3o. Nonlinear Acoustics (Experimental)

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3o-1. Fluids. In the experimental study of nonlinear acoustics, three types of quantities have been measured. These are the effective sound absorption for waves of finite amplitude, the growth of harmonic content, and the nonlinear variation terms in the isentropic expansion of the pressure in the medium in terms of the density changes.

Since the comparison of the first two of these properties with theory depends on the third, it is most effective to consider first the nonlinearity of the equation of state.

This isentropic equation of state can be expanded in a Taylor series in the condensation $s = (\rho - \rho_0)/\rho_0$:

$$p - p_0 = As + \frac{B}{2!}s^2 + \frac{C}{3!}s^3 + \cdots \tag{3o-1}$$

Here p_0 and ρ_0 are the equilibrium values of the pressure and the density. Also, $A = \rho_0 c_0^2$. By application of thermodynamics (ref. 1) the ratio B/A can be written

$$\frac{B}{A} = 2\rho_0 c_0 \left(\frac{\partial c}{\partial p}\right)_T + \frac{2c_0 T\beta}{c_p}\left(\frac{\partial c}{\partial T}\right)_P \tag{3o-2}$$

In this equation, β is the coefficient of thermal expansion, c_p the specific heat at constant pressure, and the derivatives are evaluated under condition of sound waves of infinitesimal amplitude. B/A is sometimes known as the parameter of nonlinearity.

Evaluation of C/A is more involved. It can be shown that (ref. 2)

$$\frac{C}{A} = \frac{3}{2}\left(\frac{B}{A}\right)^2 + 2\rho_0^2 c_0^3 \left(\frac{\partial^2 c}{\partial p^2}\right)_s \tag{3o-3}$$

At a hydrostatic pressure of one atmosphere, the second term on the right is generally quite small compared with the first, although it is likely to become appreciable at higher hydrostatic pressures (ref. 3).

For an ideal gas, we can expand the adiabatic equation of state

$$p = p_0\left(\frac{\rho}{\rho_0}\right)^\gamma = p_0\left[1 + \gamma s + \frac{\gamma(\gamma-1)}{2!}s^2 + \cdots\right] \tag{3o-4}$$

where γ is the ratio of specific heats.

By comparing coefficients in Eqs. (3o-1) and (3o-4) we find

$$A = \gamma p_0 \qquad B = \gamma(\gamma - 1)p_0$$

whence

$$\frac{B}{A} = \gamma - 1 \qquad \text{for an ideal gas} \tag{3o-5}$$

The ratio B/A has now been measured for a considerable number of liquids at atmospheric pressure and, in some instances, over a modest temperature range. A number

of these experimental values are given in Table 3o-1. The error in these measurements is generally of the order of 2 to 3 percent, except for the liquid metals, where the larger uncertainties are listed in the table.

The few samples of temperature dependence of B/A shown indicate that B/A can increase or decrease with temperature, depending on the material, but that the temperature variation is usually quite slight.

TABLE 3o-1. VALUES OF B/A FOR VARIOUS LIQUIDS

Liquid	T, °C	B/A	Reference	Liquid	T, °C	B/A	Reference	Estimated error, %
Acetone	20	9.2	2	Sulfur	121	9.5	6	
Alcohol								
Methyl	20	9.6	2	Water (distilled)	0	4.1	3	
					10	4.6	1	
Ethyl	20	10.5	2		20	5.0	1	
					30	5.2	3	
n-Propyl	20	10.7	2		40	5.5	3	
					50	5.55	3	
n-Butyl	20	10.7	2		60	5.6	3	
					80	5.7	3	
Benzene	30	9.0	2					
	40	9.2	2					
	50	9.3	2	Water (sea, 33 % salinity)	0	4.9	2	
	60	9.45	2		10	5.1	2	
	70	9.5	2		20	5.2	2	
					30	5.4	2	
Benzyl alcohol	30	10.2	2					
Chlorobenzene	30	9.3	2	*Liquid Metals*				
				Bismuth	318	7.1	8	15
Cyclohexane	30	10.1	2	66 Bi (wt %), 34 In	125	6.1	8	5
	40	10.1	2					
	50	10.1	2	48 Bi 52 In	125	5.1	8	5
	60	9.85	2	34 Bi 76 In	125	5.1	8	5
	70	9.75	2	17 Bi 83 In	125	4.9	8	5
Diethylamine	30	10.3	2	Indium	160	4.55	8	5
Ethylene glycol	30	9.7	5	Mercury	30	2.9	8	3
Ethyl formate	30	9.8	5	Potassium	100	2.9	7	15
Heptane	30	10.0	4	Sodium	110	2.7	8	2
Hexane	30	9.9	4	Tin	240	4.4	8	11
Methyl acetate	30	9.7	2					
Methyl iodide	30	8.2	5					

The dependence of B/A on hydrostatic pressure is shown in Table 3o-2 for several liquids. Table 3o-3 gives the few known values of the third-order ratio, C/A, all under the approximation

$$\left(\frac{\partial^2 c}{\partial p^2}\right)_S \approx \left(\frac{\partial^2 c}{\partial p^2}\right)_T$$

The general form of the acoustic wave equation for a fluid satisfying Eq. (3o-1) (with neglect of the s^3 and higher terms) is, in Lagrangian coordinates,

$$\frac{\partial^2 \xi}{\partial t^2} = \frac{c_0^2}{(1 + \partial\xi/\partial x)^{2+B/A}} \frac{\partial^2 \xi}{\partial x^2} \tag{3o-6}$$

where ξ is the particle displacement, and c_0 is the speed of sound for infinitesimal ξ.

In approximate solutions of this equation [such as Eqs. (3n-40) and (3n-92)], the ratio B/A always appears in the form

$$\beta = 1 + \frac{B}{2A} \tag{3o-7}$$

Hence distortions of the wave form of an initial sinusoid can be used to determine the ratio B/A.

Finally, the effective absorption coefficient for a finite-amplitude wave can be written for a nonrelaxing medium as

$$\frac{\alpha_{\text{eff}}}{\alpha} = 1 + \frac{3\omega^2\xi_0}{4\alpha c^2}\left(1 + \frac{B}{2A}\right)e^{-2\alpha x}(1 - e^{-2\alpha x})^2 + \text{higher-order terms} \quad (3o\text{-}8)$$

where α is the absorption coefficient for infinitesimal displacement amplitude ξ_0. The ratio B/A could therefore be obtained from this equation, although with reduced accuracy.

TABLE 3o-2. VALUES OF B/A AT VARIOUS PRESSURES

Temperature T, °C	Pressure p, kg/cm²						
	1	250	500	1,000	2,000	4,000	8,000
Water [3]							
0	4.08	4.90	5.58	6.35	6.78	6.60	
40	5.49	5.59	5.69	5.84	6.00	6.06	5.79
80	5.74	5.79	5.84	5.86	5.82	5.64	5.50
1-*Propyl alcohol* [9]							
30	10.4	....	8.9	8.0	7.3	6.4	5.7
Mercury [9]							
40.5	8.33	....	....	....	7.84	7.37	7.01

TABLE 3o-3. VALUES OF C/A

Pressure at 30°C	$\frac{3}{2}(B/A)^2$	$2\rho_0^2c_0^3(\partial^2c/\partial p^2)_T$ at 30°C	C/A
Water [3]			
1 atm	40.7	−8.7	32.0
2,000 kg/cm²	55.5	−16.9	38.6
4,000 kg/cm²	57.5	−25.0	32.5
8,000 kg/cm²	52.7	−26.7	26.0
1-*Propyl alcohol* [9]			
1 kg/cm²	162	−87	75
8,000 kg/cm²	49	−24	25

3o-2. Solids. Equation (3n-60*c*) indicates that the coefficient $\beta = 1 + B/2A$ for liquids must be replaced by $\beta = -M_3/2M_2$ for solids, where M_2 and M_3 are elastic-constant combinations that appear in the partial differential equation for purely longitudinal waves in solids (ref. 10),

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{\rho_0}\frac{\partial^2 u}{\partial x^2}\left(M_2 + M_3\frac{\partial u}{\partial x} + \text{higher-order terms}\right) \quad (3o\text{-}9)$$

where u is the displacement velocity. The constants M_2 and M_3 are often written in terms of other so-called second- and third-order elastic coefficients K_2 and K_3:

$$M_2 \equiv K_2 \qquad M_3 = K_3 + 2K_2$$

The coefficients K_2 and K_3 are in turn related to the more familiar second- and third-order elastic constants C_{ij} and C_{ijk}. The connections for the [100], [110], and [111]

directions are shown in Table 3o-4. More detailed relations of this sort are given in ref. 12.

By measurement of the distortion of an initially sinusoidal longitudinal wave through a solid, it is therefore possible to determine the third-order elastic constants. A number of these constants have been determined. Their values are given in Table 3o-5 (ref. 13).

TABLE 3o-4. K_2 AND K_3 FOR THE [100], [110], AND [111] DIRECTIONS [11]

Direction	K_2	K_3
[100]	C_{11}	C_{111}
[110]	$\frac{C_{11} + C_{12} + 2C_{44}}{2}$	$\frac{C_{111} + 3C_{112} + 12C_{166}}{4}$
[111]	$\frac{C_{11} + 2C_{12} + 4C_{44}}{3}$	$\frac{C_{111} + 6C_{112} + 12C_{144} + 24C_{166} + 2C_{123} + 16C_{456}}{9}$

TABLE 3o-5. MEASURED THIRD-ORDER ELASTIC CONSTANTS OF SOME CUBIC CRYSTALS AT ROOM TEMPERATURE [13]
($\times 10^{12}$ dynes/cm^2)

Crystal	C_{111}	C_{112}	C_{123}	C_{144}	C_{166}	C_{456}	Ref.
Ge	−7.10	−3.89	−0.18	−0.23	−2.92	−0.53	14
Si	−8.25	−4.51	−0.64	+0.12	−3.10	−0.64	14
GaAs	−6.22	−3.87	+0.57	+0.02	−2.69	−0.39	15
GaAs	−6.72	−4.02	−0.04	−0.70	−3.20	−0.69	16
InSb	3.14	−2.10	−0.48	+0.09	−1.18	+0.002	17
Cu	−15.0	−8.5	−2.5	−1.35	−6.45	−0.16	18
Cu	−12.71	−8.14	−0.50	−0.03	−7.80	−0.95	19
Ge	−7.32	−2.90	−2.2	−0.08	−3.03	−0.41	20
Ge	−7.16	−4.03	−0.18	0.53	−3.15	−0.47	21
MgO	−48.9	−0.95	−0.69	+1.13	−6.6	+1.47	21
NaCl	−8.3						22
KCl	−7.1						22
NaCl	−8.80	−0.57	0.284	0.257	−0.611	0.271	23
KCl	−7.01	−0.224	0.133	0.127	−0.245	0.118	23
BaF_2	−5.84	−2.99	−2.06	−1.21	−0.889	0.271	24
Approx. accuracy, %	±5	±10	±50	±50	±3	±15	

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