3n. Nonlinear Acoustics (Theoretical)

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Until the early 1950s most of what was known about sound waves of finite amplitude was confined to propagation, and to a lesser extent reflection, of plane waves in lossless gases. Since that time a great deal has been learned about propagation in other media, about nonplanar propagation (still chiefly in one dimension), about the effect of losses, and about standing waves. Inroads have been made on problems of refraction. Diffraction is still relatively untouched.

In this section the exact equations of motion for thermoviscous fluids will first be stated. Various retreats from the full generality of these equations will then be discussed. No attempt will be made to cover streaming and radiation pressure. See Secs. 3c-7 and 3c-8 for a discussion of those topics.

GENERAL EQUATIONS FOR FLUIDS

The basic conservation equations will be stated briefly for viscous fluids with heat flow. Other compressible media, such as solids and relaxing fluids, are discussed later in the section.

3n-1. Conservation of Mass, Momentum, and Energy. In Eulerian (spatial) coordinates the continuity and momentum equations are respectively

$$\frac{D\rho}{Dt} + \rho \frac{\partial u_i}{\partial x_i} = 0 \qquad \text{(3n-1)}$$

$$\rho \frac{Du_i}{Dt} + \frac{\partial p}{\partial x_i} = \frac{\partial}{\partial x_j} (\eta' d_{kk}\delta_{ij} + 2\eta d_{ij}) \qquad \text{(3n-2)}$$

An entropy equation is stated here in place of the usual energy equation:

$$\rho T \frac{DS}{Dt} = C_v \left[\rho \frac{D\mathfrak{I}}{Dt} - \frac{\gamma - 1}{\beta_e} \frac{D\rho}{Dt} \right] = \psi^{(\eta)} - \frac{\partial Q_i}{\partial x_i} \tag{3n-3}$$

Here ρ is the density, u_i is the ith (cartesian) component of particle velocity, p is pressure, δ_{ij} is the Kronecker delta, $d_{ij} = \frac{1}{2}(\partial u_i/\partial x_j + \partial u_j/\partial x_i)$ is the rate-of-deformation tensor, η and η' are the shear and dilatational coefficients of viscosity, C_v and C_p are the specific heats at constant volume and pressure, $\mathfrak{I}$ is absolute temperature, S is entropy per unit mass, $\gamma = C_p/C_v$ is the ratio of specific heats, $\beta_e = -\rho^{-1}(\partial\rho/\partial\mathfrak{I})_p$ is the coefficient of thermal expansion, $\psi^{(\eta)} = 2\eta d_{ij}d_{ji} + \eta' d_{kk}d_{ii}$ is the viscous energy dissipation function, and Q_i is the ith component of the total heat flux. The material derivative $D(\ \ )/Dt$ stands for $\partial(\ \ )/\partial t + u_i\partial(\ \ )/\partial x_i$. If the flow of heat is due to conduction,

$$Q_i = -\kappa \frac{\partial \mathfrak{I}}{\partial x_i} \tag{3n-4}$$

where κ is the coefficient of thermal conduction. For heat radiation the relation between q and $\mathfrak{I}$ is generally quite complicated; see, for example, Vincenti and Baldwin (ref. 1). The model used by Stokes (ref. 2) amounts to Newton's law of cooling and may be expressed by

$$\frac{\partial Q_i}{\partial x_i} = \rho C_v q(\mathfrak{I} - \mathfrak{I}_0) \tag{3n-5}$$

where $\mathfrak{I}_0$ is the ambient temperature, and q is the radiation coefficient. Although too simple to describe radiant heat transfer in a fluid adequately, this equation is worth considering because of (1) its analytical simplicity and (2) its application as a convenient model for relaxation processes.

3n-2. Equation of State. To the conservation equations must be added an equation of state.

Perfect Gas. The gas law for a perfect gas is

$$p = R\rho\mathfrak{I} \tag{3n-6}$$

where R is the gas constant. An approximate form of this equation will now be derived. For a perfect gas the small-signal sound speed c_0 is given by $c_0^2 = \gamma R\mathfrak{I}_0 = \gamma p_0/\rho_0$, where p_0 and ρ_0 are the ambient values of p and ρ. Let $\mathfrak{I} = \beta_{e0}(1 + \theta)$, $p = p_0 + \rho_0 c_0^2 P$, and $\rho = \rho_0(1 + s)$, where β_{e0} is the ambient value of β_e (for perfect gases $\beta_{e0}\mathfrak{I}_0 = 1$). Assume that θ, P, and s are small quantities of first order. Expansion of Eq. (3n-6) to second order yields

$$\theta = \gamma P - s + s^2 - \gamma P s \tag{3n-7}$$

First-order relations are now defined to be those that hold in linear, lossless acoustic theory; examples are $\rho_t = -\rho_0\nabla \cdot u$ and $p - p_0 = c_0^2(\rho - \rho_0)$. At this point we assert that any factor in a second-order term in Eq. (3n-7) may be replaced by its first-order equivalent. The justification is that any more precise substitution would result in the appearance of third- or higher-order terms, and such terms have already been excluded from Eq. (3n-7). Thus in the last second-order term in Eq. (3n-7) P may be replaced by s to give

$$\theta = \gamma P - s - (\gamma - 1)s^2 \tag{3n-8}$$

correct to second order. This is a useful approximate form of the perfect gas law.

One of the most fruitful special cases to consider is the isentropic perfect gas. When a perfect gas is inviscid and there is no heat flow, Eq. (3n-3) can be used to reduce the gas law, Eq. (3n-6), to

$$\frac{p}{p_0} = \left(\frac{\rho}{\rho_0}\right)^\gamma \tag{3n-9}$$

The square of the sound speed, which by definition is,

$$c^2 \equiv \left(\frac{\partial p}{\partial \rho}\right)_S \tag{3n-10}$$

becomes

$$c^2 = \frac{\gamma p}{\rho} = c_0^2 \left(\frac{p}{p_0}\right)^{(\gamma-1)/\gamma} \tag{3n-11}$$

An expanded form of Eq. (3n-9) is as follows:

$$P = s + \tfrac{1}{2}(\gamma - 1)s^2 + \cdots \tag{3n-12}$$

Other Fluids. For liquids and for gases that are not perfect, one can start with a general equation of state $\mathfrak{I} = \mathfrak{I}(p,\rho)$. Recognizing that $(\partial\mathfrak{I}/\partial p)_\rho = \gamma(\rho c^2 \beta_e)^{-1}$, one obtains the exact expression

$$\theta_t = \frac{\beta_{e0}}{\beta_e}(1 + s)^{-1}\left[\gamma\left(\frac{c_0}{c}\right)^2 P_t - s_t\right] \tag{3n-13}$$

In order to obtain an approximation analogous to Eq. (3n-8), it is first necessary to set down a general isentropic equation of state,

$$p - p_0 = \rho_0 c_0^2 \left(s + \frac{B}{2A}s^2 + \frac{C}{3A}s^3 + \cdots\right) \tag{3n-14}$$

where the coefficients B/A, C/A, etc., are to be determined experimentally (see Sec. 3o). With the help of this expression and some elementary thermodynamic relations, one invokes the approximation procedure described following Eq. (3n-7) and reduces Eq. (3n-13) to (ref. 3)

$$\theta = \gamma P - s - (h - 1)s^2 \tag{3n-15}$$

correct to second order, where

$$h = 1 + \frac{\gamma B}{2A} + \tfrac{1}{2}(\gamma - 1)\left(1 - \frac{B}{2A}\right) - (\gamma - 1)^2(4\beta_{e0}\mathfrak{I})^{-1} \tag{3n-16}$$

If Eqs. (3n-14) and (3n-12) are compared, it will be seen that B/A replaces the quantity $\gamma - 1$ in describing second-order nonlinearity of the $p - \rho$ relation. For a perfect gas, therefore, replace B/A by $\gamma - 1$ and β_{e0} by $\mathfrak{I}_0^{-1}$ in Eq. (3n-16). The quantity h then reduces to γ, and Eq. (3n-7) is recovered.

PROPAGATION IN LOSSLESS FLUIDS

For isentropic flow (taken here to mean that the entropy of every particle is the same and remains so) Eqs. (3n-1) and (3n-2) reduce to

$$\frac{D\rho}{Dt} + \frac{\rho\partial u_i}{\partial x_i} = 0 \tag{3n-17a}$$

$$\frac{\rho D u_i}{Dt} + \frac{\partial p}{\partial x_i} = 0 \tag{3n-17b}$$

and the equation of state may be expressed simply by $p = p(\rho)$. If the new thermodynamic quantity

$$\lambda \equiv \int_{\rho_0}^{\rho} \frac{c}{\rho'}\, d\rho' \tag{3n-18}$$

is introduced, Eqs. (3n-17) take the following symmetric form:

$$\frac{D\lambda}{Dt} + \frac{c\partial u_i}{\partial x_i} = 0 \tag{3n-19a}$$

$$\frac{Du_i}{Dt} + \frac{c\partial\lambda}{\partial x_i} = 0 \tag{3n-19b}$$

Very little has been done in the way of solving these general equations.

3n-3. Plane Waves in Lossless Fluids. For one-dimensional flow in the x direction Eqs. (3n-19) become

$$\lambda_t + u\lambda_x + cu_x = 0 \tag{3n-20a}$$

$$u_t + uu_x + c\lambda_x = 0 \tag{3n-20b}$$

where subscripts x and t now denote partial differentiation, and u represents the particle velocity in the x direction. Hyperbolic equations of this form have been studied in great detail (ref. 4). Their solutions are of two general types: (1) those representing simple waves (waves propagating in one direction only), and (2) those representing compound waves (waves propagating in both directions).

Simple Waves. Simple-wave flow is characterized by the existence of a unique relationship between the particle velocity and the local thermodynamic state of the fluid. For simple waves traveling into a medium at rest, this relationship is (ref. 5)

$$\lambda = \pm u \tag{3n-21}$$

where the (+) sign holds for outgoing waves (waves traveling in the direction of increasing x), and the (−) sign for incoming waves (waves traveling in the direction of decreasing x). Hereinafter when multiple signs are used, the upper sign pertains to outgoing waves. Equations (3n-20) now reduce to the single equation

$$u_t + (u \pm c)u_x = 0 \tag{3n-22}$$

which becomes autonomous once the equation of state is specified, since Eqs. (3n-18) and (3n-21) imply a relationship $c = c(u)$. Note that the linearized version of Eq. (3n-22), $u_t \pm c_0 u_x = 0$, possesses the familiar traveling-wave solution $u = f(x \mp c_0 t)$ of linear acoustics.

The most important nonlinear effect in simple-wave flow can be readily identified directly from Eq. (3n-22). Combine that equation with the differential expression $du = u_x\,dx + u_t\,dt$ to obtain

$$\left(\frac{dx}{dt}\right)_{u=\text{const}} = -\frac{u_t}{u_x} = u \pm c \tag{3n-23}$$

This relation states that the propagation speed of a given point on the waveform (the point being identified by the value of u there) is $u \pm c$. In linear theory the propagation speed of all points is the same, namely, $\pm c_0$. The ramifications of the variable propagation speed are discussed in Sec. 3n-4.

Compound Waves. When waves traveling in both directions are present, there is no fixed relationship between u and λ. A propagation speed can still be defined, however. New dependent variables $\mathfrak{r}$ and $\mathfrak{s}$, called "Riemann invariants," may be defined by

$$2\mathfrak{r} = \lambda + u \qquad 2\mathfrak{s} = \lambda - u \tag{3n-24}$$

If Eqs. (3n-20) are first added and then subtracted, the results are respectively

$$\mathfrak{r}_t + (u + c)\mathfrak{r}_x = 0 \tag{3n-25a}$$

$$\mathfrak{s}_t + (u - c)\mathfrak{s}_x = 0 \tag{3n-25b}$$

Thus, as first found by Riemann (ref. 6),

$$\left(\frac{dx}{dt}\right)_{r=\text{const}} = u + c \tag{3n-26a}$$

$$\left(\frac{dx}{dt}\right)_{s=\text{const}} = u - c \tag{3n-26b}$$

Despite its apparent simplicity, this result is much more complicated to apply than Eq. (3n-23).

3n-4. Plane, Simple Waves in Lossless Gases. For perfect gases the isentropic equation of state is given by Eq. (3n-9). For this case $\lambda = 2(c - c_0)/(\gamma - 1)$, and the simple-wave relation Eq. (3n-21) becomes

$$c = c_0 \pm (\beta - 1)u \tag{3n-27}$$

where $\beta = \frac{1}{2}(\gamma + 1)$. Combination of this equation with Eq. (3n-11) leads to

$$p - p_0 = p_0 \left\{\left[1 \pm (\beta - 1)\frac{u}{c_0}\right]^{2\gamma/(\gamma-1)} - 1\right\} \tag{3n-28}$$

which can be used to obtain the characteristic impedance for finite-amplitude waves. For weak waves, i.e., $u/c_0 \ll 1$, this expression reduces to the traditional one,

$$p - p_0 = \pm \rho_0 c_0 u \tag{3n-29}$$

The nonlinear differential equation for simple waves, Eq. (3n-22), becomes

$$u_t + (\beta u \pm c_0)u_x = 0 \tag{3n-30}$$

If we restrict ourselves momentarily to outgoing waves, the propagation speed is

$$\left(\frac{dx}{dt}\right)_{u=\text{const}} = \beta u + c_0 \tag{3n-31a}$$

which shows quite clearly that the peaks of the wave travel fastest, the troughs slowest. Equivalently, as the wave travels from one point to another, the peaks suffer the least delay, the troughs the most. This latter view is illustrated in Fig. 3n-1,

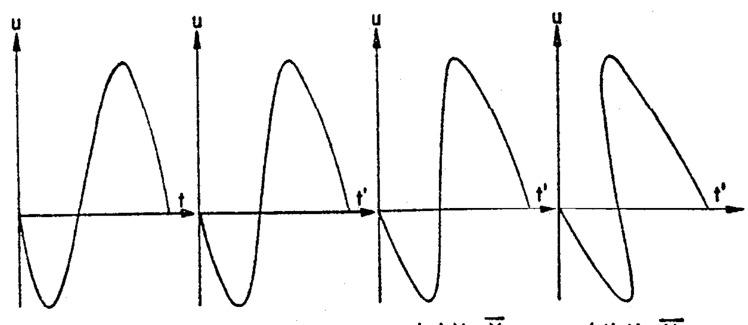

FIG. 3n-1. Progressive distortion of a finite-amplitude wave. Symbols are: u = particle velocity, x = spatial coordinate, t = time. $t' = t - x/c_0$ (delay time), $\bar{x}$ = point at which a shock begins to form.

which shows the time waveform of an outgoing disturbance at various distances from the source. The progressive distortion is quite striking, leading eventually to the curious waveform shown in Fig. 3n-1*d*. The interpretation of Fig. 3n-1*d* will be discussed presently.

Why physically does the exact propagation speed differ from c_0, the accepted value in linear theory? Two effects are at work: one kinematic, the other thermodynamic.

The sound wave travels with speed c with respect to the fluid particles. But these particles are themselves in motion, moving with velocity u. To a fixed observer, therefore, the net speed is $u + c$. This is the kinematic effect and is frequently referred to as *convection* (the fluid particles convect the wave along as a result of their own motion). The thermodynamic effect is the deviation from constancy of the sound speed c. Where the acoustic pressure is positive, the gas is a little hotter. Consequently c is greater. Conversely, in the wave troughs, where the gas is expanded and therefore colder, c is less. The variation of c from point to point along the wave can be traced to nonlinearity of the pressure-density relation. As Eq. (3n-10) shows, c would be constant if p were linearly related to ρ. This would be true, for example, for an isothermal gas.

For an incoming wave the propagation speed is

$$\left(\frac{dx}{dt}\right)_{u=\text{const}} = \beta u - c_0 \tag{3n-31b}$$

Similar arguments apply in this case. A difference is that the troughs of the particle velocity wave travel fastest (in a backward direction), the peaks slowest. Because pressure and particle velocity are out of phase in an incoming wave, however, it is still true that the peaks of the pressure wave proceed most rapidly and the troughs least so.

General Solutions. Three forms of the general solution of Eq. (3n-30) are now given. First is what might be called the "Poisson solution" (ref. 7)

$$u = f[x - (\beta u \pm c_0)t] \tag{3n-32}$$

which is implied by Eq. (3n-31); f is an arbitrary function. This result is most easily interpreted as the solution of an initial-value problem for which the spatial dependence of the particle velocity is prescribed everywhere at $t = 0$, i.e., $u(x,0) = f(x)$. The problem is somewhat artificial, however, because the progressive wave motion must already exist at $t = 0$. Of more practical interest are boundary-value problems involving a source; then simple waves arise quite naturally. If the time history of the particle velocity is known at a particular place, say $u(0,t) = g(t)$, the solution is

$$u = g\left(t - \frac{x}{\beta u \pm c_0}\right) \tag{3n-33}$$

This equation has been used to construct the waveforms in Fig. 3n-1. To make such constructions, it is convenient to use the following "inverted" form of the solution:

$$t' = g^{-1}(u) - \frac{\beta u}{c_0 \pm \beta u}\frac{x}{c_0} \tag{3n-34}$$

where $t' = t \mp x/c_0$ is the delay (for outgoing waves) or advance (for incoming waves) time appropriate for zeros of the waveform, and $g^{-1}(u)$ is the inverse function corresponding to g, i.e., $g^{-1}[g(u)] = u$.

The solution of the classic piston problem, in which a piston at rest begins at time $t = 0$ to move smoothly with a given displacement $X(t)$ in a lossless tube, is more complicated because of the moving boundary condition

$$u[X(t),t] = X'(t)H(t) \tag{3n-35}$$

where $H(t)$ is the unit step function. The solution of this problem may be given in parametric form as follows (refs. 5, 8):

$$u = X(\phi)H\left(\frac{t \mp x}{c_0}\right) \tag{3n-36a}$$

where

$$\phi = t - \frac{x - X(\phi)}{\beta X'(\phi) \pm c_0} \tag{3n-36b}$$

The parameter ϕ represents the time at which a given signal (i.e., given value of u) left the piston.

It is generally quite difficult to convert any of the three general solutions into an explicit analytical expression $u(x,t)$. One can, however, always obtain a sketch of the waveform through use of the inversion procedure indicated by Eq. (3n-34).

Shock Formation. A more far-reaching limitation, both mathematically and physically, is that these solutions contain the seeds of their own destruction. Except for a wave of pure expansion, the dependence of the propagation speed on u will cause steepening of the waveform. Steepening eventually leads to multivalued shapes like that shown in Fig. 3n-1*d*. But these must be rejected because pressure disturbances in nature cannot be multivalued, either in time or in space. In fact, once any section of the waveform attains a vertical tangent, as in Fig. 3n-1*c*, results cannot in general be continued further (ref. 9). Physically, what happens is that a shock wave begins to form. For reasons discussed in detail in Sec. 3n-8, this formally marks the end of validity of lossless, simple-wave theory. For mathematical analyses of shock formation see, for example, refs. 4 and 8.

Fubini Solution. A problem of special interest in acoustics is the propagation of a finite-amplitude wave that is sinusoidal at its point of origin. Suppose that the wave is produced by sinusoidal vibration of a piston in a lossless tube. Let the piston displacement be given by $X(t) = (u_0/\omega)(1 - \cos \omega t)$ where u_0 is the velocity amplitude of the piston, and ω is the angular frequency. The solution is given by applying Eqs. (3n-36). For the outgoing wave we have

$$\frac{u}{u_0} = \sin \omega\phi H\left(t - \frac{x}{c_0}\right) \tag{3n-37a}$$

where

$$\omega\phi = \omega t - \frac{kx - \epsilon(1 - \cos \omega\phi)}{1 + \beta\epsilon \sin \omega\phi} \tag{3n-37b}$$

Here $k = \omega/c_0$ is the wave number, and $\epsilon = u_0/c_0$ is the velocity amplitude expressed as a Mach number.

An explicit solution is now sought by writing u as a Fourier series,

$$\frac{u}{u_0} = \Sigma A_n \cos n(\omega t - kx) + \Sigma B_n \sin n(\omega t - kx) \tag{3n-38}$$

Although the exact expressions for all the coefficients A_n and B_n have not been obtained, an approximate computation is available. First expand Eq. (3n-37*b*), writing σ for $\beta\epsilon kx$, and t' for $t - x/c_0$, and rearrange as follows:

$$\omega\phi - \omega t' = \sigma \sin \omega\phi + \epsilon(1 - \cos \omega\phi - \beta\sigma \sin^2 \phi) + O(\epsilon^2)$$

If $\sigma \gg \epsilon$ (i.e., $\beta kx \gg 1$), and $\epsilon \ll 1$, this equation reduces to

$$\omega\phi = \omega t' + \sigma \sin \omega\phi \tag{3n-39}$$

Under this approximation the Fourier coefficients A_n vanish, and the B_n can be evaluated in terms of Bessel functions. The final result is (ref. 8)

$$\frac{u}{u_0} = \sum_{n=1}^{\infty} \frac{2}{n\sigma} J_n(n\sigma) \sin n(\omega t - kx) \tag{3n-40}$$

which is generally referred to as the Fubini solution (ref. 10).

The acoustic pressure signal is found by substituting the value of u given by Eq. (3n-40) in the linear impedance elation, Eq. (3n-29). Use of a more accurate expansion of Eq. (3n-28) for this purpose would not be consistent with the approximations that led to Eq. (3n-39).

The shock formation distance for this problem can be deduced by inspection of Eqs. (3n-39) [or, alternatively, the exact expression Eqs. (3n-37*b*)] and (3n-37*a*). The relationship of u to t' is one-to-one only if $\sigma < 1$. For $\sigma \geq 1$ the waveform curve $u(t')$ is multivalued. Hence a shock starts to form at $\sigma = 1$, i.e., at

$$\bar{x} = (\beta\epsilon k)^{-1} \tag{3n-41}$$

where the overbar signifies shock formation. The physical interpretation of σ is therefore that it is a spatial variable scaled in terms of the shock formation distance. The Fubini solution is not valid beyond the point $\sigma = 1$.

3n-5. An Approximate Theory of Lossless Simple Waves. The approximations leading to the Fubini solution can be used to obtain a general approximate theory of traveling waves of finite amplitude. The mathematical restrictions required are

$$\sigma \gg \epsilon \tag{3n-42a}$$

$$\epsilon \ll 1 \tag{3n-42b}$$

where the definitions of σ and ϵ are generalized to

$$\sigma = \frac{\beta\epsilon x}{x_c} \qquad \epsilon = \frac{u_0}{c_0} \tag{3n-43}$$

Here x_c is a characteristic distance defined so that significant distortion (for example, shock formation) takes place over the range $0 < \sigma < 1$, and u_0 is the maximum particle velocity that occurs in the flow. The physical implications of these restrictions are as follows:

1. The finite displacement of the source can be neglected. In other words, the exact boundary condition given by Eq. (3n-35) can be replaced by

$$u(0,t) = X'(t)H(t) \tag{3n-44}$$

Any error thus committed is made small by inequality (3n-42*a*).

2. The linear impedance relation, Eq. (3n-29), can be used to obtain the acoustic pressure, once the particle velocity waveform is known.

3. The nonlinear effect that *must* be taken into account is the nonconstancy of the propagation speed. But this effect is approximated by writing Eqs. (3n-31) as follows:

$$\left(\frac{dx}{dt}\right)_{u=\text{const}} \doteq \frac{\pm c_0}{1 \mp \beta u/c_0} \tag{3n-45}$$

Retention of nonconstancy of the propagation speed as the only important nonlinear effect gives recognition to the fact that this effect is the only *cumulative* one. It is the cause of the progressive distortion that engulfs the wave. We neglect the other nonlinear effects because they are *noncumulative*, or local. The distortion they cause does not grow with distance.

The formal theory based on these ideas will now be developed. An approximate differential equation may be derived by applying the method used earlier to convert Eq. (3n-7) to (3n-8). For simple waves the appropriate first-order relation is $u_x = \mp c_0^{-1}u_t$. When this is substituted in the nonlinear term in Eq. (3n-30), the result is

$$c_0 u_x \pm u_t - \beta c_0^{-1} u u_t = 0 \tag{3n-46}$$

This differential equation could also have been deduced from Eq. (3n-45).

Next let x and $t' = t \mp x/c_0$ be new independent variables. Equation (3n-46) reduces to

$$c_0^2 u_x - \beta u u_{t'} = 0 \tag{3n-47}$$

For the boundary condition

$$u\Big|_{x=0} = g(t)H(t) = g(t') \tag{3n-48}$$

where it is assumed that $g(t) = 0$ for $t < 0$, the solution is

$$u = g(\phi) \tag{3n-49a}$$
$$\phi = t' + \beta c_0^{-2} x g(\phi) \tag{3n-49b}$$

When the excitation is sinusoidal, i.e. $g(t) = u_0 \sin \omega t$, the Fubini solution follows exactly. It is also worth noting that within the limits of the approximate theory the difference between Lagrangian and Eulerian coordinates is negligible. As a general rule, the approximate theory is useful when $\epsilon < 0.1$ (ref. 8).

3n-6. Plane, Simple Waves in Liquids and Solids. *Liquids.* For lossless fluids whose isentropic equation of state is not given by Eq. (3n-9), we may proceed by using Eq. (3n-14). The propagation speed is (ref. 8)

$$\left(\frac{dx}{dt}\right)_{u=\text{const}} = u \pm c_0(1 + c_1U + c_2U^2 + \cdots) \tag{3n-50}$$

where $U = u/c_0$ and $c_1 = B/2A$, $c_2 = C/2A + B/4A - (B/2A)^2$, etc. Thus, in the exact solution of the piston problem [Eqs. (3n-36)], the parameter ϕ is given by

$$\phi = t - \frac{x - X(\phi)}{u \pm c_0(1 + c_1U + c_2U^2 \cdots)} \tag{3n-51}$$

where U is to be interpreted as $c_0^{-1}x_t(\phi)$.

Solids. The mathematical formalism for plane, longitudinal elastic waves in solids, either crystalline or isotropic, is very similar to that for liquids and gases (refs. 11–13). The wave equation is given in Lagrangian coordinates as

$$\xi_{tt} = c_0^2 G(\xi_a)\xi_{aa} \tag{3n-52}$$

where

$$G(\xi_a) = 1 + \left(\frac{M_3}{M_2}\right)\xi_a + \left(\frac{M_4}{M_2}\right)\xi_{aa} \cdots \tag{3n-53}$$

Here a represents the rest position of a particle; ξ is partical displacement; and M_2, M_3, M_4, etc., are quantities involving the second-, third-, fourth-, and higher-order elastic coefficients (ref. 12). The quantity c_0^2G plays the same role that $(\rho c/\rho_0)^2$ does for fluids (ref. 14). By the Lagrangian equation of continuity, $\rho_0/\rho = 1 + \xi_a$; thus replace Eq. (3n-18) by

$$\lambda = -c_0 \int_0^{\xi_a} [G(\xi_a')]^{\frac{1}{2}}\, d\xi_a' \tag{3n-54}$$
$$= -c_0[\xi_a - \tfrac{1}{4}m_3\xi_a^2 + (\tfrac{1}{8} - \tfrac{1}{6}m_4)m_3^2\xi_a^3 \cdots] \tag{3n-55}$$

where $m_3 = -M_3/M_2$, $m_4 = 1 - M_4/M_2m_3^2$, etc. Riemann invariants are defined as before by Eq. (3n-24). Note that $u = \xi_t$ in Lagrangian coordinates.

Simple-wave fields are again specified by Eq. (3n-21), which when combined with Eq. (3n-5) leads to

$$\xi_a = \mp U + \tfrac{1}{4}m_3U^2 \mp \tfrac{1}{6}m_4m_3^2U^3 \cdots \tag{3n-56}$$

The propagation speed for simple waves is

$$\left(\frac{da}{dt}\right)_{u=\text{const}} = \pm c_0 G^{\frac{1}{2}} \tag{3n-57}$$

The factor u, which appears in Eq. (3n-23), is absent here because the coordinate system is Lagrangian. Equation (3n-57) expanded in series form is

$$\left(\frac{da}{dt}\right)_{u=\text{const}} = \pm c_0[1 \pm \tfrac{1}{2}m_3U + \tfrac{1}{4}m_3^2(1 - 2m_4)U^2 \cdots] \tag{3n-58}$$

Therefore, the solution of the piston problem, given $u(0,t) = X_t(t)$, is

$$\phi = t \mp \frac{a/c_0}{1 \pm \frac{1}{2}m_3 U + \frac{1}{4}m_3^2(1 - 2m_4)U^2 \cdots} \tag{3n-59}$$

where U is to be interpreted, as in Eq. (3n-51), as $c_0^{-1}X_t(\phi)$. More complete versions of some of the series expansions given above can be found in ref. 12.

Approximate Theory. The approximate theory of simple waves described in Sec. 3n-5 is very easily generalized to apply to liquids and solids. For liquids $\gamma - 1$ is replaced by B/A, as mentioned after Eq. (3n-16). For solids $\gamma + 1$ is replaced by $-M_3/M_2$ (see ref. 12 for other useful associations). Therefore, let

$$\beta = \tfrac{1}{2}(\gamma + 1) \qquad \text{for gases} \tag{3n-60a}$$

$$\beta = 1 + \frac{B}{2A} \qquad \text{for liquids} \tag{3n-60b}$$

$$\beta = \frac{-M_3}{2M_2} \qquad \text{for solids} \tag{3n-60c}$$

and all results stated in Sec. 3n-5 become applicable for a very wide range of continuous media. For many liquids and solids the first "nonlinearity coefficient" (B/A for liquids, M_3/M_2 for solids) is known, but higher-order ones are not. In such cases it is difficult to justify using anything more precise than the approximate theory. But see ref. 12 for a discussion related to this point.

3n-7. Nonplanar Simple Waves. In this section one-dimensional nonplanar waves are considered, namely, spherical and cylindrical waves, and waves in horns. The general theory is not very highly developed. One fundamental difficulty is that simple waves of arbitrary waveform do not generally exist for nonplanar waves (ref. 15). Consider, for example, the wave motion generated by a pulsating sphere in an infinite medium. Most of the wave field consists of outgoing radiation, but there is also some backscatter (ref. 15). In the far field, however, simple waves do occur as an approximation. This is the case treated here. The results represent an extension of the approximate theory developed in Secs. 3n-5 and 3n-6.

Spherical and Cylindrical Waves. For large values of the radial coordinate r (actually large kr, where k is an appropriate wave number of the disturbance), the following approximate equation for simple waves in a fluid can be obtained (ref. 16):

$$c_0^2 w_z - \beta w w_{t'} = 0 \tag{3n-61}$$

where $t' = t \mp (r - r_0)/c_0$, r_0 is a reference distance, and β is given by Eq. (3n-60a) or (3n-60b). This equation may also apply to longitudinal waves in an isotropic solid, but so far no derivation has been given. The dependent variable w equals $(r/r_0)^{\frac{1}{2}}u$ and $(r/r_0)u$ for cylindrical and spherical waves, respectively. The independent variable z is given for the two cases by

Cylindrical: $$z = 2(\sqrt{r} - \sqrt{r_0})\sqrt{r_0} \tag{3n-62a}$$

Spherical: $$z = r_0 \ln \frac{r}{r_0} \tag{3n-62b}$$

Note that $z > 0$ for diverging waves ($r > r_0$), but $z < 0$ for converging waves ($r < r_0$).

Equation (3n-61) is solved by recognizing that it has the same form as the plane-wave equation (3n-47). For the boundary condition take $u(r_0,t) = g(t)$, which may represent either the motion of a source at r_0 or the measured time signal of a wave as it passes by the point r_0. Since $z = 0$ and $t' = t$ when $r = r_0$, the condition on w is

$$w(0,t') = g(t') \tag{3n-63}$$

Therefore, for the two kinds of waves the solution is

Cylindrical:
$$u = \left(\frac{r_0}{r}\right)^{\frac{1}{2}} g(\phi) \tag{3n-64a}$$
$$\phi = t' + 2\beta c_0^{-2} \sqrt{r_0}(\sqrt{r} - \sqrt{r_0})g(\phi) \tag{3n-64b}$$

Spherical:
$$u = \frac{r_0}{r} g(\phi) \tag{3n-65a}$$
$$\phi = t' + \beta c_0^{-2} r_0 \ln \frac{r}{r_0} g(\phi) \tag{3n-65b}$$

Some applications of these results are given in refs. 16 to 18. It has been shown (ref. 19) that Eq. (3n-65b) corresponds to a second-order approximation of results obtained using the Kirkwood-Bethe hypothesis (ref. 20).

Many special solutions for spherical and cylindrical waves have also been found. Most are of the similarity type. The most famous is Taylor's solution for the compression wave generated by a sphere that expands at a constant rate (refs. 21, 22).

Waves in Horns. For waves traveling in ducts whose cross-sectional area $A = A(x)$ does not vary rapidly, the waves may be assumed to be quasi-plane. It is assumed that the effect of variations in the cross section can be accounted for simply by correcting the continuity equation as follows:

$$\frac{D(A\rho)}{Dt} + \rho A u_x = 0 \tag{3n-66}$$

The one-dimensional formalism is thereby retained.

By the same methods used for spherical and cylindrical waves it is possible to derive an equation exactly like Eq. (3n-61). However, w and z are now defined as

$$w = \left(\frac{A}{A_0}\right)^{\frac{1}{2}} u \tag{3n-67a}$$
$$z = \int_{x_0}^{x} \left(\frac{A_0}{A}\right)^{\frac{1}{2}} dx' \tag{3n-67b}$$

where x_0 is a reference distance, $A_0 = A(x_0)$, and $t' = t \pm (x - x_0)/c_0$. The sign of z identifies the wave as outgoing ($x > x_0$) or incoming ($x < x_0$). Note that a conical horn ($A \propto x^2$) gives results identical with those for spherical waves, and a parabolic horn ($A \propto x$) gives results identical with those for cylindrical waves.

The general solution for a boundary condition of the form given by Eq. (3n-63) is (ref. 23)

$$w = \left(\frac{A}{A_0}\right)^{\frac{1}{2}} u = g(\phi) \tag{3n-68a}$$
$$\phi = t' + \beta c_0^{-2} z g(\phi) \tag{3n-68b}$$

For reference the value of the stretched coordinate z for an exponential horn ($A \propto e^{2lx}$) is

$$z = l^{-1}(1 - e^{-l(x-x_0)}) \tag{3n-69a}$$

and for a catenoidal horn ($A \propto \cosh^2 lx$) is

$$z = 2l^{-1}(\tan^{-1} e^{lx} - \tan^{-1} e^{lx_0}) \cosh lx_0 \tag{3n-69b}$$

All the results previously obtained for plane waves (approximate theory) may now be applied to nonplanar one-dimensional waves simply by replacing u and x by w and z, as given by Eqs. (3n-67). For example, for sinusoidal excitation at $x = x_0$ the shock formation distance is found by putting $\bar{z} = \pm(\beta\epsilon k)^{-1}$ and then making use of Eq. (3n-67b).

Parametric Array. An application of particular interest is the so-called parametric, end-fired array, conceived by Westervelt (ref. 53). A source such as a baffled piston emits radiation consisting of two high-frequency carrier waves into an open medium. The carriers, whose frequencies are ω_1 and ω_2, interact nonlinearly to produce a difference-frequency wave (frequency $\omega_d = \omega_2 - \omega_1$). Also produced, of course, but not of interest here, are the harmonics of the two carriers as well as the sum-frequency and other intermodulation components (ref. 54). In Westervelt's original treatment the two carrier waves were assumed to be collinear beams of collimated plane waves. More recently, Muir (ref. 55) has taken the directivity and spherical spreading of the carriers into account. In any case, however, the interaction to produce the difference-frequency wave amounts to setting into operation a line of virtual sources of frequency ω_d, all phased so as to constitute an end-fired array. The result is that the difference-frequency wave has a very high directivity. In other words, a low-frequency beam is produced that is much more highly directive than would have been the case had the source emitted the difference-frequency signal directly. Typically, too, there are no minor lobes. Absorption by the medium may be relied upon to filter out the two carrier waves and the sum-frequency component, eventually leaving the difference-frequency wave as the most prominent signal. Experiments have confirmed the remarkable properties of the parametric array (refs. 55, 56), and many further studies of it have been done (ref. 57).

WEAK-SHOCK THEORY

3n-8. General Discussion. The appearance of shocks in a flow poses a serious challenge to the theory of simple waves as developed thus far. In the first place, the waveform gradient at a shock is so high that the dissipation terms in Eqs. (3n-2) and (3n-3), heretofore deemed negligible, are in fact very large. A second problem is that since the shock is (at least approximately) a discontinuity in the medium, it can cause partial reflection of signals that catch up with it. The presence of reflected waves invalidates the simple-wave assumption. Strictly speaking, therefore, the flow cannot be simple wave, once shocks form (ref. 9).

The situation is not quite so bad as it seems, however, provided we restrict ourselves to relatively weak waves, i.e., $u_0/c_0 < 0.1$, approximately. Under this condition the signals that are reflected from a shock in the waveform are so feeble as to be negligible. The simple-wave model may therefore be retained as a good approximation. Next, triple-valued waveforms of the kind shown in Fig. 3n-1 must be avoided. This requires that provision be made for dissipation. There are two approaches. First, one can take explicit account of the dissipation terms. This leads to Burgers' equation, or variations thereof; the method is described in Sec. 3n-12. Alternatively, one can postulate mathematical discontinuities—shocks—at places where the waveform would otherwise be triple valued. The Rankine-Hugoniot relations are invoked to relate conditions on either side of each shock. In this way dissipation is accounted for indirectly. A tacit assumption, it will be noted, is that all the dissipation takes place at the shocks.

The mathematical method is more fully appreciated if the physical aspects of the process are first understood. The history of a typical waveform is depicted in Fig. 3n-2 (taken from ref. 27). Figure 3n-2*a* shows the initial waveform. Numbered dots indicate initial phase points (values of ϕ) on the wave. In the beginning, distortion takes place as described in Sec. 3n-4 (Fig. 3n-2*b* and *c*). After the shock is born (Fig. 3n-2*c*), it travels supersonically. In consequence of Eq. (3n-72), however, phase points just behind, such as number 5, travel faster. As they catch up with the shock, it grows because the top of the discontinuity is always determined by the amplitude of the phase point that just caught up with it. (Conversely, the bottom of the discontinuity always coincides with the phase point just overtaken by the shock.) The top reaches a maximum when phase point 5 catches up. After that, the top decays (Fig. 3n-2*e*). In Fig. 3n-2*f* the decay has progressed to the extent that all phase points of the original waveform between 4 and 6 have disappeared. Eventually all that remains (Fig. 3n-2*g*) is the shock and a linear section connecting it with the zero, phase point 7. This is the asymptotic shape toward which many waveforms or waveform sections tend (ref. 26).

3n-9. Mathematical Formulation of Weak-shock Theory. For the continuous sections of the waveform the most general solution from the approximate theory of simple waves is adopted, namely, Eqs. (3n-68), where w and z are given by Eqs. (3n-67). Plane, cylindrical, and spherical waves, which are not really "quasi-plane," are nevertheless included formally within the framework of this solution by taking $A = 1, x$, and x^2, respectively.

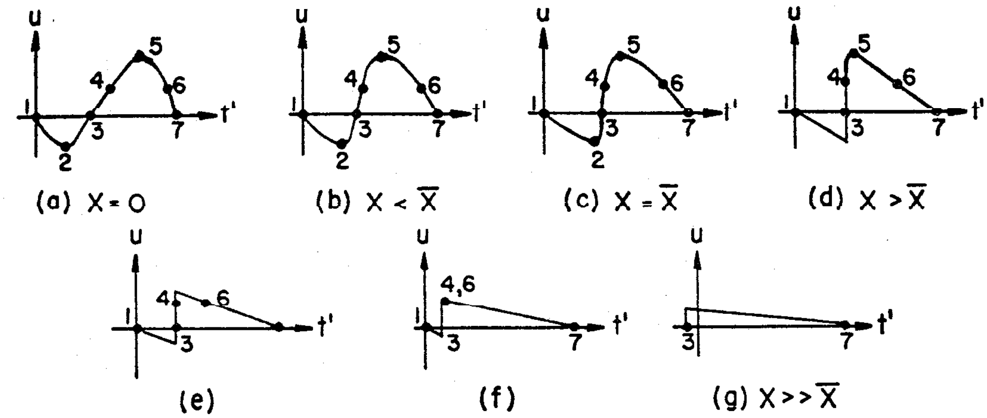

FIG. 3n-2. Development and decay of a finite-amplitude wave. Numbered points refer to initial phases (values of ϕ) of the wave. (*From ref.* 27.)

Suppose now that a shock begins to form at time $\bar{t}$ and distance $\bar{x}$. It will arrive at a subsequent point x at time t_s given by

$$t_s = \bar{t} + \int_{\bar{x}}^{x} v^{-1}\, d\mu \qquad \text{(3n-71)}$$

where v is the shock's propagation speed. The Rankine-Hugoniot relations can be combined to give v in terms of u_a and u_b, the particle velocities just ahead of and just behind the shock, respectively. An approximation of the required relation is

$$v = \pm c_0 + \tfrac{1}{2}\beta(u_a + u_b) \qquad \text{(3n-72)}$$

or, to the same order,

$$v^{-1} = \pm c_0^{-1} - \tfrac{1}{2}\beta c_0^{-2}(u_a + u_b) \qquad \text{(3n-73)}$$

Substitution of this value in Eq. (3n-71) leads to

$$t'_s = \bar{t}' - \tfrac{1}{2}\beta c_0^{-2} \int_{\bar{x}}^{x} (u_a + u_b)\, d\mu \qquad \text{(3n-74)}$$

where overbars continue to indicate values at the instant of shock formation, and primes denote retarded (or advanced) time. In terms of the generalized dependent and independent variables w and z, Eq. (3n-74) becomes

$$t'_s = \bar{t}' - \tfrac{1}{2}\beta c_0^{-2} \int_{\bar{z}}^{z} (w_a + w_b)\, d\mu \qquad \text{(3n-75)}$$

An equivalent relation is

$$\frac{dt'_s}{dz} = -\tfrac{1}{2}\beta c_0^{-2}(w_a + w_b) \tag{3n-76}$$

Once the particle velocity u has been determined, the linear impedance relation, Eq. (3n-29), is used to find the pressure signal (ref. 23).

This completes the formal solution, except for some interpretation. The waveform in the continuous sections between shocks is prescribed by Eqs. (3n-68). For each shock the path and amplitude are determined by Eq. (3n-75) or Eq. (3n-76) together with Eqs. (3n-68), which are to be evaluated just ahead of the shock ($u = u_a$, $\phi = \phi_a$, $t' = t'_s$) and just behind it ($u = u_b$, $\phi = \phi_b$, $t' = t'_s$). In principle, Eqs. (3n-68) can be combined to eliminate the parameter ϕ as follows:

$$t' = g^{-1}(w) - \beta c_0^{-2}zw \tag{3n-77}$$

Hence just ahead of the shock

$$t'_s = g^{-1}(w_a) - \beta c_0^{-2}zw_a \tag{3n-78a}$$

and just behind

$$t'_s = g^{-1}(w_b) - \beta c_0^{-2}zw_b \tag{3n-78b}$$

Equations (3n-78*a*), (3n-78*b*), and (3n-75) or (3n-76) are to be solved simultaneously for w_a, w_b, and t'_s.

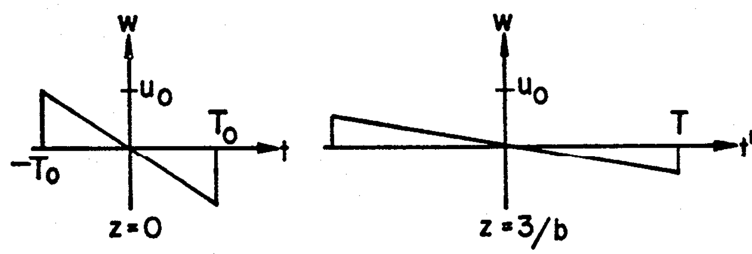

FIG. 3n-3. *N* wave.

3n-10. Applications of Weak-shock Theory. *N Wave.* Perhaps the most famous application is to the wave shaped like the letter *N*. The sonic boom is a cylindrical *N* wave in the far field. For the present consider outgoing waves only. Refer to Fig. 3n-3 for notation. At $t = 0$, $u = -u_0t/T_0$ for $-T_0 < t < T_0$. Thus $g(\phi) = -u_0\phi/T_0$, and Eq. (3n-68*b*) yields $\phi = t'/(1 + bz)$, where $b = \beta u_0/c_0^2T_0$. The solution is given by Eq. (3n-68*a*) as

$$w = -\frac{t'}{T_0}\frac{u_0}{1 + bz} \qquad -T < t' < T$$

To determine T, make use of Eq. (3n-76) for the head shock: that is,

$$\frac{dt'_s}{dz} = -\tfrac{1}{2}\beta c_0^{-2}w_b = \frac{\tfrac{1}{2}bt'_s}{1 + bz}$$

Integration gives

$$-t'_s = T = T_0(1 + bz)^{\frac{1}{2}}$$

The amplitude of the wave is therefore given by

$$u_b = \left(\frac{A_0}{A}\right)^{\frac{1}{2}} \frac{u_0}{(1 + bz)^{\frac{1}{2}}}$$

Next consider incoming waves. The major difference in the results is that z is replaced by $-z$. But z itself also changes sign [see the discussion following Eqs. (3n-67)]. The following formulas cover both incoming and outgoing waves:

$$w = \mp \frac{u_0}{1 + b|z|} \frac{t'}{T_0} \qquad -T < t' < T \tag{3n-79}$$

$$T = T_0(1 + b|z|)^{\frac{1}{2}} \tag{3n-80}$$

$$|u_b| = \left(\frac{A_0}{A}\right)^{\frac{1}{2}} \frac{u_0}{(1 + b|z|)^{\frac{1}{2}}} \tag{3n-81}$$

The growth of a converging wave ($A < A_0$) and the diminution of a diverging wave ($A > A_0$) are not comparable because the factor $(1 + b|z|)^{-\frac{1}{2}}$ acts to diminish both types of waves. Both waves spread at the same rate, however. From Eq. (3n-81) one obtains the classical results that outgoing plane, cylindrical, and spherical waves decay at great distances as $x^{-\frac{1}{2}}$, $r^{-\frac{3}{4}}$, and $r^{-1}(\ln r)^{-\frac{1}{2}}$, respectively.

Sawtooth Wave. Assume that the wave shown in Fig. 3n-3*a* is repetitive. The magnitude of the jump at the shock is now $2u_0$ to begin with. Because of the symmetry, we have $u_a = -u_b$, which means that, by Eq. (3n-72), the shocks all travel at sonic speed. Unlike the N wave, therefore, the sawtooth does not stretch out as it travels. The decay is more rapid however. Proceeding as before, we find the wave amplitude to be given by

$$|w_b| = \frac{\pi u_0}{\pi + \beta\epsilon k|z|} \tag{3n-82}$$

where k is the fundamental wave number of the wave. See ref. 28 for a discussion of power loss and related topics for sawtooth waves in an exponential horn.

Originally Sinusoidal Wave. It will be recalled that a sinusoidally vibrating piston gives rise to periodic waves whose mathematical description, for outgoing waves, is given by Eq. (3n-40), the Fubini solution. Weak-shock theory makes it possible to obtain a solution of this problem for distances beyond the point of shock formation. It turns out that after forming at $x = \bar{x} = (\beta\epsilon k)^{-1}$, the shocks reach a maximum amplitude at $x = \pi\bar{x}/2$ and thereafter decay. For distance greater than $3\bar{x}$ the wave is effectively a sawtooth of amplitude

$$u_b = \frac{\pi u_0}{1 + \sigma} \tag{3n-83}$$

where (see Sec. 3n-4) $\sigma = \beta\epsilon kx = x/\bar{x}$. This problem is treated in full in ref. 27, as is the similar one of an isolated sine-wave cycle. To generalize Eq. (3n-83) to other one-dimensional outgoing waves it is merely necessary to replace u_b by w_b and σ by $\beta\epsilon kz$.

3n-11. Limitations of Weak-shock Theory. The primary advantage of weak-shock theory over the method based on Burgers' equation (see below) is that results are obtained quickly and easily. Details of the actual profile of the wave in the neighborhood of each shock are suppressed simply by approximating the shock as a mathematical discontinuity. The method's strength is also its weakness, however. At great distances the shocks may become so weak that they become dispersed and are no longer approximate discontinuities.

As a test we may compare the shock rise time (ref. 29) τ with a characteristic period or time duration T of the wave. Thus consider the ratio

$$\frac{\tau}{T} = \frac{12\delta}{c_0|u_b|T} = \frac{12\delta}{c_0|w_b|T}\left(\frac{A}{A_0}\right)^{\frac{1}{2}} \tag{3n-84}$$

where δ is proportional to the viscosity and heat conduction coefficients of the fluid [see Eq. (3n-86)]. For an N wave $|w_b|T$ is a constant ($= u_0T_0$) so that τ/T is simply proportional to $(A/A_0)^{\frac{1}{2}}$. Therefore, if the N wave is plane, τ/T is constant, which means that the validity of the weak-shock computation does not change with distance. The wave simply spreads out as rapidly as the shock. For all other outgoing N waves, however, the shock disperses more rapidly, and eventually $\tau \sim T$, beyond which point weak-shock theory should not be trusted. Let $r_{\max}$ designate the distance at which $\tau/T = 1$. For spherical N waves we obtain,

$$\frac{r_{\max}}{r_0} = \frac{\beta u_0 c_0 T_0}{12\delta} \tag{3n-85a}$$

The comparable result for cylindrical N waves is

$$\frac{r_{\max}}{r_0} = \left(\frac{\beta u_0 c_0 T_0}{12\delta}\right)^2 \tag{3n-85b}$$

For an outgoing sawtooth wave τ/T is proportional to $(1 + \beta\epsilon k|z|)(A/A_0)^{\frac{1}{2}}$, which means that weak-shock theory is always limited, even when the wave is plane. Even for converging waves τ may approach T in certain instances (refs. 17, 18). Care must therefore be exercized in using asymptotic formulas based on Eq. (3n-82). Calculations of $r_{\max}$ for sawtooth waves based on taking $\tau = T$ are in agreement with estimates obtained by other methods (ref. 27).

The importance of the limitation on weak-shock theory varies a great deal in practice. For sonic booms the limitation is apparently not significant. Typically at ground level τ is of the order of milliseconds, whereas T is measured in tenths of a second. For long-range propagation of pulses from underwater explosions (ref. 30), however, the limitation can be crucial.

In conclusion we remark that "weak-shock theory" is in some respects a misnomer. The theory is valid for weak shocks but not, in general, for very weak ones.

BURGERS' EQUATION AND OTHER MODELS

We now consider explicitly the effects that viscosity, heat conduction, and relaxation have on the propagation of finite-amplitude waves. The full-fledged equations—(3n-1), (3n-2), (3n-3), and (3n-6) or other equation of state—must be dealt with. Successful attacks on these equations have been mainly directed at specific problems, such as the profile of a steady shock wave (ref. 29). General exact results analogous to those for lossless waves are not known. The only general approach presently available, that based on Burgers' equation, is limited to relatively weak waves. For our purposes, however, this method is a fitting companion for weak-shock theory and its predecessor, the approximate theory of lossless simple waves.

3n-12. Thermoviscous Fluids. Burgers' Equation. *Plane Waves.* By employing an approximation procedure similar to that used to change Eq. (3n-7) into (3n-8), Lighthill (ref. 29) reduced the equations of motion for outgoing plane waves in a thermoviscous perfect gas to Burgers' equation,

$$u_t + \beta u u_{x'} = \delta u_{x'x'} \tag{3n-86a}$$

Here $x' = x - c_0 t$, $\delta = \frac{1}{2}\nu[\mathcal{V} + (\gamma - 1)/\text{Pr}]$, $\nu = \eta/\rho_0$ is the kinematic viscosity, $\mathcal{V} = (\eta' + 2\eta)/\eta$ is the viscosity number, and $\text{Pr} = \eta C_p/\kappa$ is the Prandtl number. The equation applies as well to fluids of the arbitrary equation of state (refs. 31, 32); simply let β be given by Eq. (3n-60b). In certain cases it applies also to solids (ref. 33).

Equation (3n-86a) is convenient for initial-value problems because the moving coordinate x' reduces to $x' = x$ at $t = 0$. For boundary-value problems a more convenient, yet equally valid, form is (refs. 31, 3, 34)

$$c_0^2 u_x - \beta u u_{t'} = \pm \delta c_0^{-1} u_{t't'} \tag{3n-86b}$$

where $t' = t \mp x/c_0$. [To make Eq. (3n-86a) apply to incoming as well as outgoing waves, redefine x' as $x \mp c_0 t$.]

Burgers' equation has a known exact solution. The introduction of the logarithmic potential ζ by

$$u = \pm \frac{2\delta}{\beta c_0} (\ln \zeta)_t = \pm \frac{2\delta}{\beta c_0} \frac{\zeta_{t'}}{\zeta} \tag{3n-87}$$

causes Eq. (3n-86b) to be reduced to

$$\pm c_0^3 \zeta_x - \delta \zeta_{t't'} = 0 \tag{3n-88}$$

which is a diffusion equation with the usual roles of space and "time" reversed. To avoid confusion we drop the multiple-sign notation at this point and focus attention on outgoing waves. It is clear that an incoming wave can be considered simply by replacing δ with $-\delta$. The solution of Eq. (3n-88) [with the (+) sign] is

$$\zeta = \sqrt{\frac{K}{\pi}} \int_{-\infty}^{\infty} \zeta_0(\lambda) \exp\,[-K(\lambda - t')^2]\, d\lambda \tag{3n-89}$$

where $K = c_0^3/4\delta x$. The quantity $\zeta_0(t') = \zeta(0,t')$ represents the transformed boundary condition. If the original boundary condition is given by Eq. (3n-48), then, by Eq. (3n-87),

$$\zeta_0(t') = \exp\left[\int_{-\infty}^{t'} \frac{\beta c_0}{2\delta}\, g(\mu)\, d\mu\right] \tag{3n-90}$$

Normally one takes $g(t) = 0$ for $t < 0$, in which case $\zeta_0 = 1$ for $t' < 0$, and the integral's lower limit is zero. The solution of Burgers' equation has been applied to a number of specific problems (refs. 29, 32).

The only solution reviewed here is the one for which the piston motion is sinusoidal (refs. 31, 34, 35): $u(0,t) = u_0 \sin \omega t H(t)$. Equation (3n-90) gives $\zeta_0 = \exp\,[\frac{1}{2}\Gamma(1 - \cos \omega t')]$ for $t' > 0$ ($\zeta_0 = 1$ otherwise), where

$$\Gamma = \frac{\beta c_0 u_0}{\delta \omega} = \frac{\beta \epsilon}{\alpha \lambda\!\!\!^{-}} \tag{3n-91}$$

and $\alpha \lambda\!\!\!^{-} = \alpha/k$ is the dimensionless small-signal attenuation coefficient ($\alpha \lambda\!\!\!^{-} = \omega\delta/c_0^2$). The dimensionless parameter Γ characterizes the importance of nonlinear effects relative to dissipation. The value $\Gamma = 1$ roughly marks the dividing line between the importance and unimportance of nonlinearity in a periodic wave (ref. 36). When the value of ζ_0 is substituted in Eq. (3n-89), the potential ζ can be separated into transient and steady-state parts. The steady-state part, to which we restrict ourselves, may be expressed as an infinite series,

$$\zeta = I_0(\tfrac{1}{2}\Gamma) + 2 \sum_{n=1}^{\infty} (-1)^n I_n(\tfrac{1}{2}\Gamma) e^{-n^2 \alpha x} \cos n\omega t' \tag{3n-92}$$

where I_n is the Bessel function of imaginary argument.

The most interesting case is that of strong waves, i.e., $\Gamma \gg 1$. In this circumstance ζ reduces to a theta function, and the logarithmic differentiation required by Eq. (3n-87) is easy to carry out. The result is (ref. 35)

$$\frac{u}{u_0} = \frac{2}{\Gamma} \sum \frac{\sin n\omega t'}{\sinh n(1 + \sigma)/\Gamma} \tag{3n-93}$$

which is Fay's solution (ref. 37) with Fay's constant α_0 taken to be Γ^{-1}. If σ is not large, the hyperbolic sine function may be approximated by its argument, giving

$$u = \frac{2u_0}{1 + \sigma} \Sigma n^{-1} \sin n\omega t' \tag{3n-94}$$

which represents a sawtooth wave of amplitude

$$u_b = \frac{\pi u_0}{1 + \sigma}$$

This is exactly the same result found by means of weak-shock theory; see Eq. (3n-83).

For strong waves at great distances, i.e., $\sigma \gg \Gamma \gg 1$, the waveform is found, either by the Fay solution or directly by Eqs. (3n-92) and (3n-87), to be

$$u \cong 4\alpha\lambda c_0\beta^{-1}e^{-\alpha x} \sin \omega t' \tag{3n-95}$$

The simple exponential decay is expected because the wave has now become quite weak. What is remarkable is the absence of the original amplitude factor u_0. The wave amplitude at great distances is independent of the source strength. In other words saturation is reached. This result is obviously of great importance. Saturation has been observed experimentally (refs. 15, 55, 58). Note from Eq. (3n-83) that the asymptotic amplitude given by weak-shock theory is (ref. 26)

$$u_b \cong \frac{\pi c_0^2}{\beta\omega x} \tag{3n-96}$$

but this result is accurate only in the sawtooth region, which is defined roughly by $3\bar{x} < x < \alpha^{-1}$ (ref. 35).

Nonplanar Waves. For other one-dimensional waves the analog of Eq. (3n-86*b*) is

$$c_0^3 (u_x + uA_x/2A) - \beta c_0 u u_{t'} = \delta u_{t't'} \tag{3n-97}$$

(again, for incoming waves replace δ by $-\delta$). It is necessary to make the far-field assumption in deriving this equation. The transformations that have proved so helpful in previous cases, namely, Eqs. (3n-67), lead to

$$c_0^3 w_z - \beta c_0 w w_{t'} = \delta \left(\frac{A}{A_0}\right)^{\frac{1}{2}} w_{t't'} \tag{3n-98}$$

which is similar to Burgers' equation, but has one variable coefficient. No exact solutions are known.

For periodic spherical and cylindrical waves, solutions of Eq. (3n-98) have been obtained that are valid in the shock-free region ($z < \bar{z}$) and in the sawtooth region (refs. 17, 18). These solutions correspond, respectively, to the Fubini solution for spherical and cylindrical waves and to the related weak-shock solutions (ref. 27). The latter are improved upon, however, because the detailed configuration of the waveform in the vicinity of the shocks is obtained. The behavior of the shock thickness is strongly dependent upon whether the wave is a diverging or a converging one. This can be seen from the form of Eq. (3n-98). A diverging wave ($A > A_0$) is equivalent to a plane wave in a medium in which the dissipation increases with distance. Conversely, for a converging wave ($A < A_0$) the dissipation seems to decrease with distance (refs. 17, 18).

3n-13. Equations for Other Forms of Dissipation. If dissipation is due to an agency other than the thermoviscous effects discussed in the last section, it may still be possible to derive an approximate unidirectional-wave equation similar to Burgers'.

Relaxing Fluids. An elementary example of a relaxing fluid is one that radiates heat in accordance with Eq. (3n-5) (ref. 38). For simplicity take the fluid to be a perfect gas, and let it be inviscid and thermally nonconducting. At very low frequencies infinitesimal waves travel at the isothermal speed of sound, given by $b_0^2 = p_0/\rho_0$. At very high frequencies the speed is the adiabatic value, given by $b_\infty^2 =$

$\gamma p_0/\rho_0$ (the notation b_∞ is used here in place of c_0 to emphasize the role played by frequency). The dispersion m, defined by

$$m \equiv \frac{b_\infty^2 - b_0^2}{b_0^2} \tag{3n-99}$$

is equal to $\gamma - 1$ for the radiating gas. If the dispersion is very small, i.e., $m \ll 1$ (which in this case implies $\gamma \doteq 1$), the following approximate equation for plane waves can be derived:

$$\left(q + \frac{\partial}{\partial t'}\right) u_x - b_0^{-2}\left(\beta_i q + \beta_a \frac{\partial}{\partial t'}\right) u u'_t = \pm \frac{m}{2b_0} u_{t't'} \tag{3n-100}$$

where $t' = t \mp x/b_0$. It is seen that the radiation coefficient q [see Eq. (3n-5)] is the reciprocal of a relaxation time. Subscripts a and i used with β indicate adiabatic and isothermal values, respectively; that is, $\beta_a = (\gamma + 1)/2$ and $\beta_i = (1 + 1)/2 = 1$. The two values are essentially the same, since it has been assumed that $\gamma \doteq 1$. At either very low frequencies ($\omega q^{-1} \ll 1$) or very high frequencies ($\omega q^{-1} \gg 1$) the left-hand side of the equation takes on the same form as Eq. (3n-47). If the equation is linearized, a dispersion relation can be found that gives the expected behavior for a relaxation process (the actual formulas for the attenuation and phase velocity agree with the exact ones for a radiating gas only for $m \ll 1$).

Polyakova, Soluyan, and Khokhlov considered a relaxation process directly and obtained a pair of equations that can be merged to form a single equation exactly like Eq. (3n-100) except that β_i and β_a are equal (ref. 39). Some solutions (refs. 39, 40) have been found. One represents a steady shock wave. The shock profile is single-valued for very weak shocks. But when the shock is strong enough that its propagation speed [see Eq. (3n-72)] exceeds b_∞, the solution breaks down (a triple-valued waveform is predicted). This illustrates an important fact about the role of relaxation in nonlinear propagation: Relaxation absorption can stand off weak nonlinear effects, but not strong ones. In frequency terms, relaxation offers high attenuation to a broad mid-range of frequencies. If the wave is quite weak, the distortion components are easily absorbed because their frequencies fall in the range of high attenuation. But if the wave is strong, many more very high frequency components are produced, and these are not attenuated efficiently by the relaxation process. To keep the waveform from becoming triple valued, it is necessary to include a viscosity term in the approximate wave equation. In ref. 40 the problem of an originally sinusoidal wave is treated. Quantitative approximate solutions are obtained for cases in which the source frequency is either very low or very high, and a qualitative discussion is given for source frequencies in between.

Marsh, Mellen, and Konrad (ref. 30) postulated a "Burgers-like" equation for spherical waves. It is similar to Eq. (3n-100) but is corrected to take account of spherical divergence. A viscosity term is added, and β_i and β_a are the same. At either very low or very high frequencies the equation takes on the form of Eq. (3n-98) [for spherical waves $(A/A_0)^{\frac{1}{2}} = r/r_0 = e^{z/r_0}$], and some initial attempts at solving this equation were described.

Boundary-layer Effects. Consider the propagation of a plane wave in a thermoviscous fluid contained in a tube. The wave can never be truly plane because the phase fronts curve a great deal as they pass through the viscous and thermal boundary layers at the wall of the tube. If the boundary-layer thicknesses are small compared with the tube radius, however, the curvature of the phase fronts is restricted to very narrow regions, and the wave may be considered quasi-plane. The boundary layers still affect the wave, causing an attenuation that is proportional to $\sqrt{\omega}$ and a comparable dispersion. If the frequency is low, the attenuation from this source is much more important than that due to thermoviscous effects in the mainstream (central core of the fluid), and so it makes sense to find a Burgers-like equation for this case.

A one-dimensional model of time-harmonic wave propagation in ducts with boundary-layer effects treated as a body force has been given by Lamb (ref. 41). Chester (ref. 42) has generalized this model and applied it to compound flow in a closed tube. His method can be used to obtain the following equation for simple-wave flow:

$$u_x - \frac{\beta}{c_0^2} u u_{t'} = \mp \frac{1 + (\gamma - 1)/\sqrt{\mathrm{Pr}}}{c_0 D/2} \left(\frac{\nu}{\pi}\right)^{\frac{1}{2}} \int_0^\infty u_{t'}(x, t' - \mu) \frac{d\mu}{\sqrt{\mu}} \qquad \text{(3n-101)}$$

where D is the hydraulic diameter of the duct (four times the cross-sectional area divided by the circumference). No solutions are presently available. But the equation does have proper limiting forms. If the effect of the boundary layers (right-hand side) is neglected, the result is Eq. (3n-47). If the nonlinear term is dropped, the time-harmonic solution can be found, and this solution yields the correct attenuation and dispersion. Because of the relative weakness of boundary-layer attenuation (the dimensionless attenuation $\alpha\lambda$ varies as $1/\sqrt{\omega}$), the higher spectral components generated as a manifestation of steepening of the waveform are not efficiently absorbed. Thus discontinuous solutions, modified somewhat by the attenuation and dispersion, are to be expected.

REFLECTION, STANDING WAVES, AND REFRACTION

3n-14. Reflection and Standing Waves. For plane interacting waves in lossless fluids we return to Eqs. (3n-24) to (3n-26). For perfect gases the Riemann invariants are given by

$$\mathfrak{r} = \frac{c}{\gamma - 1} + \frac{u}{2} \qquad \text{(3n-102a)}$$

$$\mathfrak{s} = \frac{c}{\gamma - 1} + \frac{u}{2} \qquad \text{(3n-102b)}$$

Equations (3n-26) tell us that the quantity $\mathfrak{r}$ is forwarded unchanged with speed $u + c = \frac{1}{2}(\gamma + 1)\mathfrak{r} - \frac{1}{3}(3 - \gamma)\mathfrak{s}$. Similarly, the speed for the invariant $\mathfrak{s}$ is $u - c = \frac{1}{2}(3 - \gamma)\mathfrak{r} - \frac{1}{2}(\gamma + 1)\mathfrak{s}$. The roles of independent and dependent variables can be reversed to give the following differential equation for the flow:

$$t_{\mathfrak{r}\mathfrak{s}} + N(\mathfrak{r} + \mathfrak{s})^{-1}(t_{\mathfrak{r}} + t_{\mathfrak{s}}) = 0 \qquad \text{(3n-103)}$$

where $N = \frac{1}{2}(\gamma + 1)/(\gamma - 1)$. For monatomic and diatomic gases $N = 2$ and $N = 3$, respectively. An exact solution of this equation in terms of arbitrary functions $f(\mathfrak{r})$ and $g(\mathfrak{s})$ is known, but it is usually difficult to determine f and g from the initial conditions (ref. 4).

Reflection. Certain valuable information about reflection can be obtained without solving for the entire flow field. Consider the problem of reflection from a rigid wall. For the moment we need not be specific about the equation of state. Let the incident wave be an outgoing simple wave. The Riemann invariant $\mathfrak{r}$ for a particular signal in this wave is, by Eqs. (3n-21) and (3n-24),

$$2\mathfrak{r} = \lambda_i + u_i = 2\lambda_i$$

But $\mathfrak{r}$ can also be evaluated at the wall during the interaction of the incident and reflected waves: i.e.,

$$2\mathfrak{r} = \lambda_{\text{wall}} + u_{\text{wall}} = \lambda_{\text{wall}}$$

Elimination of $\mathfrak{r}$ between these two expressions gives

$$\lambda_{\text{wall}} = 2\lambda_i$$

This is an exact statement of the law of reflection for continuous finite-amplitude waves at a rigid wall: The quantity λ doubles, not the acoustic pressure.

To see what happens to the pressure, we must specify an equation of state. Take the case of a perfect gas, for which $\lambda = 2(c - c_0)/(\gamma - 1)$(thus $c - c_0$ doubles at a rigid wall). Using Eq. (3n-11), we obtain

$$\left(\frac{p}{p_0}\right)_{\text{wall}} = \left[2\left(\frac{p_i}{p_0}\right)^{1/\mu} - 1\right]^{\mu} \qquad \text{(3n-105)}$$

where $\mu = 2\gamma/(\gamma - 1)$. Now define a wall amplification factor $\mathcal{A}$ by

$$\mathcal{A} = \frac{p_{\text{wall}} - p_0}{p_i - p_0}$$

Substitution from Eq. (3n-105) gives

$$\mathcal{A} = \frac{[2(p_i/p_0)^{1/\mu} - 1]^{\mu} - 1}{p_i/p_0 - 1} \qquad \text{(3n-106)}$$

An analogous result in terms of the source that generated the incident simple wave is given in ref. 43; Eq. (3n-106) was first obtained by Pfriem (ref. 44). For weak waves ($p_i - p_0 \ll p_0$) $\mathcal{A} = 2$, in agreement with linear theory. The limiting value for very strong waves is $\mathcal{A} = 2^{\mu}$ ($= 2^7$ for air), a quite startling result. It is only of passing interest, however, because a wave this strong would already have deformed into a shock by the time it reached the wall [for shocks the expression for $\mathcal{A}$ is entirely different; the limiting value for strong shocks is $\mathcal{A} = 2 + (\gamma + 1)/(\gamma - 1) = 8$ for air (ref. 4)]. In fact, the deviation from pressure doubling is small even for fairly strong waves. For an originally sinusoidal wave of sound pressure level 174 dB, the maximum deviation is about 6 percent (ref. 43).

For a pressure release surface the law of reflection for finite-amplitude waves is the same as for infinitesimal waves. To see this, evaluate $\mathfrak{r}$ as before, first in the incident wave ($2\mathfrak{r} = \lambda_i + u_i = 2u_i$) and then at the pressure-release surface ($2\mathfrak{r} = \lambda_{\text{surface}} + u_{\text{surface}} = u_{\text{surface}}$, since $\lambda = 0$ when $p = p_0$, $\rho = \rho_0$). The result is

$$u_{\text{surface}} = 2u_i$$

that is, the particle velocity doubles at the surface. The reflection has an interesting effect on the wave, however. Consider a finite wave train so that after interaction the reflected signal is a simple wave. To a good approximation, the acoustic pressure wave suffers phase inversion as a result of the reflection. A wave that distorts as it travels toward the surface therefore tends to "undistort" after reflection. This effect has been observed experimentally (ref. 45).

Reflection from and transmission through other types of surfaces, such as gaseous interfaces, are considered in ref. 43.

Oblique reflection of continuous waves from a plane surface has not been solved in any general way; see ref. 46 for a perturbation treatment.

Standing Waves. First consider finite-amplitude wave motion in a tube closed at one end and containing a vibrating piston in the other end. This problem is one of the few in which much experimental evidence is available (refs. 47, 48, 50). At resonance, if the piston amplitude is sufficiently high, shocks occur traveling to and fro between the piston and the closed end. Slightly off resonance, again for high enough amplitude, the waveform exhibits cusps. Below resonance the cusps occur at the troughs of the waveform, above resonance at the peaks. It would seem that such rich phenomena would have stimulated intensive theoretical treatments of the problem.

In fact, the theoretical problem has proved a difficult nut to crack. The Riemann solution [of Eq. (3n-103)] is of no avail because of the presence of shocks. There is no well-developed weak-shock theory for compound waves as there is for simple waves. For weak waves perturbation treatments have been used (ref. 48). For strong waves one approach has been to assume the existence of shocks at the outset. The Rankine-Hugoniot relations are used to provide boundary conditions for the continuous-wave flow in between shocks (refs. 47, 49).

A more fundamental approach has been taken by Chester (ref. 42). His treatment is of general interest because of the way the effect of the boundary layer is assimilated in the one-dimensional model [see Eq. (3n-101) for an adaptation to simple waves]. An "inviscid solution" is first obtained; it contains discontinuities at and near resonance, and cusps at one point on either side of resonance. General agreement with experimental observation is thus good (ref. 50). Improved solutions are then considered in which thermoviscous effects, first in the mainstream and then in the boundary layers, are taken into account.

3n-15. Refraction. Treatments of oblique reflection and refraction at interfaces have mainly been confined to shock waves in which the flow behind the shock is basically steady. Slow, continuous refraction, such as that caused by gradual changes in the medium or by gradual variations along the phase fronts of the wave, has been treated, however (refs. 26, 51, 52). The basis of the method is ordinary ray acoustics. The propagation speed along each ray tube and the cross-sectional area of the tube are modified to take account of nonlinear effects. The approach is similar to that given in Sec. 3n-7 except that the cross-sectional area of the horn varies in a manner that depends on the wave motion.

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