal to zero. Similarly, $\beta_{1m} = 1.220$, 2.233, and 3.238 are from the Bessel function of first order and $\beta_{2n} = 1.635$, 2.679, and 3.699 are from the Bessel function of second order. The number of diametral nodes is m; the number of circular nodes is n, including the node at the boundary. The modes of vibration are not in general harmonics; the lowest characteristic frequencies are in the proportions 1.000:1.593:2.135. For a circular membrane constrained to certain radial (not diametral) nodes, harmonics are, however, possible.

The tambourine is a musical instrument that consists of a free membrane nearly of the kind discussed above. In most drums, however, the membrane closes a cavity; in the case of the kettledrum (and some kinds of capacitor microphones) this cavity is relatively rigid and airtight. If the speed of transverse waves in the membrane is significantly less than the speed of sound in the contained air, the cavity has little effect on those modes of vibration with diametral nodes. The frequencies of other modes of vibration are increased[1] by the stiffness of the contained air.

3h-6. Musical Scales. By international agreement the standard tuning frequency for musical performance is the A of 440 Hz. The frequencies of the equally tempered scale based on this frequency appear in Table 3h-5. Middle C thus has a frequency of 261.6 Hz. The C of 256 Hz, frequently used in the past for demonstrations in physics, has never been adopted for practical musical performance.

For many calculations with musical intervals it is convenient to deal with logarithmic units that can be added instead of the ratios which must be multiplied. The octave is equal to 1,200 logarithmic cents, and the equally tempered semitone is 100 cents. The interval in cents corresponding to any two frequencies f_1 and f_2 is $1{,}200 \log_2(f_2/f_1) = 3{,}986 \log_{10}(f_2/f_1)$. Table 3h-6 lists certain common intervals in cents and the corresponding ratios; the frequency ratios for intervals up to 100 cents are given in Table 3h-7.

[1] Philip M. Morse, "Vibration and Sound," 2d ed., p. 193, McGraw-Hill Book Company, New York, 1948.