31. Classical Dynamical Analogies

HARRY F. OLSON

RCA Laboratories

Analogies are useful when it is desired to compare an unfamiliar system with one that is better known. The relations and actions are more easily visualized, the mathematics more readily applied, and the analytical solutions more readily obtained in the familiar system. Analogies make it possible to extend the line of reasoning into unexplored fields. In view of the tremendous amount of study which has been directed toward the solution of circuits, particularly electric circuits, and the engineer's familiarity with electric circuits, it is logical to apply this knowledge to the solutions of vibration problems in other fields by the same theory as that used in the solution of electric circuits. The objective in this section is the establishment of analogies between electrical, mechanical, and acoustical systems.

31-1. Resistance. *Electric Resistance.* Electric energy is changed into heat by the passage of an electric current through an electric resistance. Electric resistance R_E, in abohms, is defined as

$$R_E = \frac{e}{i} \tag{31-1}$$

where e = voltage across the electric resistance, abvolts
i = current through the electric resistance, abamp

Mechanical Rectilineal Resistance. Mechanical rectilineal energy is changed into heat by a rectilinear motion which is opposed by mechanical rectilineal resistance (friction). Mechanical rectilineal resistance (termed mechanical resistance when there is no ambiguity) R_M, in mechanical ohms, is defined as

$$R_M = \frac{f_M}{u} \tag{31-2}$$

where f_M = applied mechanical force, dynes
u = velocity at the point of application of the force, cm/sec

Mechanical Rotational Resistance. Mechanical rotational energy is changed into heat by a rotational motion which is opposed by a rotational resistance (rotational friction). Mechanical rotational resistance (termed rotational resistance when there is no ambiguity) R_R, in rotational ohms, is defined as

$$R_R = \frac{f_R}{\Omega} \tag{31-3}$$

where f_R = applied torque, dyne-cm
Ω = angular velocity about the axis at the point of the torque, radians/sec

Acoustic Resistance. Acoustic energy is changed into heat either by a motion in a fluid which is opposed by acoustic resistance due to a fluid resistance incurred by viscosity or by the radiation of sound. Acoustic resistance R_A, in acoustical ohms, is defined as

$$R_A = \frac{p}{U} \tag{31-4}$$

where p = pressure, dynes/cm^2
U = volume velocity, cm^3/sec

31-2. Inductance, Mass, Moment of Inertia, Inertance. *Inductance.* Electromagnetic energy is associated with inductance. Inductance is the electric-circuit element that opposes a change in current. Inductance L, in abhenrys, is defined as

$$e = L\frac{di}{dt} \tag{31-5}$$

where e = voltage, emf, or driving force, abvolts
$\frac{di}{dt}$ = rate of change of current, abamp/sec

Mass. Mechanical rectilineal inertial energy is associated with mass in the mechanical rectilineal system. Mass is the mechanical element which opposes a change in velocity. Mass m, in grams, is defined as

$$f_M = m\frac{du}{dt} \tag{31-6}$$

where $\frac{du}{dt}$ = acceleration, cm/sec^2
f_M = driving force, dynes

Moment of Inertia. Mechanical rotational energy is associated with moment of inertia in the mechanical rotational system. Moment of inertia is the rotational element which opposes a change in angular velocity. Moment of inertia I, in gram (centimeter)^2, is defined as

$$f_R = I\frac{d\Omega}{dt} \tag{31-7}$$

where $\frac{d\Omega}{dt}$ = angular acceleration, radians/sec^2
f_R = torque, dyne-cm

Inertance. Acoustic inertial energy is associated with inertance in the acoustic system. Inertance is the acoustic element which opposes a change in volume velocity. Inertance M, in grams per (centimeter)^4, is defined as

$$p = M\frac{dU}{dt} \tag{31-8}$$

where $\frac{dU}{dt}$ = rate of change of volume velocity, cm^3/sec^2
p = driving pressure, dynes/cm^2

31-3. Electric Capacitance, Rectilineal Compliance, Rotational Compliance, Acoustic Capacitance. *Electric Capacitance.* Electric capacitance is associated with capacitance. Electric capacitance is the electric-circuit element which opposes a change in voltage. Electric capacitance C_E, in abfarads, is defined as

$$i = C_E\frac{de}{dt} \tag{31-9}$$

$$e = \frac{1}{C_E}\int i\,dt = \frac{Q}{C_E} \tag{31-10}$$

where Q = charge on the electrical capacitance, abcoulombs
e = emf, abvolts

Rectilineal Compliance. Mechanical rectilineal potential energy is associated with the compression of a spring or compliant element. Rectilineal compliance is the mechanical element which opposes a change in the applied force. Rectilineal compliance (termed compliance when there is no ambiguity) C_M, in centimeters per dyne, is defined as

$$f_M = \frac{x}{C_M} \tag{31-11}$$

where x = displacement, cm
f_M = applied force, dynes

Rotational Compliance. Mechanical rotational potential energy is associated with the twisting of a spring or compliant element. Rotational compliance is the mechanical element that opposes a change in the applied torque. Rotational compliance C_R, in radians per centimeter per dyne, is defined as

$$f_R = \frac{\phi}{C_R} \tag{31-12}$$

where ϕ = angular displacement, radians
f_R = applied torque, dyne-cm

Acoustic Capacitance. Acoustic potential energy is associated with the compression of a fluid or a gas. Acoustic capacitance is the acoustic element which opposes a change in the applied pressure. The acoustic capacitance C_A, in (centimeters)^5 per dyne, is defined as

$$p = \frac{X}{C_A} \tag{31-13}$$

where X = volume displacement, cm^3
p = pressure, dynes/cm^2

31-4. Representation of Electrical, Mechanical Rectilineal, Mechanical Rotational, and Acoustical Elements. Electrical, mechanical rectilineal, mechanical rotational, and acoustical elements have been defined in the preceding sections. Figure 31-1 illustrates schematically the three elements in each of the four systems.

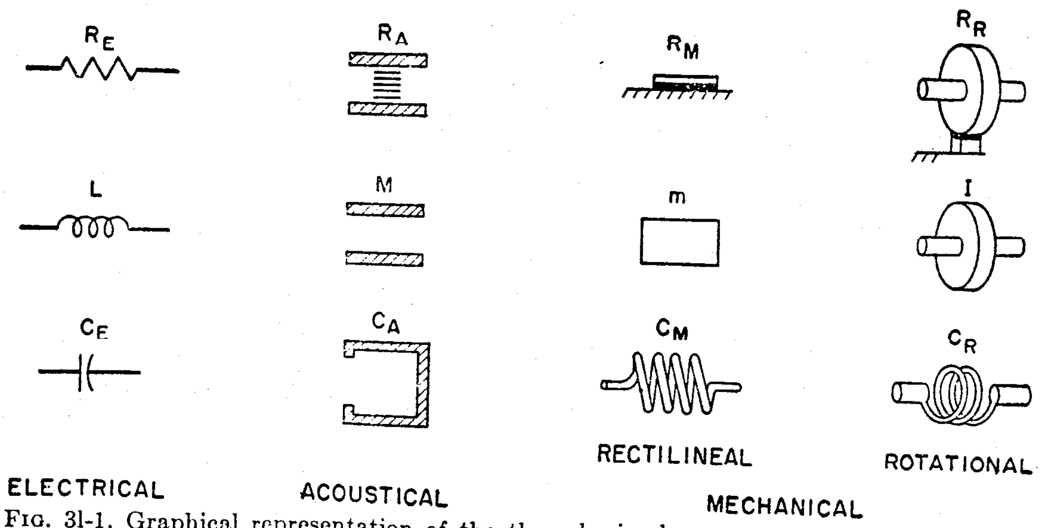


FIG. 31-1. Graphical representation of the three basic elements in electrical, mechanical rectilineal, mechanical rotational, and acoustical systems.

The electrical elements, electric resistance, inductance, and electric capacitance, are represented by the conventional symbols.

Mechanical rectilineal resistance is represented by sliding friction which causes dissipation. Mechanical rotational resistance is represented by a wheel with a sliding-friction brake which causes dissipation. Acoustic resistance is represented by narrow slits which cause dissipation due to viscosity when fluid is forced through the slits. These elements are analogous to electric resistance in the electrical system.

Inertia in the mechanical rectilineal system is represented by a mass. Moment of inertia in the mechanical rotational system is represented by a flywheel. Inertance in the acoustical system is represented as the fluid contained in a tube in which all the particles move with the same phase when actuated by a force due to pressure. These elements are analogous to inductance in the electrical system.

Compliance in the mechanical rectilineal system is represented as a spring. Rotational compliance in the mechanical rotational system is represented as a spring. Acoustic capacitance in the acoustical system is represented as a volume which acts as a stiffness or spring element. These elements are analogous to electric capacitance in the electrical system.

Table 31-1 shows the quantities, units, and symbols in the four systems.

31-5. Description of Systems of One Degree of Freedom. Electrical, mechanical rectilineal, mechanical rotational, and acoustical systems of one degree of freedom are shown in Fig. 31-2. In one degree of freedom the activity in every element of the system can be expressed in terms of one variable. In the electrical system an electromotive force e acts upon an inductance L, an electric resistance R_E, and an electric capacitance C_E connected in series. In the mechanical rectilineal system a driving force f_M acts upon a particle of mass m fastened to a spring of compliance C_M and sliding upon a plate with a frictional force which is proportional to the velocity and designated as the mechanical rectilineal resistance R_M. In the mechanical rotational system a driving torque f_R acts upon a flywheel of moment of inertia I connected to a spring or rotational compliance C_R and the periphery of the wheel sliding against a brake with a frictional force which is proportional to the velocity and designated as the mechanical rotational resistance R_R. In the acoustical system, an impinging sound wave of pressure p acts upon an inertance M and an acoustic resistance R_A comprising the air in the tubular opening which is connected to the volume or acoustical capacitance C_A. The acoustic resistance R_A is due to viscosity.

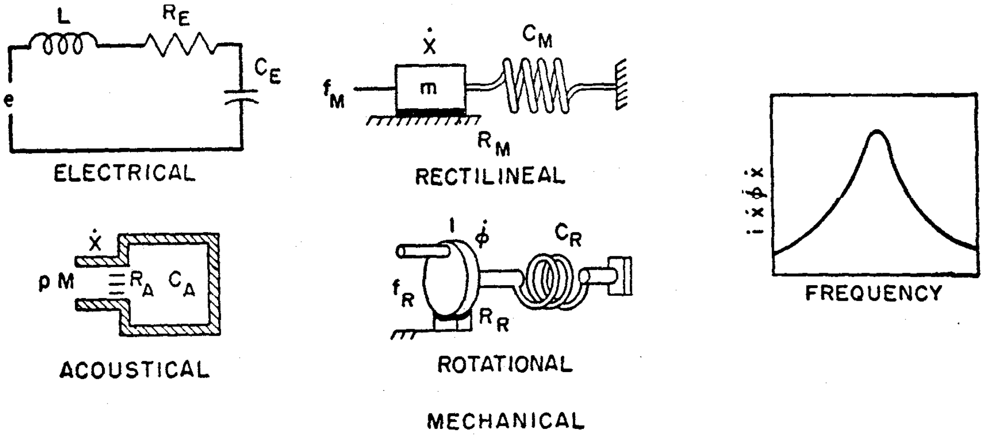


FIG. 31-2. Electrical, mechanical rectilineal, mechanical rotational, and acoustical systems of one degree of freedom and the current, velocity, angular velocity and volume velocity response characteristics.

The differential equations describing the four systems of Fig. 31-2 are as follows:

Electrical

$$L\ddot{q} + R_E\dot{q} + \frac{Q}{C_E} = E\epsilon^{j\omega t} \tag{31-14}$$

Mechanical rectilineal

$$m\ddot{x} + R_M\dot{x} + \frac{x}{C_M} = F_M\epsilon^{j\omega t} \tag{31-15}$$

Mechanical rotational

$$I\ddot{\phi} + R_R\dot{\phi} + \frac{\phi}{C_R} = F_R\epsilon^{j\omega t} \tag{31-16}$$

Acoustical

$$M\ddot{X} + R_A\dot{X} + \frac{X}{C_A} = P\epsilon^{j\omega t} \tag{31-17}$$

E, F_M, F_R, and P are the amplitudes of the driving forces in the four systems. $E\epsilon^{j\omega t} = e$, $F_M\epsilon^{j\omega t} = f_M$, $F_R\epsilon^{j\omega t} = f_R$ and $P\epsilon^{j\omega t} = p$.

The steady-state solutions of Eqs. (31-14) to (31-17) are:

Electrical

$$\dot{q} = i = \frac{E\epsilon^{j\omega t}}{R_E + j\omega L - (j/\omega C_E)} = \frac{e}{Z_E} \tag{31-18}$$

Mechanical rectilineal

$$\dot{x} = \frac{F\epsilon^{j\omega t}}{R_M + j\omega m - (j/\omega C_M)} = \frac{f_M}{Z_M} \tag{31-19}$$

Mechanical rotational

$$\dot{\phi} = \frac{F\epsilon^{j\omega t}}{R_R + j\omega I - (j/\omega C_R)} = \frac{f_R}{Z_R} \tag{31-20}$$

Acoustical

$$\dot{X} = \frac{P\epsilon^{j\omega t}}{R_A + j\omega M - (j/\omega C_A)} = \frac{p}{Z_A} \tag{31-21}$$

The vector electric impedance is

$$Z_E = R_E + j\omega L - \frac{j}{\omega C_E} \tag{31-22}$$

The vector mechanical rectilineal impedance is

$$Z_M = R_M + j\omega m - \frac{j}{\omega C_M} \tag{31-23}$$

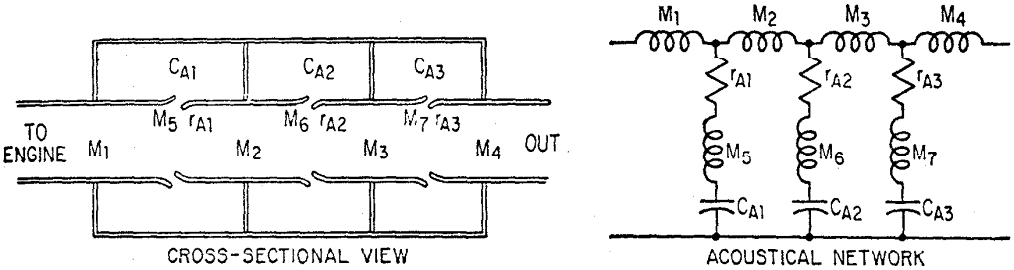
The vector mechanical rotational impedance is

$$Z_R = R_R + j\omega I - \frac{j}{\omega C_R} \tag{31-24}$$

The vector acoustic impedance is

$$Z_A = R_A + j\omega M - \frac{j}{\omega C_A} \tag{31-25}$$

31-6. Applications of Classical Electrodynamical Analogies. The fundamental principles relating to electrical, mechanical rectilineal, mechanical rotational, and acoustical analogies have been established in the preceding sections. Employing these fundamental principles, the vibrations produced in mechanical and acoustical systems owing to impressed forces can be solved as follows: Draw the electrical network which is analogous to the problem to be solved; solve the electrical network by conventional electrical circuit theory; convert the electrical answer into the original system. In this procedure any problem involving vibrating systems is reduced to the solution of an electrical network. In the illustrations in the preceding sections, the elements in the electrical network have been labeled $r_{E,L}$ and C_E. However, when analogies are used in actual practice, the conventional procedure is to label the elements in the analogous electrical network with r_M, M, and C_M for a mechanical rectilineal system; with r_R, I, and C_R for a mechanical rotational system; and with r_A, M, and C_A for an acoustical system. This procedure will be followed in this section in labeling the elements in the analogous electrical network. The customary procedure is to label the network with the caption mechanical network or rotational network or acoustical network as the case may be. When there is only one path, the term circuit will be used instead of network. A complete treatment of the examples of the use of analogies in the solution of problems in mechanical and acoustical systems is beyond the scope of this section. However, a few typical examples will serve to illustrate the principles and method.



CROSS-SECTIONAL VIEW

ACOUSTICAL NETWORK

FIG. 31-3. Cross-sectional view and acoustical network of an automobile muffler. In the acoustical network: M_1, M_2, M_3, and M_4, the inertances of the series elements; r_{A1}, r_{A2}, and r_{A3}, the acoustical resistances of the shunt elements; M_5, M_6, and M_7, the inertances of the shunt elements; C_{A1}, C_{A2}, and C_{A3}, the acoustical capacitances of the shunt elements. *(After Olson, "Dynamical Analogies," D. Van Nostrand Company, Inc., Princeton, N.J.,* 1959.)

Acoustical—Automobile Muffler. The sound output from the exhaust of an automobile engine contains all audible frequencies in addition to frequencies below and above the audible range. The purpose of a muffler is to reduce the sound output in the audible frequency range without increasing the exhaust back pressure.

By the application of acoustical principles employing analogies improved mufflers have been developed in which the following advantages have been obtained: smaller size, higher attenuation in the audible frequency range, and reduction of back pressure at the engine. A cross-sectional view of the improved muffler is shown in Fig. 31-3. The acoustical network shows that the system is essentially a low-pass acoustical filter. The main channel is of the same diameter as the exhaust pipe. Therefore, there is no increase in the direct flow of exhaust gases as compared with a plain pipe. In order not to impair the efficiency of the engine, the muffler should not increase the acoustical impedance to subaudible frequencies. The system of Fig. 31-3 can be designed so that the subaudible frequencies are not attenuated and at the same time high attenuation is introduced in the audible frequency range.

The terminations at the two ends of the network are not ideal. Therefore, it is necessary to use shunt arms tuned to different frequencies in the low-frequency range. Acoustical resistance is obtained by employing slit-type openings into the side chambers.

In a development of this kind, the frequency spectrum of the sound which issues from the exhaust is usually determined. From these data the amount of suppression required for each part of the audible frequency range can be ascertained. The acoustical network can be determined from these data and the terminating acoustical networks. In general, changes are required to compensate for the approximations. In this empirical work the acoustical network serves as the guide in directing the appropriate changes.

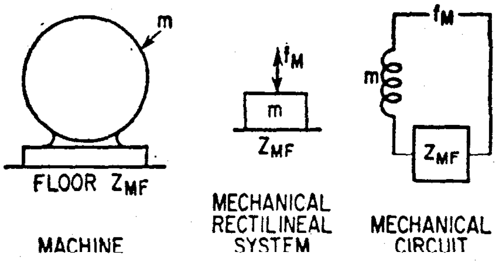


FIG. 31-4. Schematic view, mechanical rectilineal system, and mechanical circuit of a machine mounted directly upon the floor. In the mechanical circuit: f_M, the vibrating force developed by the machine; m, the mass of the machine; Z_{MF}, the mechanical impedance of the floor. (*After Olson, "Dynamical Analogies," D. Van Nostrand Company, Inc., Princeton, N.J.*, 1959.)

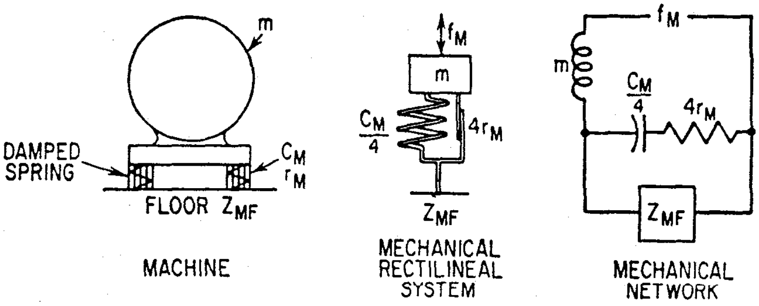


FIG. 31-5. Schematic view, mechanical rectilineal system, and mechanical network of a machine mounted upon a vibration isolating system. In the mechanical network: f_M, the vibrating force developed by the machine; m, the mass of the machine; C_M, the compliance of one of the four spring mounts; r_M, the mechanical rectilineal resistance of one of the spring mounts. (*After Olson, "Dynamical Analogies," D. Van Nostrand Company, Inc., Princeton, N.J.*, 1959.)

Mechanical Rectilineal—Machine Vibration Isolator. The vibration of a machine is transmitted from its supports to all parts of the surrounding building structure. In many cases, the vibrations are so intense as to be intolerable. The reduction of the transmission of machinery vibrations is one of the most common problems in noise control. For these conditions, the solution of the problem is to provide suitable vibrational isolation between the machine and the floor upon which it is placed.

A machine mounted directly on the floor is shown in Fig. 31-4. The mechanical rectilineal system and the mechanical circuit for vertical vibrations are shown in Fig. 31-4. The driving force f_M is due to the vibrations of the machine. The mechanical circuit shows that the only isolation in the system of Fig. 3l-4 is due to the mass of the machine.

In the simple isolating system of Fig. 3l-5 the machine is mounted on springs with mechanical resistance added to serve as damping. The compliance and mechanical resistance of each support are C_M and r_M. Since there are four supports, these values become $C_M/4$ and $4r_M$ in the mechanical rectilineal system and the mechanical network for vertical vibrations. The mechanical network depicts the action of the shunt circuit $C_M r_M$ in reducing the force of the vibration transmitted to the floor z_{MF}.

Mechanical Rotational—Vibration Damper. In reciprocating engines and other rotating machinery, rotational vibrations of large amplitudes occur at certain speeds. These rotational vibrations are sometimes of such high amplitude that the shafts will fail after a few hours of operating. A number of various rotational dampers have been developed for reducing these rotational vibrations. A typical example of a vibration damper used to control the vibrations of the flywheel is shown in Fig. 3l-6. The damper consists of a rotational element having a moment of inertia I_2 rotating on a shaft with a mechanical rotational resistance r_R between the inertia element and shaft. The inertial element is coupled to the flywheel by means of a spring of compliance C_M. The rotational compliance is $C_R = C_M/a^2$, where a is the radius at the point of attachment of the spring with respect to the center line of the shaft. Referring to the rotational network it will be seen that the rotational damper forms a shunt mechanical rotational system. The shunt rotational circuit $C_R r_R I_2$ is tuned to the frequency of the vibration. Since the mechanical rotational impedance of the shunt resonant rotational circuit is very high at the resonant frequency, the angular velocity (or amplitude) of vibration of the flywheel will be reduced. A consideration of the rotational network illustrates the principle of the device.

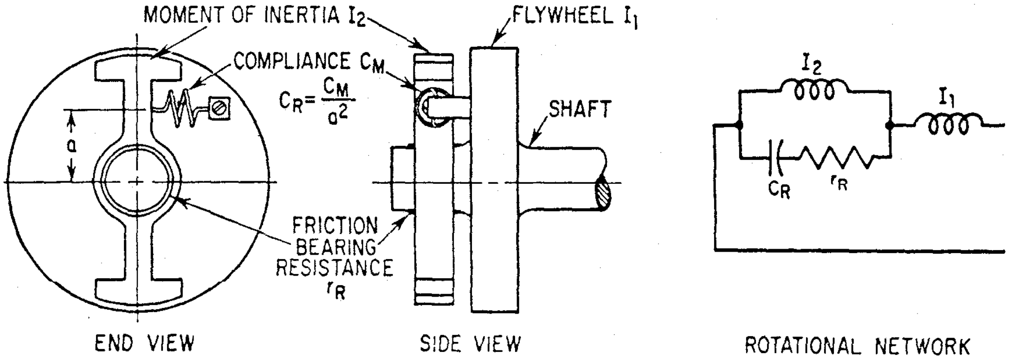


FIG. 3l-6. End and side views and the rotational network of a vibration damper. In the rotational network: I_1, the moment of inertia of the flywheel; I_2, the moment of inertia of the damper; C_R, the rotational compliance of the damper; r_R, the mechanical rotational resistance between the damper and the shaft. (*After Olson, "Dynamical Analogies," D. Van Nostrand Company, Inc., Princeton, N.J.,* 1959.)

Electrical Mechanical—Direct Radiator Dynamic Loudspeaker. The direct radiator dynamic loudspeaker shown in Fig. 3l-7 is almost universally used for radio, phonograph, television, and other small-scale sound reproductions.

The mechanical circuit of the loudspeaker is shown in Fig. 3l-7. The mechanical rectilineal impedance at the voice coil, where a force f_M is applied, can be determined from the constants of the elements of the mechanical circuit. The mass m_2 and the mechanical resistance r_{M2} of the air load can be obtained from Sec. 3i-2 on the acoustic impedance of vibrating pistons.

The electrical circuit of the loudspeaker is also shown in Fig. 3l-7. The motional

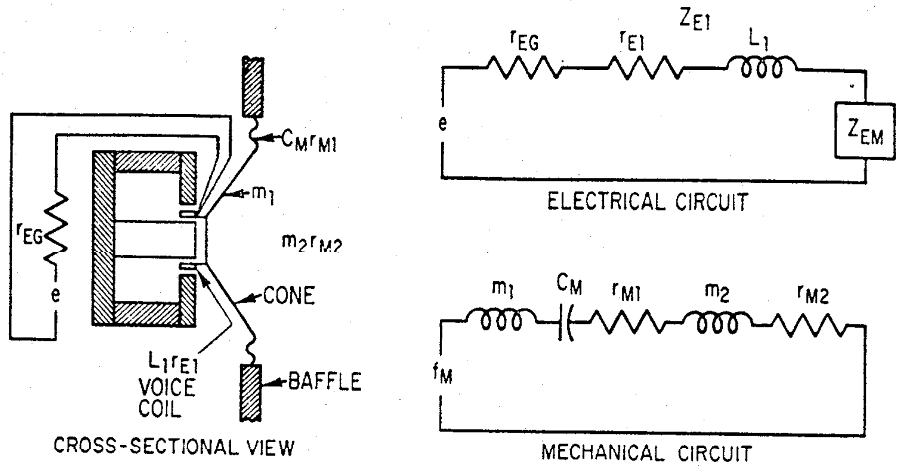


FIG. 31-7. Cross-sectional view, electrical circuit, and mechanical circuit of a direct radiator loudspeaker. In the electrical circuit: e, the open-circuit voltage of the generator or vacuum tube; r_{EG}, the electrical resistance of the voice coil; L, the inductance of the voice coil; Z_{EM}, the motional electrical impedance of the driving system. In the mechanical circuit: m_1, the mass of the cone; r_{M1}, the mechanical resistance of the suspension system; C_M, the compliance of the suspension system; m_2, the mass of the air load; r_{M2}, the mechanical rectilineal resistance of the air load. (*After Olson, "Dynamical Analogies," D. Van Nostrand Company, Inc., Princeton, N.J.*, 1959.)

electrical impedance in the electrical circuit is given by

$$z_{EM} = \frac{(Bl)^2}{z_{MT}} \tag{31-26}$$

where z_{EM} = motional electrical impedance, abohms
B = flux density in air, gauss
l = length of conductor in voice coil
z_{MT} = mechanical impedance at location f_M in mechanical circuit, mechanical ohms

The mechanical driving force is given by

$$f_M = Bli \tag{31-27}$$

where f_M = driving force, dynes
i = current in voice coil, abamp

The velocity can be determined from the mechanical circuit of Fig. 31-7 and the following equation:

$$\dot{x} = \frac{f_M}{z_{MT}} \tag{31-28}$$

where $\dot{x}$ is the velocity in centimeters per second. The sound output is given by

$$P = r_M \dot{x}^2 \tag{31-29}$$

where P = sound power output, ergs/sec
r_M = mechanical ohms
$\dot{x}$ = velocity of cone from Eq. (31-28)

The object is to select the constants so that the power output as given by Eq. (31-29) is practically independent of the frequency over the desired frequency range.

TABLE 31-1. QUANTITIES, UNITS, AND SYMBOLS FOR ELECTRICAL, MECHANICAL RECTILINEAL, MECHANICAL ROTATIONAL, AND ACOUSTICAL ELEMENTS

Electrical			Mechanical rectilineal		
Quantiy	Unit	Symbol	Quantity	Unit	Symbol
Electromotive force	Volts × 10^8	e	Force	Dynes	f_M
Charge or quantity	Coulombs × 10^{-1}	Q	Linear displacement	Centimeters	x
Current	Amperes × 10^{-1}	i	Linear velocity	Centimeters per second	$\dot{x}$ or u
Electric impedance	Ohms × 10^9	Z_E	Mechanical impedance	Mechanical ohms	Z_M
Electric resistance	Ohms × 10^9	R_E	Mechanical resistance	Mechanical ohms	R_M
Electric reactance	Ohms × 10^9	X_E	Mechanical reactance	Mechanical ohms	X_M
Inductance	Henrys × 10^9	L	Mass	Grams	m
Electric capacitance	Farads × 10^{-9}	C_E	Compliance	Centimeters per dyne	C_M
Power	Ergs per second	P_E	Power	Ergs per second	P_M

Mechanical rotational			Acoustical		
Quantity	Unit	Symbol	Quantity	Unit	Symbol
Torque	Dyne-centimeters	f_R	Pressure	Dynes per square centimeter	p
Angular displacement	Radians	ϕ	Volume displacement	Cubic centimeters	X
Angular velocity	Radians per second	$\dot{\phi}$ or Ω	Volume velocity	Cubic centimeters per second	$\dot{X}$ or U
Rotational impedance	Rotational ohms	Z_R	Acoustic impedance	Acoustic ohms	Z_A
Rotational resistance	Rotational ohms	R_R	Acoustic resistance	Acoustic ohms	R_A
Rotational reactance	Rotational ohms	X_R	Acoustic reactance	Acoustic ohms	X_A
Moment of inertia	(Gram) $(\text{centimeter})^2$	I	Inertance	Grams per $(\text{centimeter})^4$	M
Rotational compliance	Radians per dyne per centimeter	C_R	Acoustic capacitance	$(\text{Centimeter})^5$ per dyne	C_A
Power	Ergs per second	P_R	Power	Ergs per second	P_A