5b. Formulas

W. R. SMYTHE[1]

California Institute of Technology

C. YEH[2]

University of California, Los Angeles

STATIC-FIELD FORMULAS

Note. In the following formulas $\approx$ designates an approximate equality, $K(k)$ and $E(k)$ are complete elliptic integrals of modulus k, $F(\phi,k)$ and $E(\phi,k)$ are incomplete elliptic integrals, $\ln x$ is the natural logarithm of x, δ_n^m is the Kronecker delta which is zero unless m equals n when it is one, $J_n(x)$ is a Bessel function, $\Gamma(x)$ is a gamma function, $(2n-1)!!$ means $1 \cdot 3 \cdot 5 \cdots (2n-1)$, $(2n)!!$ means $2 \cdot 4 \cdot 6 \cdots (2n)$. Vectors are written boldface unless only the strength or magnitude is involved when the same symbol is used without boldface. The positive value of a difference $x - y$ is indicated by $|x - y|$.

5b-1. Capacitance Formulas in MKS Units

Single Body Remote from Earth

Sphere of radius a — $C = 4\pi\epsilon a \approx 1.1128 \times 10^{-10} a$

Oblate spheroid of semiaxes a and c, $a > c$ — $C = 4\pi\epsilon(a^2 - c^2)^{\frac{1}{2}}[\tan^{-1}(a^2c^{-2} - 1)^{\frac{1}{2}}]^{-1}$

Prolate spheroid of semiaxes a and b, $a > b$ — $C = 4\pi\epsilon(a^2 - b^2)^{\frac{1}{2}}[\tanh^{-1}(1 - b^2a^{-2})^{\frac{1}{2}}]^{-1}$

Ellipsoid of semiaxes a, b, and c, $a > b > c$

$$C = 4\pi\epsilon(a^2 - c^2)^{\frac{1}{2}}[F(k,\phi)]^{-1}$$

where $\phi = \sin^{-1}(1 - c^2a^{-2})^{\frac{1}{2}}$ and $k = (a^2 - b^2)^{\frac{1}{2}}(a^2 - c^2)^{-\frac{1}{2}}$.

Circular disk of radius a — $C = 8\epsilon a$

Elliptic disk of semiaxes a and b, $a > b$ — $C = 4\pi\epsilon a\{K[(1 - b^2a^{-2})^{\frac{1}{2}}]\}^{-1}$

Two spheres of radius a in contact — $C = 8\pi\epsilon a \ln 2$

Two spheres of radii a and b in contact

$$C = -4\pi\epsilon ab(a+b)^{-1}\{2\gamma + \psi[b(a+b)^{-1}] + \psi[a(a+b)^{-1}]\}$$

where $\psi(z) = \Gamma'(z)/\Gamma(z)$ and γ is Euler's constant 0.5772.

Circular solid cylinder of radius a and length $2b$

$$C = [8 + 6.95(b/a)^{0.76}]\epsilon a$$

This formula is accurate to 0.2 per cent when $0 \lesssim b/a \lesssim 8$.

[1] Static-field formulas.
[2] Dynamic-field formulas.

Two spheres of radius a, distance between centers c, connected by thin wire

$$C = 8\pi\epsilon a \sinh \beta \sum_{n=1}^{\infty} (-1)^{n+1} \operatorname{csch} m\beta$$

where $\cosh \beta = \frac{1}{2}ca^{-1}$.

Two spheres of radii a and b, distance between centers c, connected by thin wire

$$C = 8\pi\epsilon ab \sinh \alpha \sum_{n=1}^{\infty} \{(c \sinh n\alpha)^{-1} + [a \sinh n\alpha + b \sinh (n-1)\alpha]^{-1}\}$$

where $\cosh \alpha = \frac{1}{2}a^{-1}b^{-1}(c^2 - a^2 - b^2)$.

Two spherical caps with a common rim which meet at an external angle π/m where m is a positive integer

$$C = 4\pi\epsilon a\{1 + \sin \alpha \sum_{s=1}^{\infty} [\csc (m^{-1}s\pi + \alpha) - \csc (m^{-1}s\pi)]\}$$

The sphere of which the flatter cap is a portion has a radius a and the rim subtends an angle α at its center.

Same as above but with external angle $3\pi/2$.[1]

$$C = 4\pi\epsilon 3^{-\frac{1}{2}} a \sin \alpha \{3^{\frac{1}{2}} - 3^{-1}4 + [2 \sin \tfrac{1}{3}\alpha(\sin \tfrac{1}{3}\alpha + \sin \tfrac{1}{3}\pi)]^{-1} + [2 \cos \tfrac{1}{3}\alpha(\cos \tfrac{1}{3}\alpha + \cos \tfrac{1}{3}\pi)]^{-1}\}$$

Spherical bowl whose chord, drawn from center to rim, subtends an angle α at the center of the sphere of radius a on which it lies

$$C = 4\epsilon a(\alpha + \sin \alpha)$$

Torus formed by rotation of a circle of radius a about a coplanar line a distance b from its center

$$C = 8\pi\epsilon b(1 - a^2b^{-2})^{\frac{1}{2}} \sum_{n=1}^{\infty} (2 - \delta_n^0) \frac{Q_n}{P_n}$$

where $P_0 = 2k^{\frac{1}{2}}K(k')$, $Q_0 = 2k^{\frac{1}{2}}K(k)$, $P_1 = 2k^{-\frac{1}{2}}E(k')$, and $Q_1 = 2k^{-\frac{1}{2}}[K(k) - E(k)]$ and the moduli of the complete elliptic functions are given by

$$k = a[b + (b^2 - a^2)^{\frac{1}{2}}]^{-1} = (1 - k'^2)^{\frac{1}{2}}$$

When $n > 1$, the following recurrence formula may be used to find both P_n and Q_n

$$(2n + 1)P_{n+1} - 4na^{-1}bP_n + (2n - 1)P_{n-1} = 0$$

A capacitance table is given in *Australian J. Phys.* [**7**, 350 (1954)].

Torus formed by rotation of a circle of diameter d about a tangent line

$$C = 8\pi\epsilon d \sum_{n=1}^{\infty} [J_1(k_nd)]^{-1} S_{0,0}(k_nd) \approx 0.970 \times 10^{-10}d$$

where $S_{0,0}(k_nd)$ is a Lommel function and $J_0(k_nd) = 0$.

[1] For additional intersecting sphere-capacitance formulas, see Snow, *J. Research Natl. Bur. Standards* **43**, 377–407 (1949).

Aichi's formula for a nearly spherical surface

$$C \approx 3.139 \times 10^{-11} S^{\frac{1}{2}}$$

where S is surface area.
Cube of side a. Close lower limit

$$C \approx 0.7283 \times 10^{-10} a$$

Figure of rotation, $z = a(\cos u + k \cos 2u)$, $\rho = a(\sin u - k \sin 2u)$, $0 < k < \frac{1}{2}$

$$C \approx 1.11278 \times 10^{-10} a(1 - 0.06857k^2 - 0.00559k^4)$$

Flat circular annulus, with edges at $\rho = a$, $\rho = b$. $a < b$

$$C \approx 4.510 \times 10^{-11} b \left[\cos^{-1}\frac{a}{b} + \left(1 - \frac{a^2}{b^2}\right)^{\frac{1}{2}} \tanh^{-1}\frac{a}{b}\right]\left(1 + \frac{0.0143b}{a} \tan^3 \frac{1.28a}{b}\right)$$

Error varies from about $\pm 0.001C$ at $b = 1.1a$ to zero at $b = \infty$.

$$C \approx 17.48 \times 10^{-12}(a + b)\{\ln [16(a + b)(b - a)^{-1}]\}^{-1}$$

Error varies from about $\pm 0.001C$ at $b = 1.1a$ to zero at $b = a$.
Thin torus generated by rotation of a circle of radius a about a coplanar line a distance b from its center

$$C \approx 3.49066 \times 10^{-10} b \left(\ln \frac{8b}{a}\right)^{-1}$$

Capacitance between Two Bodies Remote from All Others and Carrying Equal and Opposite Charges

Two spheres of radii a and b with distance r between centers

$$C = (c_{11}c_{22} - c_{12}^2)(c_{11} + c_{22} + 2c_{12})^{-1}$$

where c_{11} or $c_{22} = 4\pi\epsilon ab \sinh \alpha \sum_{n=1}^{\infty} [(b \text{ or } a) \sinh n\alpha + (a \text{ or } b) \sinh (n - 1)\alpha]$.

$$c_{12} = -4\pi\epsilon abr^{-1} \sinh \alpha \sum_{n=1}^{\infty} \operatorname{csch} n\alpha \quad \text{and} \quad \cosh \alpha = \tfrac{1}{2}(r^2 - a^2 - b^2)a^{-1}b^{-1}$$

Two equal spheres of radius a with distance r between centers

$$C = 2\pi\epsilon a \sinh \beta \sum_{n=1}^{\infty} [\operatorname{csch} (2n - 1)\beta + \operatorname{csch} 2n\beta]$$

where $\cosh \beta = \frac{1}{2} r a^{-1}$.
Kirchhoff's formula for two identical plane parallel coaxial circular disks of thickness t and radius r with square edges and a distance d between adjacent faces

$$C \approx 8.855 \times 10^{-12}(\pi r^2 d^{-1} + r\{-1 + \ln [16\pi r d^{-1}(1 + t d^{-1})] + 4\pi t d^{-1} \ln (1 + t^{-1}d)\})$$

Two identical oppositely charged plane parallel coaxial infinitely thin circular disks at a distance d apart

$$C \approx 8.855 \times 10^{-12}\{\pi r^2 d^{-1} + r[\ln (16\pi r d^{-1}) - 1]\}$$

Two thin oppositely charged coaxial rings generated by rotating two coplanar circles of radius a about a line parallel to and at a distance b from the line of length c that joins their centers

$$C \approx 1.7480 \times 10^{-10} \left\{ \frac{1}{2b} \ln \frac{8b}{a} + \frac{1}{(4a^2 + c^2)^{\frac{1}{2}}} K[(1 + 4c^2b^{-2})^{-\frac{1}{2}}] \right\}^{-1}$$

Capacitance between Two Bodies, One Enclosing the Other

Concentric spheres of radii a and b, $a < b$ $\quad C = 4\pi\epsilon ab(b - a)^{-1}$
Spheres of radii a and b with distance c between centers

$$C = 4\pi\epsilon ab \sinh \alpha \sum_{s=0}^{\infty} [b \sinh n\alpha - a \sinh (n - 1)\alpha]^{-1}$$

where $\cosh \alpha = \frac{1}{2}(a^2 + b^2 - c^2)(ab)^{-1}$.
Confocal ellipsoids with semiaxes $a > b > c$, $a' > b' > c'$, and $a > a'$

$$C = 4\pi\epsilon a'(a - a')^{-1}(a^2 - c^2)^{\frac{1}{2}}\{F[(a^2 - b^2)^{\frac{1}{2}}(a^2 - c^2)^{-\frac{1}{2}}, \sin^{-1} (1 - c^2a^{-2})^{\frac{1}{2}}]\}^{-1}$$

Small sphere of radius a midway between planes a distance $2c$ apart

$$C \approx 1.1128 \times 10^{-10} \left(\frac{1}{a} - \frac{1}{c} \ln 2 \right)^{-1}$$

Sphere of radius b on axis of infinite cylinder of radius a

$$C = b[1.11285 - 0.9277r - 0.114r^2 - 0.1955r^3 + 1.8858r(1 - r)^{-0.5463}] \times 10^{-10}$$

where $r = b/a$. The error is less than 1 part in 4,000 for $0 \lesssim r \lesssim 0.95$.

Two-dimensional Formulas for Capacitance per Meter Length

Let $U + jV = f(x + jy)$; then if V_1 and V_2 form two closed curves in the xy plane such that all U lines originate inside one and terminate inside the other and are continuous in the intermediate regions, V_1 and V_2 are sections of two cylindrical conductors and the capacitance per meter between them is

$$C_1 = \epsilon[U]|V_2 - V_1|^{-1}$$

where $[U]$ is the increment in U in passing once around V_1 or V_2 in the positive direction.
Two circular cylinders of radii a and b with a distance c between centers

$$C_1 = 2\pi\epsilon \left(\cosh^{-1} \frac{|c^2 - a^2 - b^2|}{2ab} \right)^{-1}$$

One cylinder may enclose the other or they may be mutually external.
Cylinder of radius a and plane at a distance c from its center

$$C_1 = 2\pi\epsilon[\cosh^{-1} (ca^{-1})]^{-1}$$

Coaxial circular cylinders of radii a and b, $b > a$, $C_1 = 2\pi\epsilon[\ln (a^{-1}b)]^{-1}$.
Confocal elliptic cylinders semiaxes a, b and a', b', $b > a$, $b' > a'$, $a > a'$

$$C_1 = 2\pi\epsilon[\tanh^{-1} (b^{-1}a) - \tanh^{-1} (b'^{-1}a')]^{-1}$$

Rectangular prism of n width, a sides, inside coaxial circular cylinder of radius b. If $b \gg a$, $C \approx 2\pi\epsilon[\ln (a^{-1}bN)]^{-1}$, where $N = 2\pi n^{-1}\Gamma(1 + 2n^{-1})[\Gamma(1 + n^{-1})]^{-2}$.

The capacitance per unit length of conductor systems 1 to 12 given below is

$$C_1 = A\epsilon K(k)\{K[(1 - k^2)^{\frac{1}{2}}]\}^{-1}$$

where A is 2 or 4 as indicated, and $K(k)$ is a complete elliptic integral of modulus k. This is given below in terms of the arrangement of straight lines or circular arcs or both that are formed by taking a normal cross section of the two-dimensional conductor system. Any of these configurations, if used as a transmission line and perfectly conducting, has the characteristic high-frequency impedance $\eta\epsilon C_1^{-1}$ (see Table 5b-1).

1. Two collinear lines of lengths a and b with a gap c between. Also valid if $b = \infty$.

$$A = 2 \qquad k = (ab)^{\frac{1}{2}}(a + c)^{-\frac{1}{2}}(b + c)^{-\frac{1}{2}}$$

2. A circle of radius R whose center lies on an interior line of length a, or in a gap of width a between two external collinear semi-infinite lines at a distance c from the near one. Valid for an infinite line normal to a semi-infinite one if $R = \infty$.

$$A = 4 \qquad k = aR(aR - c^2 + c|2R - a|)^{-1}$$

3. A radial line of length a at a distance c from a circle of radius R inside $(-)$ with $a + c < R$, or outside $(+)$ with $a \lessgtr \infty$.

$$A = 4 \qquad k = aR[R(2c + a) \pm c(a + R)]^{-1}$$

4. A vertical line of width $2a$ bisected by a gap of width $2b$ in an infinite horizontal line whose near-end distance is c. Set $b = \infty$ to remove half of horizontal line.

$$A = 2 \qquad k^2 = \frac{2a\{(c^2 + a^2)^{\frac{1}{2}} + [(b - c)^2 + a^2]^{\frac{1}{2}}\}}{[(c^2 + a^2)^{\frac{1}{2}} + a]\{[(b - c)^2 + a^2]^{\frac{1}{2}} + a\}}$$

5. A vertical line of width $2a$ between two horizontal infinite lines a distance b apart with its center a distance c from the nearer horizontal line.

$$A = 2 \qquad k = 2\left(\sin\frac{\pi a}{b}\sin\frac{\pi c}{b}\right)^{\frac{1}{2}}\left(\sin\frac{\pi a}{b} + \sin\frac{\pi c}{b}\right)^{-1}$$

6. A line of length b on the x axis and a line of length $2a$ on the y axis centered at the origin. The gap between lines is c. Valid also for $b = \infty$.

$$A = 2 \qquad k^2 = \frac{2a\{[(c + b)^2 + a^2]^{\frac{1}{2}} - (c^2 + a^2)^{\frac{1}{2}}\}}{\{[(c + b)^2 + a^2]^{\frac{1}{2}} - a\}[(c^2 + a^2)^{\frac{1}{2}} + a]}$$

7. A line of length $2a$ which lies on a diameter of length d that bisects an opening of width s in a circle. From line center to circle is c. Use system 6 but with

$$\frac{d(c + a)}{d - c - a} \text{ for } c + b \qquad \frac{d(c - a)}{d - c + a} \text{ for } c \qquad \frac{d[d^2 + (d^2 - s^2)^{\frac{1}{2}}]}{s} \text{ for } a$$

8. A line of length a whose near end is a distance c from the point where the ends of two semi-infinite lines meet at an angle 2α and which lies on a bisector of this angle. Here α lies between 0 and π.

$$A = 2 \qquad k^2 = 1 - c^{\pi/\alpha}(c + a)^{-\pi/\alpha}$$

9. A line lying on the x axis with gaps b between its edges and the points $x = a$ and $x = -a$ at both of which the ends of two semi-infinite lines, only one to a quadrant, meet the x axis at an obtuse angle α. Here A is 4, and k is found from

$$bB\left(\frac{\alpha}{\pi}, \frac{3}{2}\right) = aB_{(1-k^2)^{\frac{1}{2}}}\left(\frac{\alpha}{\pi}, \frac{3}{2}\right)$$

where $B(m,n)$ and $B_x(m,n)$ are complete and incomplete beta functions.

10. Two lines of length $2a$, normal to the x axis, which bisects them, and a distance c apart. Here A is 4, and k is given implicitly by the equations[1]

$$E(k)F(\phi,k) - K(k)E(\phi,k) = \frac{\pi a}{c} \qquad \sin^2\phi = \frac{K(k) - E(k)}{k^2K(k)}$$

11. A line of length $2a$ midway between and parallel to two infinite lines $2b$ apart

$$A = 2 \qquad k = (1 - e^{2\pi a/b})^{\frac{1}{2}}$$

12. A $2c$ by $2b$ rectangle midway between two infinite parallel lines at a distance $2a$ apart to which the $2b$ sides are parallel. First determine the modulus h of the complete elliptic integral $K(h)$ so that it satisfies

$$(a - c)K(h) = bK[(1 - h^2)^{\frac{1}{2}}]$$

Now write N for $\pi b[2ah^2K(h)]^{-1}$ to obtain

$$A = 4 \qquad k^2 = \tfrac{1}{2}\{1 + h^2N^2 - [1 + 2N^2(h^2 - 2) + N^4]^{\frac{1}{2}}\}$$

Approximate formula for system 10 above is

$$C_1 \approx \epsilon b^{-1}a\{1 + b(\pi a)^{-1}[1 + \ln(2\pi b^{-1}a)]\}$$

Circular cylinder of radius a midway between earthed parallel plates at a distance $2b$ apart, $C_1 \approx 4\epsilon K(\sin\theta)[K(\cos\theta)]^{-1}$, where $\sin\theta = \tanh[\pi a\theta(2b\theta - \pi a)^{-1}]$. This is an upper limit which is about 0.1 percent above the true value when $a = \frac{1}{2}b$, and approaches the true value as a/b diminishes.

Square coaxial line with faces of inner square section of width $2a$ parallel to faces of outer square section of width $2b$.

$$C_1 = 2\epsilon\frac{K[(k_1^2 - k_2^2)^{\frac{1}{2}}k_1^{-1}(1 - k_2^2)^{-\frac{1}{2}}]}{K[k_2(1 - k_1^2)^{\frac{1}{2}}k_1^{-1}(1 - k_2^2)^{-\frac{1}{2}}]}$$

where k_1 and k_2 are found from

$$\frac{K(k_1)}{K[(1 - k_1^2)^{\frac{1}{2}}]} = \frac{K[(1 - k_2^2)^{\frac{1}{2}}]}{K(k_2)} = \frac{b + a}{b - a}$$

Small wire of radius a parallel to and at a distance c from the nearer of two parallel earthed plates at a distance b apart. $a \ll c$.

$$C_1 \approx 2\pi\epsilon\left[\ln\left(\frac{2b}{\pi a}\sin\frac{\pi c}{b}\right)\right]^{-1}$$

Capacitance Edge Corrections. Consider a thin, charged semi-infinite plate with straight edge parallel to and halfway between two infinite conducting plates at potential zero spaced a distance b apart. Increased capacitance per unit length of edge due to bulging of field is equivalent to adding strip of width $\pi^{-1}\ln 2$ to the edge and assuming no bulging. Same as above but infinite plates a distance $2B$ apart and central plates of thickness $2A$ with square edge. Increased capacitance per unit length due to bulging of field is equivalent to adding to central plate a strip of thickness $2A$ and width

$$\frac{2}{\pi}\left\{B\ln\frac{2B - A}{B - A} - A\ln\frac{[A(2B - A)]^{\frac{1}{2}}}{B - A}\right\}$$

and assuming no bulging or charge on edge.

[1] For other two-strip configurations see A. E. H. Love, *Proc. London Math. Soc.* **22**, **339–369 (1923)**.

Parallel-plate capacitor with rectangular step in one plate, spacing on one side of step a and on other b. $b > a$. Additional capacitance per unit length of step above that from assumption of uniform field on each side of step is

$$2\pi\epsilon\left(\frac{a^2+b^2}{ab}\ln\frac{b+a}{b-a}+2\ln\frac{b^2-a^2}{4ab}\right)^{-1}$$

Two infinite sheets, each of which has one half bent at right angles to the other, are placed with the edges of the bends parallel so that the distance between sheets on one side of the bend is a and on the other b. The additional capacitance per unit length of bend over that given by the assumption of a uniform field over each a half of the inner sheet and no field in the corner rectangle is

$$\frac{2\epsilon}{\pi}\left(\ln\frac{a^2+b^2}{4ab}+\frac{a}{b}\tan^{-1}\frac{b}{a}+\frac{b}{a}\tan^{-1}\frac{a}{b}\right)$$

Capacitance and Elastance Coefficients

In a system of n conductors the charge on conductor m is

$$Q_m = c_{1m}V_1 + c_{2m}V_2 + \cdots + c_{mm}V_m + \cdots + c_{nm}V_n$$

In a system of n conductors the potential of conductor m is

$$V_m = s_{1m}Q_1 + s_{2m}Q_2 + \cdots + s_{mm}Q_m + \cdots + s_{nm}Q_n.$$

The force or torque tending to increase distance or angle x is

$$-\frac{1}{2}\sum_{p=1}^{n}\sum_{q=1}^{n}\frac{\partial c_{pq}}{\partial x}Q_pQ_q = +\frac{1}{2}\sum_{p=1}^{n}\sum_{q=1}^{n}\frac{\partial s_{pq}}{\partial x}V_pV_q$$

The energy of a system of n conductors is

$$W = \tfrac{1}{2}\sum_{p=1}^{n}\sum_{q=1}^{n}c_{pq}V_pV_q = \tfrac{1}{2}\sum_{p=1}^{n}\sum_{q=1}^{n}s_{pq}Q_pQ_q$$

For two distant conductors, $s_{pq} = s_{qp} \approx (4\pi\epsilon r)^{-1}$. If conductor 2 encloses conductor 1 only, then, $c_{11} = -c_{12}$ and $s_{1r} = s_{2r}$, where $1 < r$. For two spheres of radii a_1 and a_2 with centers a distance c apart, far from all other bodies

$$c_{11} = 4\pi\epsilon a_1a_2\sinh\alpha\sum_{n=1}^{\infty}[a_2\sinh n\alpha \pm a_1\sinh(n-1)\alpha]^{-1}$$

where $\cosh\alpha = \frac{1}{2}|c^2 - a^2 - b^2|a^{-1}b^{-1}$ and the upper sign is used unless a_2 encloses a_1. If spheres are mutually external

$$c_{12} = -4\pi\epsilon a_1a_2c^{-1}\sinh\alpha\sum_{n=1}^{\infty}\operatorname{csch} n\alpha$$

If the capacitances to earth of two distant bodies when alone are C_1 and C_2, the capacitance coefficients are approximately

$$c_{11} \approx \frac{16\pi^2\epsilon^2r^2C_1}{16\pi^2\epsilon^2r^2 - C_1C_2} \qquad c_{12} = c_{21} \approx -\frac{C_1C_2}{4\pi\epsilon r} \qquad c_{22} \approx \frac{16\pi^2\epsilon^2r^2C_2}{16\pi^2\epsilon^2r^2 - C_1C_2}$$

5b-2. Electrostatic-force Formulas. The force in the direction of the unit vector **m** on a conductor with surface charge density σ in a dielectric of capacitivity ϵ is

$$F_m = \tfrac{1}{2}\epsilon^{-1} \int_S \sigma^2 \mathbf{m} \cdot \mathbf{n}\, dS$$

where **n** is a unit vector normal to the surface.

When a uniform isotropic dielectric body of capacitivity ϵ occupies the volume v, where, before its advent, the field due to a fixed distribution of charge was **E** and after its advent **E**′, its energy is

$$W = \tfrac{1}{2} \int_v (\epsilon_v - \epsilon)\mathbf{E} \cdot \mathbf{E}'\, dv$$

The force or torque tending to increase the distance or angle x of the above body is

$$F_x = -\frac{\partial W}{\partial x}$$

The torque tending to increase the angle α which the normal to a disk of radius a makes with a field that would be uniform and of strength E except for the disk is

$$T = \frac{8}{3}\epsilon a^3 E^2 \sin 2\alpha$$

The torque tending to increase the angle α between the field and the major axis of an oblate dielectric spheroid of capacitivity ϵ with semiaxes a and b, where $b > a$, placed in a field that would be uniform and of strength E except for the spheroid is

$$T = \frac{2\pi\epsilon_v(K-1)^2 b^2 a E^2 (3P-2)\sin 2\alpha}{3[(K-1)^2P^2 + (K-1)(2-K)P - 2K]}$$

where $P = A[(1 + A^2)\cot^{-1} A - A]$, $A = a(b^2 - a^2)^{-\frac{1}{2}}$, and $K = \epsilon\epsilon_v^{-1}$.

If the above oblate spheroid is conducting, the torque is

$$T = \frac{2\pi\epsilon_v b^2 a E^2 (3P-2)\sin 2\alpha}{3P(P-1)}$$

The torque tending to increase the angle α between the field and the major axis of a prolate dielectric spheroid of capacitivity ϵ with semiaxes a and b where $b < a$ placed in a field that would be uniform and of strength E except for the spheroid is

$$T = \frac{2\pi\epsilon_v(K-1)^2 b^2 a E^2 (2-3Q)\sin 2\alpha}{3[(K-1)^2Q^2 + (K-1)(2-K)Q - 2K]}$$

where $Q = C[(1 - C^2)\coth^{-1} C + C]$. $C = a(a^2 - b^2)^{-\frac{1}{2}}$ and $K = \epsilon\epsilon_v^{-1}$.

If the above prolate spheroid is conducting, the torque becomes[1]

$$T = \frac{2\pi\epsilon_v b^2 a E^2 (2-3Q)\sin 2\alpha}{3Q(Q-1)}$$

The axis of rotational symmetry of a right circular solid conducting cylinder of radius a and length $2b$ makes an angle θ with a field that would be uniform and of strength E except for the cylinder. The torque tending to align the axis with the field is

$$T = \pi\epsilon a^2 b E^2 \sin 2\theta(\alpha_1 - \alpha_t)$$

[1] For torque on general ellipsoid, see Stratton, "Electromagnetic Theory," p. 215, McGraw-Hill Book Company, New York, 1941.

where
$$\alpha_1 = 1 + 2.1444 \left(\frac{b}{a}\right)^{0.828} + 0.7171 \left(\frac{b}{a}\right)^{1.6752}$$
$$\alpha_t = 2 + 0.84883 \left(\frac{a}{b}\right) + 0.369 \left(\frac{a}{b}\right)^{0.548} \tanh^{0.5} \left(\frac{a}{b}\right)^{0.712}$$

The torque vanishes at $(a/b) = 1.1958$. The errors in these formulas are less than 1 part in 4,000 for $0.25 \lesssim (a/b) \lesssim \infty$.

Two parallel cylinders of radii a and b carry charges $+Q$ and $-Q$, and their axes are a distance c apart. The force per unit length tending to increase c is

$$F_1 = \frac{\pm Q^2 c}{2\pi\epsilon[(c^2 - a^2 - b^2)^2 - 4a^2b^2]^{\frac{1}{2}}}$$

The plus sign is used if one cylinder encloses the other and the minus sign if they are mutually external.

A complete elliptic integral formula given earlier (page 5-16) for capacitance per unit length used a different modulus in each of the cases 1 to 12 involving a distance c. The force tending to decrease c when the system charges are $+Q$ and $-Q$ is

$$F = \frac{\pi A Q^2}{4k(1 - k^2)K[(1 - k^2)^{\frac{1}{2}}]} \left| \frac{\partial k}{\partial c} \right|$$

Two identical infinite coplanar parallel conducting strips carry equal positive charges Q, the distance between their near edges being $2a$ and between their far edges $2b$. The repulsive force per unit length between them is

$$F_1 = \frac{Q^2}{2\pi\epsilon(a + b)}$$

The force on a point charge at a distance b from the center of a sphere of radius a at zero potential is

$$F = \frac{abQ^2}{4\pi\epsilon(a^2 - b^2)^2}$$

When $b > a$, the force is toward the center; and when $b < a$, it is away from the center. The repulsive force between a point charge q at a distance b from the center of a sphere of radius a carrying a total charge Q is, when $b > a$,

$$F = \frac{q}{4\pi\epsilon b^2}\left[Q + \frac{a^3(a^2 - 2b^2)q}{b(b^2 - a^2)^2}\right]$$

At the point x_0, y_0, z_0 inside a rectangular conducting box bounded by the planes $x = 0, a$; $y = 0, b$, $z = 0, c$, the image force on a charge Q is

$$F_z = -\frac{2Q^2}{\epsilon ab} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\sinh A_{mn}(c - 2z_0)}{\sinh A_{mn}c} \sin^2 \frac{n\pi x_0}{a} \sin^2 \frac{m\pi y_0}{b}$$

in the z direction, where $A_{mn} = \pi(ab)^{-1}(m^2a^2 + n^2b^2)^{\frac{1}{2}}$. The other force components are given by cyclic permutation of the symbols x, y, z; a, b, c; and x_0, y_0, z_0. At a distance c from one of two parallel uncharged plates at a distance b apart, the image force on a charge Q is

$$F = \frac{Q^2}{16\pi\epsilon a^2}\left[\zeta\left(2, \frac{1}{2} - \frac{c}{b}\right) - \zeta\left(2, \frac{c}{b}\right)\right]$$

where $\zeta(z,a)$ is a Riemann zeta function.

On the axis and at a distance b from the center of a conducting disk of radius a carrying a charge Q, the repulsive force on a point charge q is

$$F = \frac{q}{4\pi\epsilon(a^2 + b^2)}\left[Q - \frac{a(3b^2 + a^2)q}{2\pi b(a^2 + b^2)} + \frac{3b^2 - a^2}{2\pi b^2} q \tan^{-1}\frac{a}{b}\right]$$

At a distance c from the center of an uncharged dielectric sphere of radius a and relative capacitivity K, the attractive force on a charge Q is

$$F = \frac{(K - 1)Q^2}{4\pi\epsilon_v c^2} \sum_{n=1}^{\infty} \frac{n(n + 1)}{Kn + n + 1}\left(\frac{a}{c}\right)^{2n+1}$$

At a distance c from the plane face of an infinite block of dielectric of relative capacitivity K, the attractive force on a point charge Q is

$$F = \frac{Q^2}{16\pi\epsilon_v c^2}\frac{K - 1}{K + 1}$$

The attractive force on a point charge Q at a distance a from the plane face of a dielectric slab of thickness c and relative capacitivity K is

$$F = \frac{\beta Q^2}{16\pi\epsilon_v}\left[\frac{1}{a^2} - (1 - \beta^2) \sum_{n=1}^{\infty} \frac{\beta^{2(n-1)}}{(a + nc)^2}\right]$$

where $\beta = (K - 1)(K + 1)^{-1}$.
The attractive force per unit length on a line charge of strength λ per unit length parallel to and at a distance c from the axis of an uncharged circular cylinder of radius a and relative capacitivity K is

$$F_1 = \frac{K - 1}{K + 1}\frac{\lambda^2 a^2}{2\pi\epsilon_v c(c^2 - a^2)}$$

For a conductor, $K = \infty$; so the first factor is unity.
The force toward the wall per unit length on a line charge of strength λ per unit length parallel to and at a distance c from the axis of a circular cylindrical hole of radius a in an infinite block of dielectric of relative capacitivity K is

$$F_1 = \frac{K - 1}{K + 1}\frac{c\lambda^2}{2\pi\epsilon_v(a^2 - c^2)}$$

For a conductor, $K = \infty$; so the first factor is unity.
The attractive force per unit length on a line charge of strength λ per unit length parallel to and at a distance a from the nearer face of a dielectric slab of thickness c and relative capacitivity K is

$$F_1 = \frac{\beta\lambda^2}{4\pi\epsilon_v}\left[\frac{1}{a} - (1 - \beta^2) \sum_{n=1}^{\infty} \frac{\beta^{2(n-1)}}{a + nc}\right]$$

where $\beta = (K - 1)(K + 1)^{-1}$.
In the foregoing case, if $a = mc$ where m is an integer, the force per unit length is expressible in finite terms; thus

$$F_1 = \frac{\beta\lambda^2}{4\pi\epsilon_v c}\left\{\frac{1}{m} - \frac{1 - \beta^2}{\beta^{2(m+1)}}\left[\ln(1 - \beta^2) + \sum_{n=1}^{m} \frac{\beta^{2n}}{n}\right]\right\}$$

The attractive force per unit length on a line charge of strength λ per unit length parallel to and at a distance a from an uncharged conducting plane is

$$F_1 = \frac{\lambda^2}{4\pi\epsilon a}$$

The attractive force between a line charge of strength λ per unit length and an uncharged conducting sphere of radius a whose center is at a distance b from it is

$$F = \frac{\lambda^2 a^2}{\pi\epsilon_v b(b^2 - a^2)^{\frac{1}{2}}} \sin^{-1}\frac{a}{b}$$

The attractive force between a line charge of strength λ per unit length and an uncharged dielectric sphere of relative capacitivity K and radius a is

$$F = \frac{(K-1)\lambda^2}{\pi\epsilon_v} \sum_{n=1}^{\infty} \frac{n(2n-2)!!}{(2n-1)!!(Kn+n+1)} \left(\frac{a}{b}\right)^{2n+1}$$

5b-3. Multipole Formulas. The potential of a point charge Q is

$$V = \frac{Q}{4\pi\epsilon r}$$

where r is the distance from the charge to the field point.
The force on a point charge in a field of electric intensity $\mathbf{E}$ is

$$\mathbf{F} = Q\mathbf{E}$$

The potential of a dipole of moment $\mathbf{p}$ is

$$V = \frac{p \cos\theta}{4\pi\epsilon r^2} = \frac{\mathbf{p}\cdot\mathbf{r}}{4\pi\epsilon r^3}$$

where $\mathbf{r}$ is measured from the dipole to the field point.
The force on a dipole in a field $\mathbf{E}$ is $\mathbf{F} = (\mathbf{p}\cdot\nabla)\mathbf{E}$.
The torque on a dipole in a field $\mathbf{E}$ is $\mathbf{T} = \mathbf{p}\times\mathbf{E}$.
The mutual energy of two dipoles of moment $\mathbf{p}_1$, $\mathbf{p}_2$ which make angles θ_1 and θ_2 with the vector $\mathbf{r}$ that joins them and whose planes intersect along $\mathbf{r}$ at an angle ψ is

$$W = \frac{p_1 p_2}{4\pi\epsilon r^3}(\sin\theta_1 \sin\theta_2 \cos\psi - 2\cos\theta_1\cos\theta_2)$$

The components of force and torque between two dipoles are

$$F_r = -\frac{\partial W}{\partial r} \qquad T_\alpha = -\frac{\partial W}{\partial \alpha}$$

The potential of a multipole of the nth order and moment strength $p^{(n)}$ is

$$V_n = \frac{(-1)^n p^{(n)}}{4\pi\epsilon n!} \frac{\partial^n}{\partial l_1 \cdots \partial l_n}\left(\frac{1}{r}\right)$$
$$= \sum_{m=0}^{n} (a_{nm}\cos m\varphi + b_{nm}\sin m\varphi) r^{-n-1} P_n{}^m(\cos\theta)$$

5b-4. Dielectric-boundary Formulas. If V' and V'' are the electrostatic potentials in the dielectrics ϵ' and ϵ'', then at their uncharged interface

$$V' = V'' \qquad \text{and} \qquad \epsilon' \frac{\partial V'}{\partial n} = \epsilon'' \frac{\partial V''}{\partial n}$$

where n is a coordinate normal to the interface.
The normal stress, directed from ϵ'' to ϵ', on the above interface is

$$F_n = \frac{\epsilon'' - \epsilon'}{2\epsilon'} \left(\frac{D_t'^2}{\epsilon'} + \frac{D_n'^2}{\epsilon''} \right)$$

where D_t' and D_n' are the tangential and normal components of the displacement in ϵ'.

5b-5. Dielectric Bodies in Electrostatic Fields. A sphere of radius a and capacitivity ϵ is placed in a uniform field of intensity $\mathbf{E}$. The uniform field intensity inside and the potential outside due to its polarization are, respectively,

$$\mathbf{E}_i = \frac{3\epsilon_v \mathbf{E}}{\epsilon + 2\epsilon_v} \qquad V_p = E \frac{\epsilon_v - \epsilon}{\epsilon + 2\epsilon_v} \frac{a^3}{r^2} \cos\theta$$

where r is measured from the center of the sphere and $\mathbf{E}$ is directed along $\theta = 0$.
An oblate dielectric spheroid of capacitivity ϵ whose minor (rotational) axis on $\theta = 0$ is $2a$ and whose focal circle is of radius c is placed in a uniform electric field $\mathbf{E}$ parallel to $\theta = 0$. The uniform field inside and the potential outside due to its polarization are, respectively,

$$\mathbf{E}_i = \mathbf{E}\,\epsilon_v c^3 M \qquad V_p = M(\epsilon - \epsilon_v)a(a^2 + c^2)E(\cot^{-1}\zeta - \zeta^{-1})r\cos\theta$$

where $M = \{a(\epsilon_v - \epsilon)[(a^2 + c^2)\cot^{-1}(c^{-1}a) - ac] + \epsilon c^3\}^{-1}$

and $\zeta^2 = \frac{1}{2}c^{-2}\{r^2 - c^2 + [(r^2 - c^2)^2 + 4r^2c^2\cos^2\theta]^{\frac{1}{2}}\}$

The above spheroid is placed in a field $\mathbf{E}'$ in the $\varphi = 0$ direction, normal to the $\theta = 0$ axis. The uniform field intensity inside and the potential outside due to its polarization are, respectively,

$$\mathbf{E}_i' = 2\mathbf{E}'\epsilon_v c^3 M' \qquad V_p' = M'(\epsilon - \epsilon_v)a(c^2 + a^2)E'[\cot^{-1}\zeta - \zeta(1 + \zeta^2)^{-1}]r\sin\theta\cos\varphi$$

where $M' = \{a(\epsilon - \epsilon_v)[(a^2 + c^2)\cot^{-1}(c^{-1}a) - ac] + 2\epsilon_v c^3\}^{-1}$

The above spheroid is placed in a uniform field $\mathbf{E}_0$ which makes an angle α with its rotational $\theta = 0$ axis. The uniform field inside and the potential outside due to its polarization are, respectively,

$$\mathbf{E}_{0i} = \mathbf{E}_0\epsilon_v c^3[M\cos\alpha + M'\sin\alpha] \qquad V_{0p} = E_0[V_pE^{-1}\cos\alpha + V_p'E'^{-1}\sin\alpha]$$

where V_p', V_p, $\mathbf{E}$, and $\mathbf{E}'$ are given in the preceding formulas.
A prolate spheroid of capacitivity ϵ whose major (rotational) axis on $\theta = 0$ is $2b$ and whose focal distance is $2c$ is placed in a uniform electric field $\mathbf{E}$ parallel to $\theta = 0$. The uniform field intensity inside and the potential outside due to its polarization are, respectively,

$$\mathbf{E}_i = \mathbf{E}\epsilon_v c^3 N \qquad V_p = N(\epsilon - \epsilon_v)b(c^2 - b^2)E(\coth^{-1}\eta - \eta^{-1})r\cos\theta$$

where $N = \{b(\epsilon_v - \epsilon)[(c^2 - b^2)\coth^{-1}(c^{-1}b) + bc] + \epsilon c^3\}^{-1}$

and $\eta^2 = \frac{1}{2}c^{-2}\{r^2 + c^2 + [(r^2 + c^2)^2 - 4c^2r^2\cos^2\theta]^{\frac{1}{2}}\}$

The above spheroid is placed in a field $\mathbf{E}'$ in the $\varphi = 0$ direction normal to the $\theta = 0$ axis. The uniform field inside and the potential outside due to its polarization are, respectively,

$$\mathbf{E}_i = \mathbf{E}'\epsilon_v c^3 N' \qquad V_p' = N'(\epsilon - \epsilon_v)b(b^2 - c^2)E'[\coth^{-1}\eta - \eta(1 - \eta^2)^{-1}]r\sin\theta\cos\varphi$$

where $N = \{b(\epsilon_v - \epsilon)[(b^2 - c^2)\coth^{-1}c^{-1}b - bc] + 2\epsilon_v c^3\}^{-1}$.

The above prolate spheroid is placed in a uniform field $\mathbf{E}_0$ which makes an angle α with its rotational $\theta = 0$ axis. The uniform field inside and the potential outside due to its polarization are, respectively,

$$\mathbf{E}_{0i} = \mathbf{E}_0\epsilon_v c^3[N \cos\alpha + N' \sin\alpha] \qquad V_{0p} = E_0[V_p E^{-1}\cos\alpha + V'_p E'^{-1}\sin\alpha]$$

where V_p, V'_p, $\mathbf{E}$, and $\mathbf{E}'$ are given in the foregoing formulas.

5b-6. Static-current-flow Formulas. *Linear-circuit Formulas.* See steady-state alternating-current formulas.

Currents in Extended Media (Three Dimensions). The following formulas assume the medium to be uniform, homogeneous, and isotropic and to have a resistivity ρ which obeys Ohm's law.

The resistance between a single perfectly conducting electrode immersed in an infinite medium and the concentric infinite sphere is related to the capacitance of the same electrode by the formula

$$R = \rho\epsilon_v C^{-1}$$

where the capacitance C for a sphere, prolate or oblate spheroid, ellipsoid, circular disk, elliptic disk, two spheres in contact, two spheres connected by a wire, two spheres intersecting at an angle π/m, a spherical bowl, torus, cube, and circular plane annulus is given in the electrostatic section. The resistance between widely separated source and sink electrodes immersed in an infinite medium is

$$R_{12} \approx R_1 + R_2 - \rho(2\pi r)^{-1}$$

where R_1 and R_2 are the resistances to infinity of each alone and r, the distance between them, is large compared with their dimensions. The resistance to infinity of a single electrode, sunk into the plane surface of a semi-infinite medium such as the earth in such a way that the submerged part, if combined with its mirror image in the surface, would form one of the above electrodes, is

$$R = 2\rho\epsilon_v C^{-1}$$

When both source and sink electrodes are half submerged in the plane face just described, the resistance between them is

$$R = R_{12} \approx 2[R_1 + R_2 - \rho(2\pi r)^{-1}]$$

where R_{12}, R_1, and R_2 have the same significance as before and r, the distance between them, is much larger than the electrode dimensions. In the preceding case, if the medium has a resistivity ρ_1 to a depth a and ρ_2 below this depth, then the resistance between electrodes is

$$R \approx 2\left\{R_1 + R_2 - \frac{\rho_1}{2\pi r} + \frac{\rho_1}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-\beta)^n}{2na} - \frac{(-\beta)^n}{(4n^2a^2 + r^2)^{\frac{1}{2}}}\right]\right\}$$

where $\beta = (\rho_1 - \rho_2)(\rho_1 + \rho_2)^{-1}$ and both a and r are large compared with the electrode dimensions.

Two perfectly conducting disk electrodes of radii a and b are applied to the plane horizontal face of a semi-infinite homogeneous medium whose horizontal and vertical resistivities are ρ_1 and ρ_2. If the electrode spacing r is much greater than a and b, the resistance between them is

$$R \approx (\rho_1\rho_2)^{\frac{1}{2}}[(4a)^{-1} + (4b)^{-1} - (\pi r)^{-1}]$$

Two conical perfectly conducting electrodes of half angle β with an angle α between their axes pass normally through a spherical shell of thickness b and resistivity ρ.

The resistance between them is rigorously

$$R = \rho(\pi b)^{-1} \cosh^{-1} (\csc \beta \sin \tfrac{1}{2}\alpha)$$

A cylindrical column of length l and radius a of material of resistivity ρ connects normally the plane faces of two semi-infinite masses of the same resistivity. The resistance R between the infinite hemispherical perfectly conducting electrodes bounding the masses lies within the limits

$$\frac{\rho l}{\pi a^2} + \frac{\rho}{2a} < R < \frac{\pi \rho}{2[\pi a - l \ln (1 + \pi a/l)]}$$

This formula is most accurate for small values of l/a and is exact at $l = 0$. For large values of l

$$R \approx \rho a^{-1}(0.31831 l a^{-1} + 0.522)$$

Perfectly conducting disk electrodes of radius b are applied concentrically to the ends of a solid right circular cylinder of radius a, length $2c$, and resistivity ρ. The resistance between them is

$$R \approx 2\rho(\pi a^2)^{-1}[c + f(b)]$$

where $f(0.25a) = 2.05164a$, $f(0.50a) = 0.5336a$, and $f(0.75a) = 0.1060a$. The errors are less than 0.05 per cent if c is greater than $4a$.

Currents in Extended Media (Two Dimensions). The resistance between perfectly conducting plane electrodes covering the ends and orthogonal to the sides of a bar of rectangular section, resistivity ρ, and thickness b bent in a circular arc with inner radius a and outer radius c, which subtends an angle α at the center, is

$$R = \rho\alpha b^{-1}[\ln (a^{-1}c)]^{-1}$$

The resistance between two small cylindrical electrodes of radius r passing normally through a strip of width a, thickness b, and resistivity ρ at a distance $2c$ apart on a line midway between its edges is, if $r \ll a$ and $r \ll c$,

$$R \approx \frac{\rho}{\pi b} \ln \frac{a \sinh 2\pi a^{-1}c}{\pi r}$$

The resistance between the electrodes in the above strip when they are equidistant from its center on a line normal to its edges is

$$R \approx \frac{\rho}{\pi b} \ln \frac{2a \tan \pi a^{-1}c}{\pi r}$$

In the following six configurations the bars of resistivity ρ have rectangular cross sections and are of uniform thickness b. Perfectly conducting electrodes cover the ends which are at right angles to the sides. For 1 percent accuracy the interval between each end and the beginning of the boundary perturbation should exceed about twice the width of the intervening straight bar.

A bar of width a has an infinitely narrow cut of depth c normal to one side. The additional resistance due to the cut is

$$\Delta R = -4\rho(\pi b)^{-1} \ln \cos \tfrac{1}{2}\pi a^{-1}c$$

One side of a bar is straight and the other has a rectangular step in it. The width on one side of the step is a and on the other c where $a > c$. The additional resistance due to the distortion of the flow near the step over the sum of the resistances of the two straight portions alone is

$$\Delta R = \frac{\rho}{\pi b}\left(\frac{a^2 + c^2}{ac} \ln \frac{a + c}{a - c} + 2 \ln \frac{a^2 - c^2}{4ac}\right)$$

In the preceding case the corner of the step is cut off at 45 deg so that the width increases linearly from c to a. The additional resistance due to the tapered section over that of the two straight portions alone is

$$\Delta R = \frac{2\rho}{\pi b}\left(\frac{a^2 + c^2}{ac}\tanh^{-1}\frac{c}{a} + \frac{a^2 - c^2}{ac}\tan^{-1}\frac{c}{a} + \ln\frac{a^4 - c^4}{8a^2c^2}\right)$$

A straight rectangular bar has a right-angle bend, the width on one side of the bend being a and on the other c. The increase of resistance over the sum of the resistances of the two straight portions alone, the corner rectangle common to both being excluded, is

$$\Delta R = \frac{2\rho}{\pi b}\ln\left(\frac{a^2 + c^2}{4ac} + \frac{a}{c}\tan^{-1}\frac{c}{a} + \frac{c}{a}\tan^{-1}\frac{a}{c}\right)$$

A straight rectangular bar of width a has a hole drilled through it equidistant from its edges. The increase in resistance due to the hole is less than

$$\Delta R \approx -2\rho c(ab\theta)^{-1}\ln\cos\theta$$

where θ is a parameter chosen so that $\sin\theta = \tanh[\pi c\theta(a\theta - \pi c)^{-1}]$.
These formulas are practically exact for small holes far from the ends. When the diameter of the hole is half the strip width R is about 0.1 per cent too large. For small values of c/a the parameter is given by

$$\theta \approx \frac{2\pi c}{a}\left(1 - \frac{\pi^2c^2}{3a^2} + \frac{\pi^4c^4}{3a^4}\right)$$

The value of ΔR given above is unchanged if the hole is replaced by two semicircular notches of the same radius in opposite edges of the strip.

Perfectly conducting electrodes are applied to a block of thickness b, width a, length c, and resistivity ρ in such a way as to cover the full thickness over a band of width w at the center of opposite ends. The resistance between the electrodes lies between the limits

$$\frac{2\rho}{\pi b}\cosh^{-1}\frac{\cosh\frac{1}{2}\pi a^{-1}c}{\sin\frac{1}{2}\pi a^{-1}w} > R > \frac{2}{\pi b}\sinh^{-1}\frac{\sinh\frac{1}{2}\pi a^{-1}c}{\sin\frac{1}{2}\pi a^{-1}w}$$

5b-7. Static-magnetic-field Formulas. *Magnetic Field of Various Circuit Configurations.* The magnetic induction due to a current density **i** flowing in a volume v is

$$\mathbf{B} = \frac{\mu}{4\pi}\nabla\times\int_v\frac{\mathbf{i}\,dv}{r}$$

The magnetic induction of a thin linear circuit with total current I is

$$B = \frac{\mu I}{4\pi}\oint\frac{\sin\theta\,ds}{r^2}$$

where θ is the angle between **ds** and **r** and **B** is normal to the plane of **ds** and **r**.
The magnetic induction due to a long straight cylinder carrying current parallel to its axis, when both current density and permeability are independent of the azimuth angle θ, is $B_\theta = \mu_a I_a(2\pi a)^{-1}$ where a is distance of field point from axis, I_a is current inside radius a, and μ is the permeability at the field point.
The edges of a flat strip lie at $x = a$ and $x = -a$ and it carries a uniformly distributed current I in the z direction. The distances of a field point in the positive quadrant from the near and far edges are, respectively, r_1 and r_2 and the angle between r_1 and r_2

is α. The magnetic induction components are

$$B_y = \frac{\mu I}{4\pi a} \ln \frac{r_2}{r_1} \qquad B_x = -\frac{\mu I}{4\pi a} \alpha$$

A conductor of rectangular section of area A is bounded by the planes $x = a$, $x = -a$, $y = b$, and $y = -b$ and carries a uniformly distributed current I in the z direction. The distances from a field point in the positive quadrant to the corners, starting with the nearest and proceeding clockwise about the z axis, are r_1, r_2, r_3, and r_4. The angles between successive r's are α_1, α_2, α_3, and α_4, and the x and y components of r_1 and r_3 are x_1, y_1 and x_3, y_3. If all the above quantities are taken positive, the magnetic-induction components are

$$B_x = -\frac{1}{2}\mu I(\pi A)^{-1}\left(y_3\alpha_4 - y_1\alpha_1 + x_3 \ln \frac{r_3}{r_2} - x_1 \ln \frac{r_4}{r_1}\right)$$

$$B_y = \frac{1}{2}\mu I(\pi A)^{-1}\left(x_3\alpha_2 - x_1\alpha_3 + y_3 \ln \frac{r_3}{r_4} - y_1 \ln \frac{r_2}{r_1}\right)$$

The space inside and outside the conductor has the same permeability μ.

The magnetic induction outside the conductors of a long bifilar line that consists of a cylinder whose axis is $y = a$ which carries a uniformly distributed x-directed current I and another cylinder whose axis is $y = -a$ that carries the same current in the opposite direction is

$$B_y = \tfrac{1}{2}\pi^{-1}\mu I z(r_2^{-2} - r_1^{-2}) \qquad B_z = -\tfrac{1}{2}\pi^{-1}\mu I[r_2^{-2}(y + a) - r_1^{-2}(y - a)]$$

where r_1 and r_2 are the distances from positive and negative wire axes, respectively, and μ is the permeability of the conductors and surrounding space.

The magnetic induction of bifilar lines composed of flat strips or rectangular bars can be found by taking the vector sum of the inductions already given for each conductor alone.

A long circular conducting cylinder of radius b has a longitudinal hole of radius a whose axis is displaced a distance c from the cylinder axis. If a longitudinal current I is uniformly distributed over the conducting area, the induction **B** in the hole is uniform and normal to c and its magnitude is

$$B = \mu c I[2\pi(b^2 - a^2)]^{-1}$$

A circular loop of wire lies at $z = 0$, $\rho = a$ and carries a current clockwise about the z axis. The magnetic-induction components are

$$B_z = A(I_1 - a^{-1}\rho I_2) \qquad B_\rho = A a^{-1} z I_2$$

where[1] $I_1 = \pi^{-1}\int_0^\pi (1 - b\cos\theta)^{-\frac{3}{2}}\,d\theta$, $I_2 = \pi^{-1}\int_0^\pi (1 - b\cos\theta)^{\frac{3}{2}}\cos\theta\,d\theta$, $A = \frac{1}{2}\mu I a^2(a^2 + z^2 + \rho^2)^{-\frac{3}{2}}$, and $b = 2a\rho(a^2 + z^2 + \rho^2)^{-1}$.

Two coaxial wire loops of radius a at a distance a apart carry currents I in the same direction and constitute a Helmholtz coil which gives a nearly uniform field on the axis midway between them. For a small distance r around this point the field varies as $(r/a)^4$. The induction there is

$$B = 8\mu I 5^{-\frac{3}{2}} a^{-1}$$

Accurate values of B may be found by a superposition of the fields calculated separately by the preceding formula for a single loop.

[1] Six-place tables of I_1 and I_2 suitable for linear interpolation are given by C. L. Bartberger, *J. Appl. Phys.* **21**, 1108 (1950).

The magnetic-induction components at a great distance from a small loop of wire at $\theta = \frac{1}{2}\pi$, $r = a$ which carries a current I are

$$B_r = \tfrac{1}{2}\mu I r^{-3} a^2 \cos\theta \qquad B_\theta = \tfrac{1}{4}\mu I r^{-3} a^2 \sin\theta$$

A rectangular loop of wire lies at $x = \pm a$, $y = \pm b$ and carries a current I clockwise about the z axis. The distances of the field point at x, y, z in the positive octant from successive corners, starting with the nearest, are r_1, r_2, r_3, and r_4 and the components of r_1 and r_3 are x_1, y_1, z and x_3, y_3, z. The components of the magnetic induction are

$$B_x = \tfrac{1}{4}\pi^{-1}\mu I z\{[r_1(r_1 - y_1)]^{-1} + [r_3(r_3 + y_3)]^{-1} - [r_4(r_4 + y_3)]^{-1} - [r_2(r_2 - y_1)]^{-1}\}$$
$$B_y = \tfrac{1}{4}\pi^{-1}\mu I z\{[r_3(r_3 + x_3)]^{-1} + [r_1(r_1 - x_1)]^{-1} - [r_4(r_4 - x_1)]^{-1} - [r_2(r_2 + x_3)]^{-1}\}$$
$$B_z = \tfrac{1}{4}\pi^{-1}\mu I\{x_1[r_1(r_1 - y_1)]^{-1} - x_1[r_4(r_4 + y_3)]^{-1} + x_3[r_2(r_2 - y_1)]^{-1} - x_3[r_3(r_3 + y_3)]^{-1}$$
$$+ y_1[r_1(r_1 - x_1)]^{-1} - y_1[r_2(r_2 + x_3)]^{-1} + y_3[r_4(r_4 - x_1)]^{-1} - y_3[r_3(r_3 + x_3)]^{-1}\}$$

All lengths are to be taken positive. If the single wire of the preceding formulas is replaced by N wires, the fields may be found rigorously by superimposing N solutions of the type given, one for each wire, or by integration over the section. In case the area of this section is small compared with other coil dimensions, a sufficiently accurate result is often given by substitution of NI for I in these formulas and the use of the dimensions of the center turn for that of the loop.

A helix of pitch α is wound on a cylinder of radius a. The angles between the positive axis and vectors drawn from the field point to the ends of the helix wire are β_1 and β_2. The axial component of the induction is then given rigorously by

$$B_a = \tfrac{1}{4}\mu I \cot\alpha(\pi a)^{-1}(\cos\beta_2 - \cos\beta_1)$$

There is also a component normal to the axis which becomes negligible when α is small. The axial component of the induction on the axis of a solenoid with n turns per unit length is, using the notation of the preceding formula,

$$B_a = \tfrac{1}{2}\mu n I(\cos\beta_2 - \cos\beta_1)$$

The induction approaches uniformity everywhere inside an infinitely long solenoid as the pitch decreases and its limiting value is $B_2 = n\mu I$.

When any figure, such as a torus, generated by the rotation of a closed curve about a coplanar external line, is closely and uniformly wound with N turns of wire so that each turn nearly coincides with one position of the generating curve, then, when carrying a current I, the exterior induction is zero and the interior induction is

$$B_\varphi = \tfrac{1}{2}\mu N I(\pi r)^{-1}$$

A coil of N circular turns wound closely over the entire surface of an oblate spheroid whose major and minor semiaxes are a and b will give a uniform induction B inside, provided that the projections of these turns on the b axis are uniformly spaced. The total number of ampere-turns needed is

$$NI = -\frac{2B}{\mu}\left[\frac{a^2 - b^2}{b - a^2(a^2 - b^2)^{-\frac{1}{2}}\cos^{-1}(b/a)}\right]$$

When $b = a$, this becomes $NI = 3bB/\mu$.

A coil of N circular turns wound closely over the entire surface of a prolate spheroid whose major and minor semiaxes are b and a will give a uniform induction B inside, provided that the projections of these turns on the b axis are uniformly spaced. The total number of ampere-turns needed is

$$NI = \frac{2B}{\mu}\left[\frac{b^2 - a^2}{b - a^2(b^2 - a^2)^{-\frac{1}{2}}\cosh^{-1}(b/a)}\right]$$

Self- and Mutual Inductance for Static Fields. The mutual inductance between two circuits is given by the formulas

$$M = L_{12} = 10^{-7} \oint_1 \oint_2 r^{-1}\, \mathbf{ds}_1 \cdot \mathbf{ds}_2 = \tfrac{1}{4}\pi^{-1}10^7 \int_v \mathbf{B}_1 \cdot \mathbf{B}_2\, dv$$

where $\mathbf{ds}_1$ and $\mathbf{ds}_2$ are elements of circuit 1 and circuit 2 and $\mathbf{B}_1$ and $\mathbf{B}_2$ are their separate magnetic inductions for unit current. One line integral covers each circuit and the volume integral covers the whole field region.
The self-inductance of a circuit is a special case of the above formula

$$L = \tfrac{1}{4}\pi^{-1}10^7 \int_v B^2\, dv$$

where B is the magnetic induction per unit current and v includes the entire field region.
The energy in the field of n circuits carrying currents $I_1, I_2, \ldots, I_n$ is

$$W = \tfrac{1}{2} \sum_{p=0}^{n} \sum_{q=0}^{n} L_{pq} I_p I_q$$

Note. In the following material there are many references to Grover. These refer to F. W. Grover, "Inductance Calculations," Dover Publications, Inc., New York, 1962. In this book most inductances are given in microhenrys and lengths in centimeters. In the following formulas mks units are used; so the inductances are in henrys and the lengths in meters. Unless otherwise stated, the permeability throughout is that of a vacuum.

The self-inductance of a round wire of relative permeability K_m and length l in a vacuum is

$$L \approx 2l[\ln (2a^{-1}l) - 1 + \tfrac{1}{4}K_m] \times 10^{-7}$$

The self-inductance of a rectangular bar of perimeter p is

$$L \approx 2l[\ln (4p^{-1}l) + \tfrac{1}{2} + 0.1118l^{-1}p] \times 10^{-7}$$

The self-inductance of a bar of elliptical section, semiaxes a and b, is

$$L \approx 2l\{\ln [2l(a + b)^{-1}] - 0.05685\} \times 10^{-7}$$

The self-inductance of a tube of external and internal radii a and b is

$$L \approx 2l\left[\ln \frac{2l}{a} + \frac{b^4}{(a^2 - b^2)^2} \ln \frac{a}{b} + \frac{7b^2 - 5a^2}{4(a^2 - b^2)}\right] \times 10^{-7}$$

Note. In the following formulas for bifilar lines the inductance per unit length is found by setting $l = 1$. In all cases l is supposed to be much greater than the pair spacing. The current densities are taken uniform. The current goes out on one element and returns on the other.

The self-inductance of two parallel cylinders of radii a and b and length l with a distance d between axes is

$$L = l\{1 + 2 \ln [(ab)^{-1}d^2]\} \times 10^{-7}$$

The self-inductance of two similar parallel wires of radius a and relative permeability K_m with a distance d between axes is

$$L \approx l[4 \ln (a^{-1}d) + K_m - 4d] \times 10^{-7}$$

The self-inductance of two similar parallel rectangular wires of perimeter p with a distance d between centers is

$$L \approx [4l \ln (2p^{-1}d) + 6l + 0.447p - 4d] \times 10^{-7}$$

The self-inductance of two similar parallel tubes, external radius a, internal radius b, with a distance d between centers is

$$L \approx l\left[4 \ln \frac{d}{a} + \frac{4b^4}{(a^2 - b^2)^2} \ln \frac{a}{b} + \frac{3b^2 - a^2}{a^2 - b^2}\right] \times 10^{-7}$$

The self-inductance of a coaxial line when the external radii of the inside conductor, insulation space, and outside conductor are c, b, and a, respectively, and the relative permeabilities K_m, K'_m, and K''_m and when the length l is great compared with a is

$$L = 2l\left\{\frac{1}{4} K_m + K'_m \ln \frac{b}{c} + \frac{1}{4} K''_m \left[\frac{4a^4}{(a^2 - b^2)^2} \ln \frac{a}{b} - \frac{3a^2 - b^2}{a^2 - b^2}\right]\right\} \times 10^{-7}$$

If $K_m = K'_m = K''_m$, this formula also holds for a noncoaxial line provided the axes are parallel.

The self-inductance of a wire of radius r and relative permeability K_m which is bent into a circular loop of mean radius a, neglecting small terms in r^4/a^4, is

$$L \approx 4\pi a\left[\left(1 + \frac{r^2}{8a^2}\right) \ln \frac{8a}{r} + \frac{r^2}{24a^2} - 2 + \frac{1}{4} K_m\right] \times 10^{-7}$$

The self-inductance of a wire of radius r and relative permeability K_m which is bent into a rectangular loop with sides a and b and diagonal $d = (a^2 + b^2)^{\frac{1}{2}}$ is[1]

$$L \approx 4\left[a \ln \frac{2ab}{r(a + d)} + b \ln \frac{2ab}{r(b + d)} + 2d - \left(2 - \frac{1}{4} K_m\right)(a + b)\right] \times 10^{-7}$$

The self-inductance of a wire with rectangular section of perimeter p which is bent into a rectangular loop with sides a and b and diagonal d is

$$L \approx 4\left[a \ln \frac{4ab}{p(a + d)} + b \ln \frac{4ab}{p(b + d)} + 2d + \frac{1}{2}(a + b) + 0.223p\right] \times 10^{-7}$$

The self-inductance of a thin band of radius a and width b is

$$L \approx 4\pi a[\ln (8b^{-1}a) - \tfrac{1}{2}] \times 10^{-7}$$

The mutual inductance of two thin coaxial circular loops of radii a and b, when r_1 and r_2 are the farthest and nearest distances between the loops, is given in terms of complete elliptic integrals by[2]

$$\begin{aligned} M &= 8\pi k^{-1} a^{\frac{1}{2}} b^{\frac{1}{2}}[(1 - \tfrac{1}{2}k^2)K(k) - E(k)] \times 10^{-7} \\ &= 8\pi k_1^{-1} a^{\frac{1}{2}} b^{\frac{1}{2}}[K(k_1) - E(k_1)] \times 10^{-7} \end{aligned}$$

where $k^2 = r_1^{-2}(r_1^2 - r_2^2)$ and $k_1^2 = (r_1 - r_2)(r_1 + r_2)^{-1}$.

The mutual inductance between a long straight wire and a loop of radius a whose diameter it intersects at right angles at a distance c from the loop center is

$$M = 4\pi[c \sec \alpha - (c^2 \sec^2 \alpha - a^2)^{\frac{1}{2}}] \times 10^{-7} \qquad c > a$$
$$M = 4\pi c \tan (\tfrac{1}{4}\pi - \tfrac{1}{2}\alpha) \times 10^{-7} \qquad c < a$$

[1] Tables are given by Grover, pp. 59–65.
[2] Grover gives tables on pp. 77–87.

where α is the acute angle between the plane of the loop and the plane defined by its center and the straight wire.

The mutual inductance of two parallel coaxial identical rectangular loops whose sides are a and b and which are spaced so that the distance from any corner of one loop to the most distant corner of the other is d is[1]

$$M = 4\left[a \ln \frac{(a + A)B}{(a + D)d} + b \ln \frac{(b + B)A}{(b + D)d} + 8(D - A - B + d)\right] \times 10^{-7}$$

where $A^2 = a^2 + d^2$, $B^2 = b^2 + d^2$ and $D^2 = a^2 + b^2 + d^2$.

The mutual inductance between two circular loops of wire whose axes intersect at an angle γ at a point where the radius a of one loop subtends an angle α and the radius b of the other an angle β is

$$M = 4\pi^2 a \sum_{n=1}^{\infty} \frac{a^n \sin\alpha \sin\beta}{n(n+1)b^n} P_n^1(\cos\alpha) P_n^1(\cos\beta) P_n(\cos\gamma) \times 10^{-7}$$

where the last terms include two associated Legendre functions and one polynomial.[2] The mutual inductance of two circular loops with parallel axes can be calculated from tables in Grover, pages 177 to 192.

Note. The self- or mutual inductance of thin coils whose cross section is small compared with other dimensions is given approximately by insertion of the factor N^2 or N_1N_2, respectively, where N is the total number of turns, in the corresponding loop formula and the use of the mean coil dimensions for the corresponding loop dimensions.

A circular ring encircles or is encircled by a coaxial helix, the larger radius being A and the smaller a. The distances from the plane of the ring to the farther and nearer ends of the helix are b_1 and b_2 and n is the number of turns per meter on the helix. The mutual inductance is

$$M = 2\pi n(A + a)\{c[k_1^{-1}(K_1 - E_1) \pm k_2^{-1}(K_2 - E_2)] + (A - a)(b_1^{-1}\psi_1 \pm b_2^{-1}\psi_2)\} \times 10^{-7}$$

where the subscript 1 or 2 indicates the use of b_1 or b_2 for b in the following formulas:

$$k^2 = 4Aa[(A + a)^2 + b^2]^{-1} \qquad k' = (1 - k^2) \qquad c^2 = 4Aa(A + a)^{-2} \qquad k' \sin\beta = (1 - c^2)^{\frac{1}{2}}$$

$$\psi = K(k)E(k',\beta) - [K(k) - E(k)]F(k',\beta) - \tfrac{1}{2}\pi$$

The upper sign in the $\pm$ is taken when the plane of the ring cuts the helix; otherwise the lower sign is used. Complete elliptic integrals of modulus k are indicated by K or $K(k)$ and E or $E(k)$ and $E(k',\beta)$ and $F(k',\beta)$ are incomplete elliptic integrals of modulus k' and amplitude β.[3]

Note. The following current-sheet formulas assume that the current density on the shell is uniform and flows around the cylinder normal to the axis in an infinitely thin sheet. A correction may be added to take account of the fact that the current is actually concentrated in wires of definite radius and spacing as in Grover, pages 148 to 150, but is often not needed for close windings. By a process equivalent to integration of the preceding formula, an exact formula for the mutual inductance between a cylindrical current sheet or helix and a coaxial concentric current sheet can be derived.[4]

[1] For tables, see Grover, pp. 66–69.
[2] For tables, see Grover, pp. 193–208.
[3] For tables, see Grover, pp. 114–118.
[4] Louis Cohen, *Bull. Natl. Bur. Standards* **3**, 298 (1907). For practical purposes, tables given in Grover, pp. 122–141, are better.

The self-inductance of a current sheet of radius a, length b, and diagonal $d = (4a^2 + b^2)^{\frac{1}{2}}$ having a total number of turns N is[1]

$$L = \tfrac{4}{3}\pi b^{-2}N^2[d(4a^2 - b^2)E(k) - b^2 dK(k) - 8a^3] \times 10^{-7}$$

where $k = 2d^{-1}a$.

A current sheet is wound on the surface of the toroid formed by the rotation in the φ direction of a plane area S about an external line. If there are N turns and if the current density is independent of φ and has no φ component, then the self-inductance is

$$L = 2K_m N^2 \int_s r^{-1}\, dS \times 10^{-7}$$

where K_m is the relative permeability inside the current sheet and r is the distance of the area element dS from the rotational axis. The self-inductance in the above case, if S is a circle of radius a whose center is at a distance b from the rotational axis, is

$$L = 4\pi K_m N^2[b - (b^2 - a^2)^{\frac{1}{2}}] \times 10^{-7}$$

The self-inductance, if S is a rectangular section with sides parallel to the axis of length a and sides normal to it of length b and with the inside surface a distance R from the axis, is

$$L = 2N^2 a K_m \ln (1 + R^{-1}b) \times 10^{-7}$$

The self-inductance of a circular coil of N turns and circular section is

$$L \approx 4\pi N^2 a[(1 + \tfrac{1}{8}r^2a^{-2}) \ln (8r^{-1}a) + r^2(24a^2)^{-1} - 1.75] \times 10^{-7}$$

where r is the radius of the section, a the radius of the axis of the section, and $(r/a)^n$ is neglected when $n > 2$. The self-inductance of the above coil if it has a square section of side c is, if $c \ll a$,

$$L \approx 4\pi a N^2\{\tfrac{1}{2}[1 + c^2(24a^2)^{-1}] \ln (32c^{-2}a^2) - 0.84834 + 0.051a^{-2}c^2\} \times 10^{-7}$$

The self-inductance of coils of rectangular section can be calculated from tables given in Grover, pages 94 to 113.

The mutual inductance of coils of rectangular section and parallel axes can be calculated from tables given in Grover, pages 225 to 235. The mutual inductance of coils of rectangular section with inclined axes can be found from tables given by Grover on pages 209 to 214.

The increase in self-inductance of a circuit due to the placement of a sphere of radius a and relative permeability K_m in a position near it where the induction B per unit current is nearly uniform is

$$\Delta L \approx a^3B^2(K_m - 1)(K_m + 2)^{-1} \times 10^7$$

The increase of self-inductance of a loop of radius a due to the insertion concentrically of a sphere of radius b and infinite permeability is

$$\Delta L = 8\pi a^{-2}b^3K(a^{-2}b^2) \times 10^{-7}$$

The mutual inductance between two coaxial loops of radii a and b when the distance between centers is c and there is an infinite slab of thickness t and relative permeability K_m between and parallel to them is

$$M = 8\pi(ab)^{\frac{1}{2}}(1 - \beta^2) \sum_{n=0}^{\infty} k_n^{-1}\beta^{2n}[(1 - k_n^2)K(k_n) - E(k_n)]$$

$$k_n^2 = 4ab[(a + b)^2 + (c + 2nt)^2]^{-1} \qquad \beta = (K_m - 1)(K_m + 2)^{-1}$$

[1] For most purposes the tables given in Grover, pp. 142–162, are more practical than the formula.

Magnetic Forces on Circuits. The component of force in newtons tending to displace one of a pair of circuits in the x direction, the other being fixed, is

$$F_x = I_1 I_2 \frac{\partial M}{\partial x}$$

where I_1 and I_2 are the currents and M is the mutual inductance. The torque in newton meters tending to rotate one of a pair of circuits through an angle α, the other being fixed, is

$$T_\alpha = I_1 I_2 \frac{\partial M}{\partial \alpha}$$

Thus any desired forces or torques may be computed from the mutual-inductance formulas of the last few pages by differentiation, provided that it is possible to express M explicitly in terms of x or α. When this is not possible the difference in the mutual-inductance values calculated for the position x or α and the position $x + dx$ or $\alpha + d\alpha$ using the Grover tables may be multiplied by $I_1 I_2$ and divided by dx or $d\alpha$. In many cases the tabular intervals are small enough so this will give adequate accuracy; in other cases careful interpolation will be needed. Notice that in Grover's tables distances are in centimeters.

The force per unit length between two long parallel circular cylinders or tubes carrying uniformly distributed currents I_1 and I_2 is

$$F_1 = 2 I_1 I_2 a^{-1} \times 10^{-7}$$

The force is attractive when I_1 and I_2 have the same direction; otherwise it is repulsive.

The force per unit length between two parallel strips[1] of width a symmetrically placed with their faces a uniform distance b apart and carrying currents I_1 and I_2 is

$$4 I_1 I_2 a^{-1} [\tan^{-1} (b^{-1} a) - \tfrac{1}{2} a^{-1} b \ln (1 + b^{-2} a^2)] \times 10^{-7}$$

The force is attractive when I_1 and I_2 have the same direction; otherwise it is repulsive.

The force between two coaxial loops of radii a and b with centers at a distance c apart that carry currents I_1 and I_2 is

$$F = I_1 I_2 \pi c k [a^{\frac{1}{2}} b^{\frac{1}{2}} (1 - k^2)]^{-1} [(2 - k^2) E(k) - 2(1 - k^2) K(k)] \times 10^{-7}$$

where $k^2 = 4ab[(a + b)^2 + c^2]^{-1}$. The force is attractive when I_1 and I_2 encircle the axis in the same direction.

The axial force between a circular loop of radius a and a coaxial helix of radius b (a may be greater or less than b) and n turns per meter is

$$F = I_1 I_2 n (M - M') \times 10^{-7}$$

The loop center may lie inside or outside the helix. Here M and M' are the mutual inductances between a loop of radius a and coaxial loops of radius b whose planes pass through the extreme near end and extreme far end of the helix, respectively. The force is toward the center of the helix if the currents circle the axis in the same direction.

The force between a helix and a coaxial circular coil of mean radius a, square section of side c, and N turns is given approximately by the foregoing formula if $N I_1$ is used for I_1 and $a[1 + c^2 (24 a^2)^{-1}]$ for a. The force between two coaxial single-layer coils may be calculated by a formula in Grover on page 258 and a table on page 115.

The torque on a circular coil of rectangular section with internal and external radii a and b and any length which carries a current I, has N turns, and whose axis makes

[1] The force between two parallel rectangular bus bars is given by B. Hague, "Electromagnetic Problems in Electrical Engineering," p. 338, Oxford University Press, New York, 1929.

an angle α with a uniform field of induction **B** is

$$T = \tfrac{1}{3}\pi BNI(a^2 + ab + b^2)\sin\alpha$$

The torque on the above coil if it has a circular section of radius b whose center is at a distance a from the axis is

$$T = \tfrac{1}{4}\pi BNI(4a^2 - b^2)\sin\alpha$$

The torque on one of two concentric circular loops of wire of radii a and b which carry currents I_1 and I_2 is

$$T = 4\pi^2 aI_1I_2 \times 10^{-7} \sum_{n=0}^{\infty} \frac{2n+2}{2n+1}\left[\frac{(2n+1)!!}{(2n+2)!!}\right]^2 \left(\frac{a}{b}\right)^{2n+1} P_{2n+1}{}^1(\cos\alpha)$$

where α is the angle between their axes and $P_{2n+1}{}^1(\cos\alpha)$ is a Legendre function. It is directed so as to set one current parallel to the other.

The force on any circuit near the plane face of a semi-infinite block of material having a uniform relative permeability K_m which is independent of field strength equals the force between the circuit carrying a current I and its mirror-image circuit in the plane face carrying a current $I' = (K_m - 1)(K_m + 1)^{-1}I$. The direction of I', if K_m is greater than one, is such that the projections of I and I' on the interface coincide in position and direction. It is evident that if $K_m \gg 1$ then $I \approx I'$ and the exact value of K_m need not be known.

The force per unit length on an infinite wire carrying a current I parallel to the walls of an infinite evacuated rectangular conduit of infinite permeability is

$$F_x = 4\pi b^{-1}I^2 \times 10^{-7} \sum_{m=1}^{\infty} \operatorname{csch}(m\pi ab^{-1}) \sinh[m\pi b^{-1}(2c - a)] \cos^2(m\pi db^{-1})$$

where the walls of the conduit are at $x = 0$, $x = a$ and $y = 0$, $y = b$. The wire lies at $x = c$, $y = d$. To get F_y, interchange a with b and c with d. The series converges very rapidly unless the wire is near the wall. The force per unit length toward the nearest wall on an infinite wire parallel to and at a distance c from the axis of an evacuated cylindrical hole of radius a in a block of material of relative permeability K_m is

$$F_1 = 2(a^2 - c^2)^{-1}cI^2(K_m - 1)(K_m + 1)^{-1} \times 10^{-7}$$

Permeable Bodies in Magnetic Fields. The energy of an unmagnetized body of volume v when placed in a field of induction **B** produced by fixed sources in a region of constant permeability μ is

$$W = \tfrac{1}{2}\int_v (\mu^{-1} - \mu_i^{-1})\mathbf{B}\cdot\mathbf{B}_i\,dv$$

where $\mathbf{B}_i$ and μ_i are the final values of the magnetic induction and permeability in the volume element dv inside the body and the integration is over the volume of the body. The torque tending to decrease the angle α between **B** and the major axis of an oblate permeable spheroid of relative permeability K_m with semiaxes a and b, where $b > a$, placed in a uniform field of induction **B** produced by fixed sources in a vacuum is

$$T = \frac{(K_m - 1)^2 b^2 a B^2 (3P - 2)\sin 2\alpha}{6[(K_m - 1)^2P^2 + (K_m - 1)(2 - K_m)P - 2K_m]} \times 10^7$$

where $P = A[(1 + A^2)\cot^{-1}A - A]$ and $A = a(b^2 - a^2)^{-\frac{1}{2}}$.

The torque tending to decrease the angle α between **B** and the major axis of a prolate permeable spheroid of relative permeability K_m with semiaxis a and b where $b < a$ placed in a uniform field of induction **B** produced by fixed sources in a vacuum is

$$T = \frac{(K_m - 1)^2 b^2 a B^2 (2 - 3Q) \sin 2\alpha}{6[(K_m - 1)^2 Q^2 + (K_m - 1)(2 - K_m)Q - 2K_m]} \times 10^7$$

where $Q = c[(1 - c^2)\coth^{-1} c + c]$ and $c = a(a^2 - b^2)^{-\frac{1}{2}}$.

The attractive force between a long cylinder carrying a uniformly distributed current I and an external sphere of relative permeability K_m and radius a whose center is at a distance b from the cylinder axis is

$$F = 4I^2 \times 10^{-7} \sum_{n=1}^{\infty} \frac{(2n-2)!!\,n(K_m - 1)}{(2n-1)!!(nK_m + n + 1)} \left(\frac{a}{b}\right)^{2n+1}$$

If the permeability is very large in the above case, the force is

$$F = 4I^2 a^2 b^{-1} (b^2 - a^2)^{-\frac{1}{2}} \sin^{-1}(b^{-1}a) \times 10^7$$

Magnetic Shielding. Two long wires of a bifilar lead at $\rho = c$, $\varphi = 0$ and $\rho = c$, $\varphi = \pi$ carry currents I and $-I$ and are shielded by a cylinder of relative permeability K_m of internal and external radius a and b. The components of the induction outside the shield are

$$B_\rho = -16I \times 10^{-7} \sum_{n=0}^{\infty} \frac{b^{4n+2} c^{2n+1} \rho^{-2n-2} \sin(2n+1)\theta}{(K_m + 1)^2 b^{4n+2} - (K_m - 1)^2 a^{4n+2}}$$

$$B_\varphi = 16I \times 10^{-7} \sum_{n=0}^{\infty} \frac{b^{4n+2} c^{2n+1} \rho^{-2n-2} \cos(2n+1)\theta}{(K_m + 1)^2 b^{4n+2} - (K_m - 1)^2 a^{4n+2}}$$

A long cylindrical shield of internal and external radius a and b and relative permeability K_m is placed across a uniform field of induction **B**. The induction $\mathbf{B}_i$ inside is uniform and of magnitude

$$B_i = \frac{4K_m b^2 B}{4K_m b^2 + (K_m - 1)^2 (b^2 - a^2)}$$

A spherical shield of internal and external radius a and b and relative permeability K_m is placed in a uniform field of induction **B**. The induction $\mathbf{B}_i$ inside is uniform and its magnitude is

$$B_i = \frac{9K_m b^3 B}{9K_m b^3 + 2(K_m - 1)^2 (b^3 - a^3)}$$

The Magnetic Circuit. The reluctance $\mathcal{R}$ of a magnetic circuit is well defined only when all the magnetic flux Φ links all N turns of the magnetizing coils which when carrying a current I generate the magnetomotance $\mathfrak{F}$. Then

$$\mathfrak{F} = \mathcal{R}\Phi = NI$$

The reluctance of a toroid of such high and uniform relative permeability K_m that there is no flux leakage can be calculated regardless of the position of the magnetizing coil from the current-sheet self-inductance formulas for N turns already given for toroids of various sections. Thus

$$\mathcal{R} = N^2 L^{-1}$$

The change in reluctance of a closed magnetic plane circuit of thickness b, rectangular section and uniform relative permeability K_m so high that leakage is negligible due to the presence of corners, steps, tapered sections, and circular holes can be calculated from the formulas already given for resistance change ΔR for two-dimensional current flow in media of resistivity ρ. Thus

$$\Delta\mathcal{R} = 4\pi \times 10^7 K_m \rho^{-1}\,\Delta R$$

If a gap of uniform width a is cut out of a magnetic circuit of high relative permeability K_m, normal to the induction $\mathbf{B}$, and if a is small compared with all dimensions of the section of area A cut, then the increase in reluctance is

$$\Delta\mathcal{R} \approx 4\pi a A^{-1}(K_m - 1) \times 10^{-7}$$

where the surrounding space is empty and the fringing field at the edge of the gap is neglected.

The fringing field may be calculated when the region of negative x is filled with an infinitely permeable medium except for a gap bounded by $y = \frac{1}{2}a$ and $y = -\frac{1}{2}a$ which extends to $x = -\infty$. A magnetomotance is applied across the gap so that far from the edge the induction is B_0. The induction B_y anywhere on the x axis is then given implicitly by

$$x = \pi^{-1}a[B_0 B_y^{-1} - \tanh^{-1}(B_y B_0^{-1})]$$

where $0 \lesseqgtr B_y \lesseqgtr B_0$.

If the magnetomotance across a gap with faces at $z = \frac{1}{2}b$ and $z = -\frac{1}{2}b$ in an infinitely permeable cylinder bounded by $\rho = a$ is $\mathfrak{F}_0$, then the magnetomotance in the gap when $\rho < a$ is

$$\mathfrak{F} \approx \mathfrak{F}_0\left[\frac{z}{b} + \sum_{n=1}^{\infty} C_n \frac{I_0(\frac{1}{2}n\pi\rho/b)}{I_0(\frac{1}{2}n\pi a/b)} \sin\frac{n\pi z}{2b}\right]$$

where $C_1 = -0.17232$ and when $n > 1$.

$$C_n = \frac{(-1)^n}{n}\left[0.5836\,\frac{0.1775 \cdot 1.1775 \cdots (n - 0.8225)}{0.8225 \cdot 1.8225 \cdots (n - 0.1775)} - 0.0201n^{-2}\right]$$

The induction is $\mathbf{B} = -4\pi \times 10^{-7}\nabla\mathfrak{F}$. This formula assumes that the field across the edge of the gap is two-dimensional. If this is the only gap in an infinitely permeable circuit, then $\mathfrak{F}_0 = NI$ where N is the number of turns of the magnet coil and I is its current.[1]

Permanent Magnets. In the following formulas it is assumed that the magnetization M of a permanent magnet is absolutely rigid and that any magnetization induced in it by external fields is negligible compared with M. The energy of such a magnet when placed in an external field of induction $\mathbf{B}$ in a vacuum is $W = -\int \mathbf{M}\cdot\mathbf{B}\,dv$, where the integration is over the volume of the magnet and the "loop" definition of $\mathbf{M}$ is used rather than the "pole" definition. The forces and torques acting on the magnet are

$$F_x = \frac{\partial W}{\partial x} \qquad T = \frac{\partial W}{\partial \theta}$$

The moment of a magnet is $\mathbf{m} = \int \mathbf{M}\,dv$ where the integration is over the magnet volume.

The mutual (apparently potential) energy of two thin needles magnetized lengthwise at a distance a apart large compared with their length and having loop moments of magnitude m_1 and m_2, when immersed in a medium of relative permeability K_m, is

$$W = m_1 m_2 K_m^{-1} r^{-3}(\sin\theta_1 \sin\theta_2 \cos\psi - 2\cos\theta_1\cos\theta_2) \times 10^{-7}$$

[1] Tables of $\frac{1}{2}C_n$ are given by W. R. Smythe, *Revs. Modern Phys.* **20**, 176 (1948).

where θ_1 and θ_2 are the angles between $\mathbf{m}_1$ and $\mathbf{m}_2$, respectively, and $\mathbf{r}$. The angle between the planes that contain $\mathbf{m}_1$ and $\mathbf{m}_2$ and intersect in $\mathbf{r}$ is ψ. The repulsive force between two needles is $-\partial W/\partial r$ and if α is the azimuth angle about any line the torque on either magnet about that line is $-\partial W/\partial\alpha$, the other magnet being fixed. In a vacuum where K_m is unity this formula applies to magnets of moments $\mathbf{m}_1$ and $\mathbf{m}_2$ of any shape provided their dimensions are small compared with $\mathbf{r}$. In other media the mutual energy depends on the shape.

Uniformly magnetized bodies may be replaced by their equivalent current sheets for the purpose of calculating fields and mutual torques in a vacuum. The current sheet coincides with the surface of the body and the current density encircles the body in a path normal to the direction x of magnetization and is uniform in terms of x and numerically equal to M. Thus the fields of thin disks magnetized normal to their faces and the torques and forces between them are identical with those between circular loops already given, if I_1 and I_2 are replaced by M_1 and M_2. Similarly, in a vacuum the fields and forces involving uniformly magnetized bars may be calculated from the formulas already given for solenoids provided nI, where n is the number of turns per meter, is replaced by M. The mutual-inductance tables given by Grover and already referred to may be used.

A right circular cylinder of length b and radius a uniformly magnetized lengthwise with an intensity M, when placed with its flat end against an infinitely permeable flat surface, adheres with a force

$$F = 8\pi abM^2\{k^{-1}[K(k) - E(k)] - k_1^{-1}[K(k_1) - E(k_1)]\} \times 10^{-7}$$

where the moduli of the complete elliptic integrals are $k = 2a(4a^2 + b^2)^{-\frac{1}{2}}$ and $k_1 = a(a^2 + b^2)^{-\frac{1}{2}}$. If M is very large, this gives approximately

$$F \approx 2\pi^2a^2M^2 \times 10^{-7}$$

The same force is experienced by two identical cylindrical magnets placed N to S. The same force, but repulsive, appears if they are placed N to N or S to S.

A long straight bar of uniform cross-sectional area S has a uniform lengthwise magnetization M. The flat end, when placed in contact with an infinitely permeable flat block, adheres with a force

$$F \approx 2\pi SM^2 \times 10^{-7}$$

The above bar bent in the shape of a horseshoe with coplanar ends will, if the magnetization remains uniform, adhere with twice this force.

The torque on a sphere with uniform magnetization M immersed in a medium of relative permeability K_m in a field of induction B such that the angle between B and M is α is

$$T = \frac{4\pi a^3MB\sin\alpha}{2K_m + 1}$$

The torque on any body of volume v with a uniform magnetization M when placed in a uniform field of induction B in a vacuum so that the angle between B and M is α is

$$T = BMv\sin\alpha$$

DYNAMIC-FIELD FORMULAS

5b-8. The Electromagnetic Field Equations. In this section some basic relations and concepts of the classic electromagnetic field are given. The mks or Giorgi system of units will be used throughout.

Maxwell's Equations. The basic equations governing the field vectors are Maxwell's equations.

DIFFERENTIAL FORMS

$$\nabla \times \mathbf{E}(\mathbf{r},t) = -\frac{\partial}{\partial t}\mathbf{B}(\mathbf{r},t) \tag{5b-1}$$

$$\nabla \times \mathbf{H}(\mathbf{r},t) = \mathbf{J}(\mathbf{r},t) + \frac{\partial}{\partial t}\mathbf{D}(\mathbf{r},t) \tag{5b-2}$$

$$\nabla \cdot \mathbf{B}(\mathbf{r},t) = 0 \tag{5b-3}$$

$$\nabla \cdot \mathbf{D}(\mathbf{r},t) = \rho(\mathbf{r},t) \tag{5b-4}$$

where **E** is the electric field intensity vector in volts/meter, **H** is the magnetic field intensity vector in amperes/meter, **B** is the magnetic-induction vector in webers/meter², **D** is the electric displacement vector in coulombs/meter², **J** is the current density vector in amperes/meter², ρ is the volume density of charge in coulombs/meter³, **r** is the position vector in meters, and t is the time in seconds. The vector **J** and the volume density of charge ρ are source quantities, and the vectors **E**, **H**, **B**, **D** are field quantities. The conservation of charge is expressed by the equation of continuity

$$\nabla \cdot \mathbf{J}(\mathbf{r},t) = -\frac{\partial}{\partial t}\rho(\mathbf{r},t) \tag{5b-5}$$

which is a corollary of Eq. (5b-4) and the divergence of Eq. (5b-2).

INTEGRAL FORMS. Integral forms of Maxwell's equations follow readily from Eqs. (5b-1) to (5b-4) with the aid of Stokes' theorem and the divergence theorem. They are:

$$\oint_\Gamma \mathbf{E} \cdot dl = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{n}\, dS \qquad \text{(Faraday's emf law)} \tag{5b-6}$$

$$\oint_\Gamma \mathbf{H} \cdot dl = \int_S \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}\right) \cdot \mathbf{n}\, dS \qquad \text{(Generalized Ampères' law)} \tag{5b-7}$$

$$\oint_S \mathbf{D} \cdot \mathbf{n}\, dS = \int_V \rho\, dV \qquad \text{(Gauss' law)} \tag{5b-8}$$

$$\oint_S \mathbf{B} \cdot \mathbf{n}\, dS = 0 \qquad \text{(magnetic flux conservation law)} \tag{5b-9}$$

where Γ is a closed curve spanned by an arbitrary surface S, both stationary in the observer's frame of reference; **n** is a unit vector normal to S; and V is the volume enclosed by a closed surface S.

DUALITY, MAGNETIC SOURCES. For a "simple" medium in which $\mathbf{D}(\mathbf{r},t) = \epsilon\mathbf{E}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t) = \mu\mathbf{H}(\mathbf{r},t)$, where ϵ and μ are respectively the dielectric constant and the permeability of the medium, Maxwell's equations possess a certain duality in **E** and **H** provided that the mathematical artifice of magnetic charge and magnetic current are introduced. Hence, the generalized Maxwell's equations are:

$$\nabla \times \mathbf{E} = -\mathbf{J}_m - \mu\frac{\partial \mathbf{H}}{\partial t} \tag{5b-10}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \epsilon\frac{\partial \mathbf{E}}{\partial t} \tag{5b-11}$$

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon}\rho \tag{5b-12}$$

$$\nabla \cdot \mathbf{H} = \frac{1}{\mu}\rho_m \tag{5b-13}$$

where $\mathbf{J}_m$ and ρ_m are respectively the fictitious magnetic current source and magnetic charge source. Substituting the duality transformation,

$$\mathbf{E}^d = \pm\left(\frac{\mu}{\epsilon}\right)^{\frac{1}{2}}\mathbf{H} \qquad \mathbf{H}^d = \mp\left(\frac{\epsilon}{\mu}\right)^{\frac{1}{2}}\mathbf{E} \qquad \mathbf{J}^d = \pm\left(\frac{\epsilon}{\mu}\right)^{\frac{1}{2}}\mathbf{J}_m$$

$$\mathbf{J}_m{}^d = \mp\left(\frac{\mu}{\epsilon}\right)^{\frac{1}{2}}\mathbf{J} \qquad \rho^d = \pm\left(\frac{\epsilon}{\mu}\right)^{\frac{1}{2}}\rho_m \qquad \rho_m{}^d = \mp\left(\frac{\mu}{\epsilon}\right)^{\frac{1}{2}}\rho$$

into Eqs. (5b-10) to (5b-13) gives

$$\nabla \times \mathbf{E}^d = -\mathbf{J}_m{}^d - \mu\frac{\partial \mathbf{H}^d}{\partial t} \qquad \nabla \times \mathbf{H}^d = \mathbf{J}^d + \epsilon\frac{\partial \mathbf{E}^d}{\partial t}$$

$$\nabla \cdot \mathbf{E}^d = \frac{1}{\epsilon}\rho^d \qquad \nabla \cdot \mathbf{H}^d = \frac{1}{\mu}\rho_m{}^d$$

Thus to every electromagnetic field (**E**,**H**) produced by electric current **J**, there is a dual field ($\mathbf{H}^d,\mathbf{E}^d$) produced by a fictive magnetic current[1] $\mathbf{J}_m{}^d$.

TIME-PERIODIC FIELD. If all quantities have time dependence $e^{-i\omega t}$, the time factor can be suppressed and Maxwell's equations in simple, linear, time-independent media become relations between complex amplitudes. The differential forms of Maxwell's equation are:

$$\nabla \times \mathbf{E}(\mathbf{r}) = i\omega\mathbf{B}(\mathbf{r}) \tag{5b-14}$$

$$\nabla \times \mathbf{H}(\mathbf{r}) = \mathbf{J}(\mathbf{r}) - i\omega\mathbf{D}(\mathbf{r}) \tag{5b-15}$$

$$\nabla \cdot \mathbf{B}(\mathbf{r}) = 0 \tag{5b-16}$$

$$\nabla \cdot \mathbf{D}(\mathbf{r}) = \rho(\mathbf{r}) \tag{5b-17}$$

It is understood that $\mathbf{E}(\mathbf{r},t) = \text{Re}\,[\mathbf{E}(\mathbf{r})e^{-i\omega t}]$, $\mathbf{H}(\mathbf{r},t) = \text{Re}[H(\mathbf{r})e^{-i\omega t}]$, . . . , etc. Re is shorthand for "the real part of."

Covariance of Maxwell's Equations. According to the theory of relativity, the Maxwell's equations are covariant under the Lorentz transformation. In other words, Maxwell's equations have the same form in all inertial frames of reference.

LORENTZ TRANSFORMATIONS. The Lorentz transformations between an inertial frame $S(\mathbf{r},t)$ and another inertial frame $S'(\mathbf{r}',t')$ which is moving at a uniform velocity $\mathbf{v}$ with respect to S can be written in the general form

$$\mathbf{r}' = \mathbf{r} - \gamma\mathbf{v}t + (\gamma - 1)\frac{\mathbf{r}\cdot\mathbf{v}}{v^2}\mathbf{v} \tag{5b-18}$$

$$t' = \gamma\left(t - \frac{\mathbf{r}\cdot\mathbf{v}}{c^2}\right) \tag{5b-19}$$

where $\gamma = (1 - \beta^2)^{-\frac{1}{2}}$, $\beta = v/c$, $\mathbf{r} = x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z$, and c is the velocity of light in vacuum.

FIELD AND SOURCE TRANSFORMATIONS. To assure the covariance of Maxwell's equations between S and S' systems, the following transformations for the field vectors, the current density vector, and the charge density must be used:

$$\mathbf{E}' = \gamma(\mathbf{E} + \mathbf{v} \times \mathbf{B}) + (1 - \gamma)\frac{\mathbf{E}\cdot\mathbf{v}}{v^2}\mathbf{v} \tag{5b-20}$$

$$\mathbf{B}' = \gamma\left(\mathbf{B} - \frac{1}{c^2}\mathbf{v} \times \mathbf{E}\right) + (1 - \gamma)\frac{\mathbf{B}\cdot\mathbf{v}}{v^2}\mathbf{v} \tag{5b-21}$$

$$\mathbf{D}' = \gamma\left(\mathbf{D} + \frac{1}{c^2}\mathbf{v} \times \mathbf{H}\right) + (1 - \gamma)\frac{\mathbf{D}\cdot\mathbf{v}}{v^2}\mathbf{v} \tag{5b-22}$$

$$\mathbf{H}' = \gamma(\mathbf{H} - \mathbf{v} \times \mathbf{D}) + (1 - \gamma)\frac{\mathbf{H}\cdot\mathbf{v}}{v^2}\mathbf{v} \tag{5b-23}$$

$$\mathbf{J}' = \mathbf{J} - \gamma\mathbf{v}\rho + (\gamma - 1)\frac{\mathbf{J}\cdot\mathbf{v}}{v^2}\mathbf{v} \tag{5b-24}$$

$$\rho' = \gamma\left(\rho - \frac{1}{c^2}\mathbf{J}\cdot\mathbf{v}\right) \tag{5b-25}$$

[1] This duality property is intimately related to Babinet's principle discussed in Sec. 5b-12.

Constitutive Relations. Only two of the four Maxwell's equations (5b-1) to (5b-4) are independent, since the two divergence equations (5b-3) and (5b-4) can be obtained from the two curl equations (5b-1) and (5b-2) and the continuity equation (5b-5). Therefore, the number of field vectors required to describe an electromagnetic field must be reduced to two from the original four. This reduction is accomplished by the introduction of constitutive parameters which provide a mathematical description of the macroscopic electromagnetic properties of matter.

ELECTRIC AND MAGNETIC POLARIZATION VECTORS. The behavior of a material medium in an electromagnetic field can be described in terms of distributions of electric and magnetic dipoles. The medium can be characterized by two polarization density functions: **P**, electric dipole moment per unit volume, and **M**, magnetic dipole moment per unit volume. The polarization may be induced under action of the field from other sources, or it may be virtually permanent and independent of external fields. The permanent polarizations will be designated by $\mathbf{P}_0$ and $\mathbf{M}_0$. The relationships between the field vectors and the polarization vectors are

$$\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P} + \mathbf{P}_0 \qquad (5b\text{-}26)$$
$$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M} + \mathbf{M}_0) \qquad (5b\text{-}27)$$

where ϵ_0 and μ_0 are respectively the permittivity and permeability of free space.

ISOTROPIC MEDIA. In simple isotropic media, the polarization vectors are proportional to the field (i.e., $\mathbf{P} = \epsilon_0\chi\mathbf{E}$ and $\mathbf{M} = \chi_m\mathbf{H}$), and the constitutive parameters are scalar quantities:

$$\mathbf{D} = \epsilon_0(1 + \chi)\mathbf{E} = K\epsilon_0\mathbf{E} = \epsilon\mathbf{E} \qquad (5b\text{-}28)$$
$$\mathbf{B} = \mu_0(1 + \chi_m)\mathbf{H} = K_m\mu_0\mathbf{H} = \mu\mathbf{H} \qquad (5b\text{-}29)$$

where χ is the electric susceptibility, K is the relative permittivity of the medium (or the dielectric constant), ϵ is its absolute permittivity, χ_m is the magnetic susceptibility, K_m is the relative permeability of the medium, and μ is its absolute permeability. For isotropic inhomogeneous media, ϵ and μ may be functions of positions. Strictly speaking, the relations (5b-28) and (5b-29) are definably only for time-periodic phenomena, since in general ϵ and μ are functions of the frequency. (The frequency dependence of the constitutive parameters is known as the *dispersive property* of the medium.) Hence, these relations are applicable to other than time-periodic, time-varying fields only when over the significant part of the frequency spectrum covered by the Fourier components of the time dependence the constitutive parameters ϵ and μ are sensibly independent of frequency.

ANISOTROPIC MEDIA. The constitutive relations for an anisotropic medium have the form

$$\mathbf{D} = \boldsymbol{\varepsilon}\cdot\mathbf{E} \qquad (5b\text{-}30)$$
$$\mathbf{B} = \boldsymbol{\mu}\cdot\mathbf{H} \qquad (5b\text{-}31)$$

where $\boldsymbol{\varepsilon}$ and $\boldsymbol{\mu}$ are second-rank tensors having ϵ_{ij} and μ_{ij} as their components. For inhomogeneous and anisotropic medium, ϵ_{ij} and μ_{ij} are functions of positions. For anisotropic and dispersive medium, ϵ_{ij} and μ_{ij} are functions of the frequency; the relationships (5b-30) and (5b-31) then become relationships between complex amplitudes.

CONDUCTING MEDIA. A conducting medium is characterized by a linear relation between current density and the electric vector: For isotropic conducting medium

$$\mathbf{J} = \sigma\mathbf{E} \qquad (5b\text{-}32)$$

where σ is a scalar. For anisotropic conducting medium

$$\mathbf{J} = \boldsymbol{\sigma}\cdot\mathbf{E} \qquad (5b\text{-}33)$$

where σ is a second-rank tensor having components σ_{ij}. Again σ may be position-dependent or frequency-dependent.

UNIFORMLY MOVING MEDIA. Assume that an inertial frame $S'(\mathbf{r}',t')$ is moving at a uniform velocity $\mathbf{v}$ with respect to an observer's inertial frame $S(\mathbf{r},t)$. If the constitutive relations in S' frame are $\mathbf{D}' = \epsilon'\mathbf{E}'$ and $\mathbf{B}' = \mu'\mathbf{H}'$, then with the aid of Eqs. (5b-20) to (5b-23) we may find the constitutive relations in the observer's S frame from the following equations:

$$\mathbf{D} - \epsilon'(\mathbf{v} \times \mathbf{B}) = \epsilon'\mathbf{E} - \frac{1}{c^2}\mathbf{v} \times \mathbf{H} \tag{5b-34}$$

$$\mathbf{v} \times \mathbf{D} + \frac{1}{\mu'}\mathbf{B} = \frac{1}{\mu'}\left(\frac{1}{c^2}\mathbf{v} \times \mathbf{E}\right) + \mathbf{H} \tag{5b-35}$$

Note that in uniformly moving medium **D** is linearly related to **E** as well as **H**, and **B** is also linearly related to **H** as well as **E**.[1]

NONLINEAR MEDIA. The constitutive relations for a nonlinear medium have the form

$$\mathbf{D} = \epsilon(\mathbf{E})\mathbf{E} \tag{5b-36}$$

$$\mathbf{B} = \mu(\mathbf{H})\mathbf{H} \tag{5b-37}$$

where $\epsilon(\mathbf{E})$ and $\mu(\mathbf{H})$ are functions of the field strengths. Substituting these constitutive relations into Eqs. (5b-1) and (5b-2) gives a set of equations that are nonlinear. Because of the field-dependent characteristics of the permittivity and the permeability of the medium, there is energy exchange between a number of electromagnetic fields of different frequencies.[2]

Boundary Conditions. BOUNDARIES BETWEEN STATIONARY MEDIA. Let Γ be a smooth surface separating two media, 1 and 2; let the unit vector normal to the boundary be **n**, pointing from medium 1 into medium 2.

1. Media 1 and 2 are dielectrics having constitutive parameters ϵ_1, μ_1, σ_1, and ϵ_2, μ_2, σ_2, respectively. The boundary conditions are:

Tangential components of the electric field vector **E** are continuous:

$$\mathbf{n} \times (\mathbf{E}_2 - \mathbf{E}_1) = 0 \tag{5b-38}$$

Tangential components of the magnetic field vector **H** are continuous:

$$\mathbf{n} \times (\mathbf{H}_2 - \mathbf{H}_1) = 0 \tag{5b-39}$$

Normal component of **D** is discontinuous by an amount equal to the electric surface-charge density:

$$\mathbf{n} \cdot (\mathbf{D}_2 - \mathbf{D}_1) = \rho_s \tag{5b-40}$$

Normal component of **B** is continuous:

$$\mathbf{n} \cdot (\mathbf{B}_2 - \mathbf{B}_1) = 0 \tag{5b-41}$$

Note that Eqs. (5b-40) and (5b-41) can be derived from Eqs. (5b-38) and (5b-39) with the aid of Maxwell's equations and the continuity equation. Hence, either Eqs. (5b-38) and (5b-39) or Eqs. (5b-40) and (5b-41) are sufficient to specify the boundary conditions.

2. Medium 1 is a perfect conductor, and medium 2 is a dielectric. The boundary conditions are

$$\mathbf{n} \times \mathbf{E}_2 = 0 \qquad \mathbf{n} \times \mathbf{H}_2 = K \qquad \mathbf{n} \cdot \mathbf{B}_2 = 0 \qquad \mathbf{n} \cdot \mathbf{D}_2 = \rho_s \tag{5b-42}$$

[1] C. Moller, "The Theory of Relativity," Oxford University Press, London, 1952.
[2] N. Bloembergen, "Nonlinear Optics," W. A. Benjamin, Inc., New York, 1965.

where K is the electric surface current density. A surface having these boundary conditions is said to be an "electric wall." By duality a surface displaying the boundary conditions

$$\mathbf{n} \times \mathbf{E}_2 = -K_m \qquad \mathbf{n} \times \mathbf{H}_2 = 0 \qquad \mathbf{n} \cdot \mathbf{D}_2 = 0 \qquad \mathbf{n} \cdot \mathbf{B}_2 = \rho_{sm} \tag{5b-43}$$

is said to be a "magnetic wall." K_m is the magnetic surface current density, and ρ_{sm} is the magnetic charge density.

3. Medium 1 has a surface impedance Z_s which is defined as the ratio of the tangential electric field to the tangential magnetic field at the surface, and electromagnetic fields are impenetrable into medium 1. Medium 2 is a dielectric. The boundary condition is

$$\mathbf{n} \times \mathbf{E}_2 = Z_s \mathbf{H}_2 \tag{5b-44}$$

If medium 1 is a good conductor, in which σ_1 is larger but finite,[1] then $Z_s = (1 - i)(\omega\mu_1/2\sigma_1)^{\frac{1}{2}}$ with $i = \sqrt{-1}$. The surface-impedance boundary condition is valid only for time-harmonic fields.

BOUNDARIES BETWEEN MOVING MEDIA. Let medium 1 be moving with respect to medium 2 with a velocity $\mathbf{v}$; let $S'(\mathbf{r}',t')$ and $S(\mathbf{r},t)$ be inertial frames for medium 1 and medium 2, respectively. The boundary conditions are

$$\mathbf{n} \times (\mathbf{E}_1 + \mathbf{v} \times \mathbf{B}_1) = \mathbf{n} \times (\mathbf{E}_2 + \mathbf{v} \times \mathbf{B}_2) \tag{5b-45}$$
$$\mathbf{n} \times (\mathbf{H}_1 - \mathbf{v} \times \mathbf{D}_1) = \mathbf{n} \times (\mathbf{H}_2 - \mathbf{v} \times \mathbf{D}_2) \tag{5b-46}$$

where $\mathbf{n}$ is a unit vector, in inertial frame S, normal to the boundary and pointing from medium 1 into medium 2. $\mathbf{E}_1$, $\mathbf{B}_1$, $\mathbf{H}_1$, and $\mathbf{D}_1$, and $\mathbf{E}_2$, $\mathbf{B}_2$, $\mathbf{H}_2$, and $\mathbf{D}_2$ are respectively field vectors in medium 1 and medium 2 as observed from the inertial frame[2] $S(\mathbf{r},t)$.

RADIATION CONDITIONS. The field associated with a finite distribution of sources or the field scattered from obstacles must satisfy conditions at infinity which pertain to the finiteness of the energy radiated by the sources or scattered by obstacles as well as the assurance that the field at infinity represents an outgoing wave. For time-periodic field in a homogeneous medium the condition at infinity take the form

$$\lim_{r \to \infty} r \left[\mathbf{H} - \left(\frac{\epsilon}{\mu}\right)^{\frac{1}{2}} (\mathbf{e}_r \times \mathbf{E}) \right] = 0 \tag{5b-47}$$
$$\lim_{r \to \infty} r\mathbf{E} \text{ is finite} \tag{5b-48}$$

where r is the radial distance from an arbitrary origin in the neighborhood of the sources or the scattering bodies, and $\mathbf{e}_r$ is a unit vector directed from the origin in the radial direction.

EDGE CONDITION. At sharp edges the field vectors may become infinite. But the order of this singularity is restricted by the Bonwkamp-Meixner edge condition: The energy density must be integrable over any finite domain even if this domain happens to include field singularities: i.e., the energy in any finite region of space must be finite. For example, when applied to a perfectly conducting sharp edge, this condition states that the singular components of the electric and magnetic vectors are of order $\xi^{-\frac{1}{2}}$, where ξ is the distance from the edge, whereas the parallel components are always finite.

UNIQUENESS THEOREM. A field in a lossy region is uniquely specified by the sources within the region plus the tangential components of $\mathbf{E}$ over the boundary, or the tangential components of $\mathbf{H}$ over the boundary, or the former over part of the bound-

[1] M. A. Leontovich, "Investigation of Propagation of Radiowaves," part II, Moscow, 1948; T. B. A. Senior, *Appl. Sci. Research* **B-8,** 418 (1960).

[2] A recent example of the interaction of electromagnetic waves with moving media was given by C. Yeh, *J. Appl. Phys.*, **36,** 3513 (1965); **38,** 5194 (1967).

ary and the latter over the rest of the boundary. The uniqueness theorem for the lossy case can be carried over to the lossless case if we consider the field in a lossless medium to be the limit of the corresponding field in a lossy medium as the loss goes to zero.

Energy Relations. POYNTING'S VECTOR THEOREM. Taking the scalar product of Maxwell's equations (5b-1) and (5b-2) with **H** and **E**, respectively, and subtracting the resultant equations gives the following energy relation:

$$\nabla \cdot \mathbf{S}(t) + \mathbf{E}(t) \cdot \mathbf{J}(t) = -\frac{\partial}{\partial t}(w_e(t) + w_m(t)) \tag{5b-49}$$

where $\mathbf{S}(t) = \mathbf{E}(t) \times \mathbf{H}(t)$, defined as the instantaneous Poynting's vector representing the flow of energy associated with an electromagnetic field; $w_e(t) = \frac{1}{2}\mathbf{E}(t) \cdot \mathbf{D}(t)$, defined as the electric energy per unit volume; and $w_m(t) = \frac{1}{2}\mathbf{H}(t) \cdot \mathbf{B}(t)$, defined as the magnetic energy per unit volume. Equation (5b-49) is the differential form of Poynting's vector theorem. Taking the volume integral of Eq. (5b-49) gives the integral form of Poynting's vector theorem:

$$\int \mathbf{S}(t) \cdot \mathbf{n}\, dA + \int (\mathbf{E} \cdot \mathbf{J})\, dv = -\frac{\partial}{\partial t}\int (w_e + w_m)\, dv \tag{5b-50}$$

The first integral represents the electromagnetic energy flowing out or in per second from a volume v bounded by a surface A. The second integral represents power generated within the volume v; or, if $\mathbf{J} = \sigma\mathbf{E}$, it represents power dissipated as Joules heat in the volume v. The third integral represents the time rate of change of electric and magnetic energy in the volume v.

For time-harmonic fields, we have the following relations in terms of complex quantities:

$$\tfrac{1}{2}\nabla \cdot \mathbf{S} + \tfrac{1}{2}\mathbf{J} \cdot \mathbf{E}^* = 2i\omega(\bar{w}_m - \bar{w}_e) \tag{5b-51}$$

and

$$\tfrac{1}{2}\int \mathbf{S} \cdot \mathbf{n}\, dA + \tfrac{1}{2}\int (\mathbf{J} \cdot \mathbf{E}^*)\, dv = 2i\omega \int (\bar{w}_m - \bar{w}_e)\, dv \tag{5b-52}$$

where $\mathbf{S} = \mathbf{E} \times \mathbf{H}^*$ is the complex Poynting's vector, $\bar{w}_e$ which is $\frac{1}{4}\mathbf{E} \cdot \mathbf{D}^*$ and $\bar{w}_m$ which is $\frac{1}{4}\mathbf{H} \cdot \mathbf{B}^*$ are time-average energy densities. The asterisks denote complex-conjugate values. Real part of $\frac{1}{2}\int \mathbf{S} \cdot \mathbf{n}\, dA$ gives the time-averaged power generated within the volume v, or, if $\mathbf{J} = \sigma\mathbf{E}$, it represents the time-averaged power dissipated as Joules heat within the volume v.

MAXWELL'S STRESS TENSOR. Neglecting the contribution due to electrostriction and magnetostriction, components of the second-rank Maxwell's stress tensor in a material medium are

$$T_{\alpha\beta} = E_\alpha D_\beta - \tfrac{1}{2}\delta_{\alpha\beta}E_\gamma D_\gamma + H_\alpha B_\beta - \tfrac{1}{2}\delta_{\alpha\beta}H_\gamma B_\gamma \tag{5b-53}$$

where $\delta_{\alpha\beta}$ is the Kronecker delta; $\alpha = x, y, z$; $\beta = x, y, z$; $\gamma = x, y, z$. The volume density of force in the medium is

$$\mathbf{f}_\alpha = \frac{\partial T_{\alpha\beta}}{\partial x_\beta} - \frac{\partial}{\partial t}(\mathbf{D} \times \mathbf{B}) \tag{5b-54}$$

where $\partial T_{\alpha\beta}/\partial x_\beta$ is the tensor divergence of $T_{\alpha\beta}$. The total force on a volume element V is given by

$$\mathbf{F}_\alpha = \int_V \mathbf{f}_\alpha\, dv \tag{5b-55}$$

The above expression is particularly useful for computation of forces acting on dielectric or magnetic materials by electromagnetic waves.

The Wave Equations for the Field Vectors. By combining Maxwell's equations with the constitutive relations, equations for the field vectors **E** and **H** can be derived:

IN HOMOGENEOUS ISOTROPIC MEDIA

$$\nabla \times \nabla \times \mathbf{E} + \mu\epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = -\mu \frac{\partial \mathbf{J}}{\partial t} - \nabla \times \mathbf{J}_m \qquad (5b\text{-}56)$$

$$\nabla \times \nabla \times \mathbf{H} + \mu\epsilon \frac{\partial^2 \mathbf{H}}{\partial t^2} = -\epsilon \frac{\partial \mathbf{J}_m}{\partial t} + \nabla \times \mathbf{J} \qquad (5b\text{-}57)$$

IN INHOMOGENEOUS ISOTROPIC MEDIA

$$\nabla \times \nabla \times \mathbf{E} - \frac{\nabla \mu(\mathbf{r})}{\mu(\mathbf{r})} \times \nabla \times \mathbf{E} + \epsilon(\mathbf{r})\mu(\mathbf{r}) \frac{\partial^2 \mathbf{E}}{\partial t^2} = -\mu(\mathbf{r}) \frac{\partial \mathbf{J}}{\partial t} - \nabla \times \mathbf{J}_m \qquad (5b\text{-}58)$$

$$\nabla \times \nabla \times \mathbf{H} - \frac{\nabla \epsilon(\mathbf{r})}{\epsilon(\mathbf{r})} \times \nabla \times \mathbf{H} + \epsilon(\mathbf{r})\mu(\mathbf{r}) \frac{\partial^2 \mathbf{E}}{\partial t^2} = -\epsilon(\mathbf{r}) \frac{\partial \mathbf{J}_m}{\partial t} + \nabla \times \mathbf{J} \qquad (5b\text{-}59)$$

$\epsilon(\mathbf{r})$ and $\mu(\mathbf{r})$ are respectively the inhomogeneous dielectric permittivity and the inhomogeneous magnetic permeability of the medium.[1]

IN HOMOGENEOUS ANISOTROPIC MEDIA. In a general anisotropic medium with $\mathbf{B} = \boldsymbol{\mu} \cdot \mathbf{H}$ and $\mathbf{D} = \boldsymbol{\varepsilon} \cdot \mathbf{E}$, the wave equations are expressible only as two coupled second-order differential equations. These equations are usually very involved; only in the special case of a gyromagnetic ferrite medium[2] or a gyroelectric plasma medium[3] have the solutions for these wave equations been found.

IN HOMOGENEOUS ISOTROPIC MOVING MEDIA. In the observer's S system in which a simple homogeneous isotropic medium is moving at velocity $\mathbf{v}$, the wave equations are

$$\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - K\gamma^2\left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla\right)^2\right] \mathbf{A}(\mathbf{r},t) = -\mu'\mathbf{J} - \frac{\mu' K}{n'^2}\gamma\mathbf{v}(\gamma\mathbf{J} \cdot \mathbf{v} - \gamma c^2 \rho) \qquad (5b\text{-}60)$$

$$\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - K\gamma^2\left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla\right)^2\right] \phi(\mathbf{r},t) = -\mu' c^2 \rho - \frac{\mu' K}{n'^2}\gamma c^2(\gamma\mathbf{J} \cdot \mathbf{v} - \gamma c^2 \rho) \qquad (5b\text{-}61)$$

$$\mathbf{E} = -\nabla\phi - \frac{\partial}{\partial t}\mathbf{A}$$

$$\mathbf{B} = \nabla \times \mathbf{A}$$

where $K = (c^2\epsilon'\mu' - 1)/c^2$, $n' = c\sqrt{\epsilon'\mu'}$, $\gamma = (1 - \beta^2)^{-\frac{1}{2}}$, and $\beta = v/c$. ϵ' and μ' are respectively the dielectric constant and the permeability of the medium in the S' system which is stationary with respect to the medium.

The Vector and Scalar Potentials. THE **A** AND ϕ POTENTIALS. The electromagnetic field can, in general, be divided into two parts, one associated with electric-type sources **J** and ρ, the other associated with magnetic-type sources $\mathbf{J}_m$ and ρ_m. Each part can be developed by means of vector and scalar potentials as follows:

$$\mathbf{E}_e = -\nabla\phi_e - \frac{\partial \mathbf{A}_e}{\partial t} \qquad (5b\text{-}62)$$

$$\mathbf{B}_e = \nabla \times \mathbf{A}_e \qquad (5b\text{-}63)$$

$$\mathbf{D}_m = -\nabla \times \mathbf{A}_m \qquad (5b\text{-}64)$$

$$\mathbf{H}_m = -\nabla\phi_m - \frac{\partial \mathbf{A}_m}{\partial t} \qquad (5b\text{-}65)$$

[1] J. R. Wait, "Electromagnetic Waves in Stratified Media," Pergamon Press, New York, 1964.

[2] B. Lax, and K. J. Button, "Microwave Ferrites and Ferrimagnetics," McGraw-Hill Book Company, New York, 1962.

[3] M. A. Heald, and C. B. Wharton, "Plasma Diagnostics with Microwaves," John Wiley & Sons, Inc., New York, 1965.

The general representation of the field in terms of potentials is accordingly

$$\mathbf{E} = -\nabla\phi_e - \frac{\partial \mathbf{A}_e}{\partial t} - \frac{1}{\epsilon}\nabla \times \mathbf{A}_m \tag{5b-66}$$

$$\mathbf{H} = \frac{1}{\mu}\nabla \times \mathbf{A}_e - \nabla\phi_m - \frac{\partial \mathbf{A}_m}{\partial t} \tag{5b-67}$$

For homogeneous isotropic media the differential equations relating the potentials to the source functions are:

$$\nabla^2\mathbf{A}_e - \mu\epsilon\frac{\partial^2\mathbf{A}_e}{\partial t^2} = -\mu\mathbf{J} \tag{5b-68}$$

$$\nabla^2\phi_e - \mu\epsilon\frac{\partial^2\phi_e}{\partial t^2} = -\frac{\rho}{\epsilon} \tag{5b-69}$$

$$\nabla^2\mathbf{A}_m - \mu\epsilon\frac{\partial^2\mathbf{A}_m}{\partial t^2} = -\epsilon\mathbf{J}_m \tag{5b-70}$$

$$\nabla^2\phi_m - \mu\epsilon\frac{\partial^2\phi_m}{\partial t^2} = -\frac{\rho_m}{\mu} \tag{5b-71}$$

with the auxiliary conditions

$$\nabla\cdot\mathbf{A}_e + \mu\epsilon\frac{\partial\phi_e}{\partial t} = 0 \tag{5b-72}$$

$$\nabla\cdot\mathbf{A}_m + \mu\epsilon\frac{\partial\phi_m}{\partial t} = 0 \tag{5b-73}$$

THE DEBYE POTENTIALS. For source-free region, it is sometimes more convenient to derive the time-harmonic electromagnetic fields from two scalar potentials as follows:[1]

$$\mathbf{E} = \nabla \times (\mathbf{a}\Psi) + \frac{i}{\omega\epsilon}\nabla \times \nabla \times (\mathbf{a}\Phi) \tag{5b-74}$$

$$\mathbf{H} = \nabla \times (\mathbf{a}\Phi) + \frac{1}{i\omega\mu}\nabla \times \nabla \times (\mathbf{a}\Psi) \tag{5b-75}$$

where $\mathbf{a}$ is a unit vector or the position vector $\mathbf{r}$. For example, in spherical coordinates $\mathbf{a} = \mathbf{r}$, the radial position vector; in cylindrical coordinates $\mathbf{a} = \mathbf{e}_z$, the axial vector; in rectangular coordinates $\mathbf{a} = \mathbf{e}_x$ or $\mathbf{e}_y$ or $\mathbf{e}_z$, the unit vector in x or y or z direction, respectively. The two scalar functions Ψ and Φ are the Debye potentials which satisfy a pair of second-order differential equations. These differential equations are obtained by substituting Eqs. (5b-74) and (5b-75) into the wave equations. In homogeneous isotropic medium, Ψ and Φ satisfy the scalar Helmholtz equation,

$$(\nabla^2 + k^2)\begin{Bmatrix}\Psi\\ \Phi\end{Bmatrix} = 0 \tag{5b-76}$$

with $k^2 = \omega^2\mu\epsilon$. By choosing $\mathbf{a}$ appropriately, one may also apply Eqs. (5b-74) and (5b-75) to the case of an inhomogeneous medium.[2]

Basic Wave Types

1. Transverse electromagnetic waves (*TEM* waves)—containing neither an electric nor a magnetic field component in the direction of propagation.
2. Transverse magnetic waves (*TM* or *E* waves)—containing an electric field component but not a magnetic field component in the direction of propagation.

[1] The vector $\mathbf{a}\Phi$ may be identified as the electric Hertz vector and the vector $\mathbf{a}\Psi$ may be identified as the magnetic Hertz vector. $\mathbf{a}$ in this case is a constant vector.

[2] C. Yeh, *Phys. Rev.* **131**, 2350 (1963).

3. Transverse electric waves (*TE* or *H* waves)—containing a magnetic field component but not an electric field component in the direction of propagation.

4. Hybrid waves (*HE* waves)—containing all components of electric and magnetic fields. These hybrid waves are obtainable by linear superposition of *TE* and *TM* waves.

Formal Solutions for the Time-harmonic Vector Wave Equation. INTEGRAL REPRESENTATIONS. Upon direct integration of the wave equation in homogeneous isotropic medium, integral solutions in terms of the sources can be obtained. A harmonic time dependence of $e^{-i\omega t}$ is assumed and suppressed in this section.

Direct integration of Eqs. (5b-68) to (5b-71) gives

$$\mathbf{A}_e = \frac{\mu}{4\pi}\int_V \mathbf{J}(\mathbf{r}')\,\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\,dv' \tag{5b-77}$$

$$\phi_e = \frac{1}{4\pi\epsilon}\int_V \rho(\mathbf{r}')\,\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\,dv' \tag{5b-78}$$

$$\mathbf{A}_m = \frac{\epsilon}{4\pi}\int_V \mathbf{J}_m(\mathbf{r}')\,\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\,dv' \tag{5b-79}$$

$$\phi_m = \frac{1}{4\pi\mu}\int_V \rho_m(\mathbf{r}')\,\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\,dv' \tag{5b-80}$$

where $\mathbf{r}$ is the coordinate of the observation point and $\mathbf{r}'$ is the coordinate of the source point. The integration with respect to the primed coordinates extends throughout the volume V occupied by the source.

Direct integration of Eqs. (5b-56) and (5b-57) gives

$$\mathbf{E} = i\omega\mu\int_V \Gamma(\mathbf{r},\mathbf{r}')\cdot\mathbf{J}(\mathbf{r}')\,dv' - \int_V \nabla G(\mathbf{r},\mathbf{r}')\times\mathbf{J}_m(\mathbf{r}')\,dv' \tag{5b-81}$$

$$\mathbf{H} = i\omega\epsilon\int_V \Gamma(\mathbf{r},\mathbf{r}')\cdot\mathbf{J}_m(\mathbf{r}')\,dv' + \int_V \nabla G(\mathbf{r},\mathbf{r}')\times\mathbf{J}(\mathbf{r}')\,dv' \tag{5b-82}$$

with

$$\Gamma(\mathbf{r},\mathbf{r}') = \left(u + \frac{1}{k^2}\nabla\nabla\right) G(\mathbf{r},\mathbf{r}') \tag{5b-83}$$

$$G(\mathbf{r},\mathbf{r}') = \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} \tag{5b-84}$$

The properties of u and $\nabla\nabla$ are $u\cdot\mathbf{c} = \mathbf{c}$ and $(\nabla\nabla)\cdot\mathbf{c} = \nabla(\nabla\cdot\mathbf{c})$ where $\mathbf{c}$ is any vector function. The gradient operator ∇ is defined as

$$\nabla = \mathbf{e}_x\frac{\partial}{\partial x} + \mathbf{e}_y\frac{\partial}{\partial y} + \mathbf{e}_z\frac{\partial}{\partial z}$$

where $\mathbf{e}_x$, $\mathbf{e}_y$, $\mathbf{e}_z$ are unit vectors for the unprimed coordinates. It is noted that the radiation condition has been met by Eqs. (5b-77) to (5b-82). Hence, these equations represent the integral expressions for the fields in unbounded homogeneous region.

More general integral expressions for the field vectors in terms of the current sources and the field values over the bounding surfaces S_i are also available:

$$\begin{aligned}\mathbf{E}(\mathbf{r}) = i\omega\mu\int_V \Gamma(\mathbf{r},\mathbf{r}')\cdot\mathbf{J}(\mathbf{r}')\,dv' + \int_V \nabla' G(\mathbf{r},\mathbf{r}')\times\mathbf{J}_m(\mathbf{r}')\,dv' \\ + \int_{S_1\cdots S_n} [(\mathbf{n}'\cdot\mathbf{E}(\mathbf{r}'))\nabla' G(\mathbf{r},\mathbf{r}') + (\mathbf{n}'\times\mathbf{E}(\mathbf{r}'))\times\nabla' G(\mathbf{r},\mathbf{r}') \\ + i\omega\mu G(\mathbf{r},\mathbf{r}')(\mathbf{n}'\times\mathbf{H}(\mathbf{r}'))]\,dS'\end{aligned} \tag{5b-85}$$

$$\begin{aligned}\mathbf{H}(\mathbf{r}) = i\omega\epsilon\int_V \Gamma(\mathbf{r},\mathbf{r}')\cdot\mathbf{J}_m(\mathbf{r}')\,dv' - \int_V \nabla' G(\mathbf{r},\mathbf{r}')\times\mathbf{J}(\mathbf{r}')\,dv' \\ + \int_{S_1\cdots S_n} \{[\mathbf{n}'\times\mathbf{H}(\mathbf{r}')]\times\nabla' G(\mathbf{r},\mathbf{r}') + [\mathbf{n}'\cdot\mathbf{H}(\mathbf{r}')]\nabla' G(\mathbf{r},\mathbf{r}') \\ - i\omega\epsilon G(\mathbf{r},\mathbf{r}')[\mathbf{n}'\times\mathbf{E}(\mathbf{r}')]\}\,dS\end{aligned} \tag{5b-86}$$

where region V is assumed to be bounded by the surfaces $S_1 \cdots S_n$, S_n is a surface enclosing V, and $\mathbf{n}'$ is the inward-drawn normal from any boundary surface S_i into the volume V. The gradient operator ∇' is with respect to the primed coordinates. $G(\mathbf{r},\mathbf{r}')$ is the scalar Green's function. For an unbounded region, S_n recedes to infinity, $G(\mathbf{r},\mathbf{r}')$ is given by Eq. (5b-84), and Eqs. (5b-85) and (5b-86) represent fields in unbounded region. For a bounded region, the scalar Green's function is

$$G(\mathbf{r},\mathbf{r}') = \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} + g(\mathbf{r},\mathbf{r}') \tag{5b-87}$$

where $g(\mathbf{r},\mathbf{r}')$ satisfies

$$\nabla'^2 g + k^2 g = 0 \tag{5b-88}$$

everywhere in the volume V and over the boundary surfaces $S_1 \cdots S_n$.

SEPARATION OF VARIABLES. Only in five coordinate systems is the method of separation of variables applicable to the source-free vector-wave equation in homogeneous medium [i.e., Eq. (5b-56) or (5b-57) with $\mathbf{J} = \mathbf{J}_m = 0$]. For some specific variation of $\epsilon(\mathbf{r})$ and $\mu(\mathbf{r})$, the method of separation of variables may also be used to solve the source-free vector-wave equation in inhomogeneous medium.[1]

1. Rectangular Coordinates. In the rectangular coordinates x, y, z the three distinct types of basic rectangular wave functions, characterized by the relationship between the field vectors and the z axis (the direction of propagation), are:

TEM Waves

$$\mathbf{E} = \mathbf{E}_0 e^{ikz} \tag{5b-89}$$
$$\mathbf{H} = \left(\frac{\epsilon}{\mu}\right)^{\frac{1}{2}} (\mathbf{e}_z \times \mathbf{E}_0) e^{ikz}$$

TM Waves

$$\mathbf{H} = \nabla \times (\mathbf{e}_z\Phi) \qquad \mathbf{E} = \frac{i}{\omega\epsilon} \nabla \times \nabla \times (\mathbf{e}_z\Phi) \tag{5b-90}$$
$$\Phi = e^{\pm ik_x x} e^{\pm ik_y y} e^{\pm i\gamma z}$$
$$\gamma = (k^2 - k_x^2 - k_y^2)^{\frac{1}{2}}$$

TE Waves

$$\mathbf{E} = \nabla \times (\mathbf{e}_z\Psi) \qquad \mathbf{H} = \frac{1}{i\omega\mu} \nabla \times \nabla \times (\mathbf{e}_z\Psi) \tag{5b-91}$$
$$\Psi = e^{\pm ik_x x} e^{\pm ik_y y} e^{\pm i\gamma z}$$
$$\gamma = (k^2 - k_x^2 - k_y^2)^{\frac{1}{2}}$$

2. Circular Cylindrical Coordinates. In the circular cylindrical coordinates ρ, ϕ, z the three distinct types of basic circular cylindrical wave functions, characterized by the relationship between the field vectors and the z axis (the direction of propagation), are:

TEM Waves

$$E_z = H_z = 0$$
$$E_\rho = \frac{\partial U_n}{\partial \rho} e^{ikz} \qquad H_\phi = \frac{1}{\rho}\frac{\partial U_n}{\partial \phi} e^{ikz} \tag{5b-92}$$

or

$$H_\rho = \frac{\partial V_n}{\partial \rho} e^{ikz} \qquad E_\phi = \frac{1}{\rho}\frac{\partial V_n}{\partial \phi} e^{ikz} \tag{5b-93}$$

where U_n or V_n are solutions of Laplace's equation in two dimensions; explicitly

$$U_n \text{ or } V_n = \begin{Bmatrix} \rho^n \\ \rho^{-n} \end{Bmatrix} e^{in\phi} \tag{5b-94}$$

$$U_0 \text{ or } V_0 = \begin{Bmatrix} \ln \rho \\ \text{constant} \end{Bmatrix} \begin{Bmatrix} \theta \\ \text{constant} \end{Bmatrix} \tag{5b-95}$$

[1] C. Yeh, and K. F. Casey, *IEEE Trans. Microwave Theory and Tech.* **MTT-13**, 297 (1965).

TM Waves

$$\mathbf{H} = \nabla \times (\mathbf{e}_z\Phi) \qquad \mathbf{E} = \frac{i}{\omega\epsilon} \nabla \times \nabla \times (\mathbf{e}_z\Phi) \tag{5b-96}$$

$$\Phi = \{Z_\nu(\Lambda\rho)\}\{e^{\pm i\nu\phi}\}\{e^{\pm i\gamma z}\} \tag{5b-97}$$

$$\gamma = (k^2 - \Lambda^2)^{\frac{1}{2}} \tag{5b-98}$$

where $Z_\nu(\Lambda\rho)$ are two linearly independent solutions to the Bessel differential equation of order ν.

TE Waves

$$\mathbf{E} = \nabla \times (\mathbf{e}_z\Psi) \qquad \mathbf{H} = \frac{1}{i\omega\mu} \nabla \times \nabla \times (\mathbf{e}_z\Psi) \tag{5b-99}$$

$$\Psi = \{Z_\nu(\Lambda\rho)\}\{e^{\pm i\nu\phi}\}\{e^{\pm i\gamma z}\} \tag{5b-100}$$

$$\gamma = (k^2 - \Lambda^2)^{\frac{1}{2}} \tag{5b-101}$$

where $Z_\nu(\Lambda\rho)$ have been defined earlier.

3. Spherical Coordinates. In the spherical coordinates r, θ, ϕ the three distinct types of basic spherical wave functions, characterized by the relationship between the field vectors and the radial r direction (the direction of propagation), are:

TEM Waves

$$E_r = H_r = 0$$

$$E_\theta = \frac{e^{\pm ikr}}{r \sin \theta} \qquad H_\phi = \pm \sqrt{\frac{\epsilon}{\mu}}\, E_\theta \tag{5b-102}$$

TM Waves

$$\mathbf{H} = \nabla \times (\mathbf{e}_r r\Phi) \qquad \mathbf{E} = \frac{i}{\omega\epsilon} \nabla \times \nabla \times (\mathbf{e}_r r\Phi) \tag{5b-103}$$

$$\Phi = \{z_n(kr)\} \begin{Bmatrix} P_n{}^m \\ Q_n{}^m \end{Bmatrix} (\cos \theta) \{e^{\pm im\phi}\} \tag{5b-104}$$

where $z_n(kr)$ are two linearly independent solutions to the spherical Bessel differential equation and are related to the cylinder function by the expression

$$z_n(kr) = \left(\frac{\pi}{2kr}\right)^{\frac{1}{2}} Z_{n+\frac{1}{2}}(kr)$$

$P_n{}^m(\cos \theta)$ and $Q_n{}^m(\cos \theta)$ are two linearly independent solutions to the associated Legendre differential equation.[1]

TE Waves

$$\mathbf{E} = \nabla \times (\mathbf{e}_r r\Psi) \qquad \mathbf{H} = \frac{1}{i\omega\mu} \nabla \times \nabla \times (\mathbf{e}_r r\Psi) \tag{5b-105}$$

$$\Psi = \{z_n(kr)\} \begin{Bmatrix} P_n{}^m \\ Q_n{}^m \end{Bmatrix} (\cos \theta) \{e^{\pm im\phi}\} \tag{5b-106}$$

4. Elliptical Cylinder Coordinates. In the elliptical coordinates[2] ξ, η, z the two distinct types of basic elliptical cylindrical wave functions, characterized by the relationship between the field vectors and the z axis (the direction of propagation), are

TM Waves

$$\mathbf{H}^{(e,0)} = \nabla \times (\mathbf{e}_z\Phi^{(e,0)}) \qquad \mathbf{E}^{(e,0)} = \frac{i}{\omega\epsilon} \nabla \times \nabla \times (\mathbf{e}_z\Phi^{(e,0)}) \tag{5b-107}$$

$$\Phi^{(e)} = \{Mc_n{}^{(1),(2)}(\xi,q)\}\{ce_n(\eta,q)\}\{e^{\pm i\gamma z}\} \tag{5b-108}$$

$$\Phi^{(0)} = \{Ms_n{}^{(1),(2)}(\xi,q)\}\{se_n(\eta,q)\}\{e^{\pm i\gamma z}\} \tag{5b-109}$$

$$q = \frac{c}{2}(k^2 - \gamma^2)^{\frac{1}{2}}$$

[1] W. Magnus and F. Oberhettinger, "Formulas and Theorems for the Functions of Mathematical Physics," Chelsea Publishing Company, New York, 1954.

[2] In terms of the rectangular coordinates x, y, the elliptical coordinates ξ, η are defined by the following relations: $x = c \cosh \xi \cos \eta$, $y = c \sinh \xi \sin \eta$ $(0 \leq \xi < \infty,\ 0 \leq \eta \leq 2\pi)$, where c is the semifocal length.

where $ce_n(\eta,q)$ and $se_n(\eta,q)$ are respectively even and odd periodic angular Mathieu functions. The radial Mathieu functions corresponding to the even function $ce_n(\eta,q)$ having the same characteristic values are $Mc_n^{(1),(2)}(\xi,q)$, and those corresponding to the odd function[1] $se_n(\eta,q)$ are $Ms_n^{(1),(2)}(\xi,q)$.

TE Waves

$$\mathbf{E}^{(e,0)} = \nabla \times (\mathbf{e}_z\Psi^{(e,0)}) \qquad \mathbf{H}^{(e,0)} = \frac{1}{i\omega\mu}\nabla \times \nabla \times (\mathbf{e}_z\Psi^{(e,0)}) \tag{5b-110}$$

$$\Psi^{(e)} = \{Mc_n^{(1),(2)}(\xi,q)\}\{ce_n(\eta,q)\}\{e^{\pm i\gamma z}\}$$

$$\Psi^{(0)} = \{Ms_n^{(1),(2)}(\xi,q)\}\{se_n(\eta,q)\}\{e^{+i\gamma z}\}$$

$$q = \frac{c}{2}(k^2 - \gamma^2)^{\frac{1}{2}}$$

5. Parabolic Cylinder Coordinates. In the parabolic coordinates[2] ξ, η, z the two distinct types of basic parabolic cylindrical wave functions, characterized by the relationship between the field vectors and the z axis (the direction of propagation), are:

TM Waves

$$\mathbf{H} = \nabla \times (\mathbf{e}_z\Phi) \qquad \mathbf{E} = \frac{i}{\omega\epsilon}\nabla \times \nabla \times (\mathbf{e}_z\Phi) \tag{5b-111}$$

$$\Phi = \{U_m^{(1),(2)}(\xi)\}\{V_m^{(1),(2)}(\eta)\}\{e^{\pm i\gamma z}\} \tag{5b-112}$$

where U and V satisfy Weber's equation of the confluent hypergeometric type,[3]

$$\left[\frac{\partial^2}{\partial\xi^2} + (q^2\xi^2 + m)\right]U(\xi) = 0 \tag{5b-113}$$

$$\left[\frac{\partial^2}{\partial\eta^2} + (q^2\eta^2 - m)\right]V(\eta) = 0 \tag{5b-114}$$

$$q^2 = k^2 - \gamma^2$$

TE Wave

$$\mathbf{E} = \nabla \times (\mathbf{e}_z\Psi) \qquad \mathbf{H} = \frac{1}{i\omega\mu}\nabla \times \nabla \times (\mathbf{e}_z\Psi) \tag{5b-115}$$

$$\Psi = \{U_m^{(1),(2)}(\xi)\}\{V_m^{(1),(2)}(\eta)\}\{e^{\pm i\gamma z}\} \tag{5b-116}$$

Polarization of Waves. Consider a plane wave in free space propagating in the z direction and having the following components:

$$\mathbf{E} = \mathbf{e}_xE_1e^{ikz-i\omega t} + \mathbf{e}_yE_2e^{ikz-i\omega t} \tag{5b-117}$$

$$\mathbf{B} = -\mathbf{e}_xE_2\sqrt{\mu_0\epsilon_0}\,e^{ikz-i\omega t} + \mathbf{e}_yE_1\sqrt{\mu_0\epsilon_0}\,e^{ikz-i\omega t} \tag{5b-118}$$

with $k = \omega\sqrt{\mu_0\epsilon_0}$. Note that (E_x,B_y) and (E_y,B_x) are linearly independent fields, and E_1 and E_2 are complex constants.

LINEARLY POLARIZED WAVE. E_1 and E_2 have the same phase. In this case $\mathbf{E}$ at any point in space oscillates along a directional line which makes a constant angle ϕ with the x axis, this angle being given by $\phi = \tan^{-1}(E_2/E_1)$.

CIRCULARLY POLARIZED WAVE. E_1 and E_2 have the same magnitude but their phases differ by 90°. Hence

$$\begin{aligned}\mathbf{E} &= \mathrm{Re}\ (\mathbf{e}_x \pm i\mathbf{e}_y)E_1e^{ikz-i\omega t}\\ &= E_1[\mathbf{e}_x\cos(\omega t - kz) \pm \mathbf{e}_y\sin(\omega t - kz)]\end{aligned}$$

Hence $\mathbf{E}$ at any point in space does not oscillate. Its magnitude is constant, but its direction rotates at the angular velocity ω. When $E_2 = -iE_1$, the wave is said to be

[1] J. Meixner and F. W. Schäfke, "Mathieu-funktionen und Spharoid-funktionen," Springer-Verlag OHG, Berlin, 1954.

[2] In terms of the rectangular coordinates x, y, the parabolic coordinates ξ, η are defined by the following relations: $x = \frac{1}{2}(\xi^2 - \eta^2)$, $y = \xi\eta$ $(-\infty < \xi < \infty, 0 \leq \eta < \infty)$.

[3] S. O. Rice, *Bell System Tech. J.* **33**, 417 (1954).

right-handed circularly polarized. When $E_2 = iE_1$, the wave is said to be left-handed circularly polarized.

ELLIPTICALLY POLARIZED WAVE. E_1 and E_2 have arbitrary relative amplitudes and phases. At any point in space the tip of **E** describes a locus which is an ellipse.

References

1. Stratton, J. A.: "Electromagnetic Theory," McGraw-Hill Book Company, New York, 1941.
2. Smythe, W. R.: "Static and Dynamic Electricity," 3d ed., McGraw-Hill Book Company, New York, 1968.
3. Elliott, R. S.: "Electromagnetics," McGraw-Hill Book Company, New York, 1966.
4. Papas, C. H.: "Theory of Electromagnetic Wave Propagation," McGraw-Hill Book Company, New York, 1965.
5. Van Bladel, J.: "Electromagnetic Fields," McGraw-Hill Book Company, New York, 1964.
6. Panofsky, W. K. H., and M. Phillips: "Classical Electricity and Magnetism," 2d ed., Addison-Wesley Publishing Company, Inc., Reading, Mass., 1962.
7. Kraichman, M. B., "Handbook of Electromagnetic Propagation in Conducting Media," NAVMAT P-2302, 1970, U.S. Government Printing Office, Washington, D.C. 20402.

5b-9. Guided Waves. In this section some basic properties of guided waves are given. These properties are found from the solutions that satisfy the source-free Maxwell's equations and the appropriate boundary conditions. When the guided waves propagate along a straight-line path, one may assume that every component of the electromagnetic wave may be represented in the form

$$f(u,v)e^{i\gamma z}e^{-i\omega t} \qquad (5b\text{-}119)$$

in which z is chosen as the propagation direction and u, v are generalized orthogonal coordinates in a transverse plane.[1] γ is the propagation constant. Under this assumption, the transverse field components in homogeneous isotropic medium (ϵ,μ) are

$$E_u = \frac{1}{\omega^2\mu\epsilon - \gamma^2}\left(\frac{i\gamma}{h_1}\frac{\partial E_z}{\partial u} + \frac{i\omega\mu}{h_2}\frac{\partial H_z}{\partial v}\right) \qquad (5b\text{-}120)$$

$$E_v = \frac{1}{\omega^2\mu\epsilon - \gamma^2}\left(\frac{i\gamma}{h_2}\frac{\partial E_z}{\partial v} - \frac{i\omega\mu}{h_1}\frac{\partial H_z}{\partial u}\right) \qquad (5b\text{-}121)$$

$$H_u = \frac{1}{\omega^2\mu\epsilon - \gamma^2}\left(\frac{-i\omega\epsilon}{h_2}\frac{\partial E_z}{\partial v} + \frac{i\gamma}{h_1}\frac{\partial H_z}{\partial u}\right) \qquad (5b\text{-}122)$$

$$H_v = \frac{1}{\omega^2\mu\epsilon - \gamma^2}\left(\frac{i\omega\epsilon}{h_1}\frac{\partial E_z}{\partial u} + \frac{i\gamma}{h_2}\frac{\partial H_z}{\partial v}\right) \qquad (5b\text{-}123)$$

and the longitudinal field components satisfy the following equation:

$$\left[\frac{1}{h_1h_2}\left(\frac{\partial}{\partial u}\frac{h_2}{h_1}\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\frac{h_1}{h_2}\frac{\partial}{\partial v}\right) + \Gamma^2\right]\left\{\begin{matrix}E_z\\H_z\end{matrix}\right\} = 0 \qquad \Gamma^2 = \omega^2\mu\epsilon - \gamma^2 \qquad (5b\text{-}124)$$

Only discrete values of Γ^2 will satisfy the boundary conditions. These allowed Γ^2 values are called *eigenvalues;* and corresponding to these eigenvalues are the *eigenfunctions.* The orthogonality properties of the field components can therefore be found according to the well-known orthogonality properties of the eigenfunction.

It will be recalled from Sec. 5b-8 (Basic Wave Types) that *TM* modes refer to waves having $H_z = 0$, *TE* modes having $E_z = 0$, HE modes having all field components $\neq 0$, and *TEM* modes having $E_z = 0$ and $H_z = 0$.

Propagation Characteristics. Propagation characteristics of guided waves refer to the behavior of the propagation constant γ as a function of frequency. In general,

[1] J. A. Stratton, "Electromagnetic Theory," chap. 1, McGraw-Hill Book Company, New York, 1941.

γ may be complex: $\gamma = i\alpha + \beta$, where α is the attenuation constant and β is a phase constant. Several commonly used terms to describe guided waves are defined as follows:

$$
\begin{aligned}
&\text{Cutoff frequency,} && f_c = \frac{\Gamma}{2\pi\sqrt{\mu\epsilon}} \\
&\text{Cutoff wavelength,} && \lambda_c = \frac{1}{f_c\sqrt{\mu\epsilon}} \\
&\text{Guide wavelength,} && \lambda_g = \frac{2\pi}{\gamma} = \frac{2\pi}{(k^2 - \Gamma^2)^{\frac{1}{2}}} \qquad k = \omega\sqrt{\mu\epsilon} \\
&\text{Phase velocity,} && v_p = \frac{\omega}{\gamma} \\
&\text{Group velocity,} && v_g = \frac{d\omega}{d\gamma} = \frac{\gamma}{\omega\epsilon\mu} \cdot \frac{1}{1 - (\Gamma/k)(d\Gamma/dk)}
\end{aligned}
\tag{5b-125}
$$

The above considerations are applicable for TE, TM, or HE waves only. For TEM modes, we have $\gamma = k$, $f_c = 0$, $v_p = \omega/k$, and $\lambda_g = 2\pi/k$ with $k = \omega\sqrt{\mu\epsilon}$.

Bounded Waveguides. Only TM waves and TE waves are physically possible in a cylindrical region bounded by a simply connected conducting region. However, in a coaxial region with perfectly conducting walls, a TEM as well as TM and TE waves can be present.

The propagation parameters for cylindrical waveguides bounded by good (but not perfectly) conducting walls are summarized as follows:

FOR PROPAGATING MODES, $f > f_c$

$$
\begin{aligned}
\alpha &= 0 \\
\gamma &= \beta = k\left[1 - \left(\frac{f_c}{f}\right)^2\right]^{\frac{1}{2}} \\
v_p &= \frac{\omega}{\beta} = \frac{1}{\sqrt{\mu\epsilon(1 - (f_c/f)^2)}} \\
v_g &= \frac{1}{\mu\epsilon v_p}
\end{aligned}
\tag{5b-126}
$$

FOR NONPROPAGATING MODES (THE EVANESCENT WAVES), $f < f_c$

$$
\begin{aligned}
\beta &= 0 \\
\gamma &= i\alpha = ik\left[\left(\frac{f_c}{f}\right)^2 - 1\right]^{\frac{1}{2}}
\end{aligned}
\tag{5b-127}
$$

ATTENUATION DUE TO IMPERFECTLY CONDUCTING WALLS

$$\alpha_w = \frac{\text{power loss}}{2\text{ power transfer}} = \frac{W_L}{2W_T} \qquad \text{nepers/m} \tag{5b-128}$$

$$W_L = \frac{R_s}{2}\oint_L [\|H_t\|^2 + \|H_z\|^2]\,dl \tag{5b-129}$$

$$W_T = \frac{1}{2}\int_A \|E_t\|\,\|H_t\|\,dS \tag{5b-130}$$

TM modes:

$$W_L \approx \frac{R_s}{8\pi^2\mu^2 f_c^2}\left(\frac{f}{f_c}\right)^2 \oint_L \left(\frac{\partial E_z}{\partial n}\right)^2 dl \tag{5b-131}$$

$$W_T \approx \frac{1}{2(\mu/\epsilon)^{\frac{1}{2}}}\left[1 - \left(\frac{f_c}{f}\right)^2\right]^{\frac{1}{2}} \left(\frac{f}{f_c}\right)^2 \int_A E_z^2\,dS \tag{5b-132}$$

TE modes:

$$W_L \approx \frac{R_s}{2}\oint_L \left[H_z^2 + \left(\frac{f}{f_c}\right)^2 \frac{1 - (f_c/f)^2}{4\pi^2 f_c^2\mu\epsilon}\left(\frac{\partial H_z}{\partial l}\right)^2\right] dl \tag{5b-133}$$

$$W_T \approx \left(\frac{\mu}{\epsilon}\right)^{\frac{1}{2}} \left(\frac{f}{f_c}\right)^2 \left[1 - \left(\frac{f_c}{f}\right)^2\right]^{\frac{1}{2}} \int_A H_z^2\,dS \tag{5b-134}$$

where $|H_t|^2$ is the square of the total transverse magnetic field, $\partial/\partial n$ is the normal derivative at the boundary conducting wall, $\partial H_z/\partial l$ is the derivative of H_z tangent to curve L along the cross-sectional bounding wall, A is the cross-sectional area of the guide, $R_s = (\pi f \mu/\sigma_c)^{\frac{1}{2}}$ is the surface resistance, and σ_c is the conductivity of the boundary conductor.

ATTENUATION DUE TO IMPERFECT DIELECTRIC

$$\alpha_d \approx \frac{(\mu/\epsilon_r)^{\frac{1}{2}}\sigma_d}{2[1 - (f_c/f)^2]^{\frac{1}{2}}} \qquad \frac{\sigma_d}{\omega\epsilon_r} \ll 1 \tag{5b-135}$$

where $\epsilon = \epsilon_r(1 - \sigma_d/i\omega\epsilon_r)$, σ_d is the conductivity of the dielectric in the guide, and ϵ_r is the real part of the dielectric constant ϵ. α_d is in nepers/meter.

The above approximate expressions for the attenuation constant are not valid for frequencies very close to the cutoff frequencies and for very high frequencies. It has also been assumed that the field configurations are not affected by the presence of small wall and dielectric losses.

Field components and propagation parameters for waves guided in rectangular and circular tubes are summarized in Table 5b-1. Table 5b-2 provides the field configurations for several lower-order modes in rectangular and circular waveguides. In a bounded waveguide, an arbitrary field **E** or **H** within the waveguide may be expanded in terms of the mode functions as follows:

$$\mathbf{E} = \sum_p A_p \mathbf{E}^{TM} + B_p \mathbf{E}^{TE}$$

$$\mathbf{H} = \sum_p B_p \mathbf{H}^{TE} + A_p \mathbf{H}^{TM}$$

i.e., the mode functions for TE and TM waves are a complete set.

For details concerning bounded waveguides of other simple shapes (such as elliptical, parabolic, triangular, etc.), the reader is referred to the literature.[1] For waveguides of arbitrary cross-sectional shape for which solutions in terms of known and tabulated eigenfunctions are not available, one must resort to numerical means[2] or to approximations based on variational techniques.[3] Numerically speaking, the problem reduces to finding the eigenvalue Γ which satisfies the Helmholz equation $(\nabla^2 + \Gamma^2)F = 0$ and the boundary condition $F = 0$ on C for TM waves and $\partial F/\partial n = 0$ on C for TE waves by the use of a computer. The well-known difference method has been used successfully for this type of problem.[3] The variational method offers a way to obtain a rather accurate value for the eigenvalue Γ which is related to the propagation constant γ by the relation $\gamma = \sqrt{\omega^2\mu\epsilon - \Gamma^2}$, from the knowledge of an approximate field configuration (i.e., a trial function). Specifically, for a TM modes, if a trial function $u(x,y)$ vanishes on the boundary and satisfies the conditions

$$\int_A uE_z^{(0)}\,dx\,dy = 0, \int_A uE_z^{(1)}\,dx\,dy = 0, \ldots, \int_A uE_z^{(n-1)}\,dx\,dy = 0 \tag{5b-136}$$

where $E_z^{(0)}, E_z^{(1)}, \ldots, E_z^{(n-1)}$ are the eigenfunctions for the equation

$$(\nabla^2 + \Gamma_n^2)E_z^{(n)} = 0$$

[1] F. E. Borgnis and C. H. Papas, Electromagnetic Waveguides and Resonators, "Handbuch der Physik," vol. 16, Springer-Verlag OHG, Berlin, 1958.

[2] R. F. Harrington, Field Computation by Moment Methods, the Macmillan Company, New York, 1968.

[3] F. E. Borgnis and C. H. Papas, "Randwertproblems der Mikrowellenphysik," Springer-Verlag OHG, Berlin, 1955.

TABLE 5b-1. FORMULAS FOR RECTANGULAR AND CIRCULAR WAVEGUIDES

TM waves		*TE* waves	
Field components	Propagation parameters	Field components	Propagation parameters
Rectangular waveguides			
TM_{mn} modes		*TE_{mn} modes*	
$E_z = E_0 \sin\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right)$	$(f_c)_{mn} = \frac{1}{2\sqrt{\mu\epsilon}}\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]^{\frac{1}{2}}$	$H_z = H_0 \cos\left(\frac{m\pi}{a}x\right) \cos\left(\frac{n\pi}{b}y\right)$	$(f_c)_{mn}$, $(\lambda_g)_{mn}$, γ for *TE* waves are the same as those for *TM* waves
$H_x = \frac{-i\omega\epsilon}{4\pi^2 f_c^2}\frac{\partial E_z}{\partial y}$		$E_x = \frac{i\omega\mu}{4\pi^2 f_c^2}\frac{\partial H_z}{\partial y}$	
$H_y = \frac{i\omega\epsilon}{4\pi^2 f_c^2}\frac{\partial E_z}{\partial x}$	$(\lambda_g)_{mn} = \frac{1/(f\sqrt{\mu\epsilon})}{\{1 - [(f_c)_{mn}/f]^2\}^{\frac{1}{2}}}$	$E_y = \frac{-i\omega\mu}{4\pi^2 f_c^2}\frac{\partial H_z}{\partial x}$	$(\alpha_w)_{mn} = \frac{2R_s}{b(\mu/\epsilon)^{\frac{1}{2}}[1 - (f_c/f)^2]^{\frac{1}{2}}}$
$E_x = \frac{i\gamma}{4\pi^2 f_c^2}\frac{\partial E_z}{\partial x}$	$\gamma = \frac{2\pi}{(\lambda_g)_{mn}}$	$H_x = \frac{i\gamma}{4\pi^2 f_c^2}\frac{\partial H_z}{\partial x}$	$\times \left\{\left(1 + \frac{b}{a}\right)\left(\frac{f_c}{f}\right)^2 + \left[1 - \left(\frac{f_c}{f}\right)^2\right]\right.$
$E_y = \frac{i\gamma}{4\pi^2 f_c^2}\frac{\partial E_z}{\partial y}$	$(\alpha_w)_{mn} = \frac{2R_s}{b(\mu/\epsilon)^{\frac{1}{2}}[1 - (f_c/f)^2]^{\frac{1}{2}}}$	$H_y = \frac{i\gamma}{4\pi^2 f_c^2}\frac{\partial H_z}{\partial y}$	$\left.\cdot\frac{(b/a)[(b/a)m^2 + n^2]}{(b^2m^2/a^2) + n^2}\right\}$
$H_z = 0$	$\times\left[\frac{m^2(b/a)^3 + n^2}{m^2(b/a)^2 + n^2}\right]$	$E_z = 0$	

Notes: 1. a = length of waveguide along x axis; b = length of the waveguide along y axis
2. m, n are integers; for *TM* waves $m \neq 0$, $n \neq 0$; For *TE* waves, either m or n may be zero, but not both.
3. Dominant mode is the TE_{10} wave having the lowest cutoff frequency.

TM waves		*TE* waves	
Circular waveguides			
TM_{nl} modes		*TE_{nl} modes*	
$\left\{E_z = E_0 J_n(\Gamma_c\rho)\begin{Bmatrix}\cos\\ \sin\end{Bmatrix} n\phi\right.$	$(\Gamma_c)_{nl} = \frac{p_{nl}}{a}$; p_{nl} = lth root of $J_n(p_{nl}) = 0$	$H_z = H_0 J_n(\Gamma_c\rho)\left\{\begin{matrix}\cos\\ \sin\end{matrix} n\phi\right\}$	$(\Gamma_c)_{nl} = \frac{p'_{nl}}{a}$; p'_{nl} = lth roots of $J_n'(p'_{nl}) = 0$
$H_z = 0$	$(\lambda_c)_{nl} = \frac{2\pi a}{p_{nl}}$	$E_z = 0$	$(\lambda_c)_{nl} = \frac{2\pi a}{p'_{nl}}$
$H\rho = \frac{-i\omega\epsilon}{4\pi^2 f_c^2\rho}\frac{\partial E_z}{\partial\phi}$	$(f_c)_{nl} = \frac{p_{nl}}{2\pi a\sqrt{\mu\epsilon}}$	$E\rho = \frac{i\omega\mu}{4\pi^2 f_c^2\rho}\frac{\partial H_z}{\partial\phi}$	$(f_c)_{n\mu} = \frac{p'_{nl}}{2\pi a\sqrt{\mu\epsilon}}$
$H\phi = \frac{i\omega\epsilon}{4\pi^2 f_c^2}\frac{\partial E_z}{\partial\rho}$	$(\lambda_g)_{nl} = \frac{1/(f\sqrt{\mu\epsilon})}{\{1 - [f_c/f]^2\}^{\frac{1}{2}}}$	$E\phi = \frac{-i\omega\mu}{4\pi^2 f_c^2}\frac{\partial H_z}{\partial\rho}$	$(\lambda_g)_{nl} = \frac{1/(f\sqrt{\mu\epsilon})}{\{1 - [f_c/f]^2\}^{\frac{1}{2}}}$
$E\rho = \frac{i\gamma}{4\pi^2 f_c^2}\frac{\partial E_z}{\partial\rho}$	$\gamma = \frac{2\pi}{(\lambda_g)_n}$	$H\rho = \frac{i\gamma}{4\pi^2 f_c^2}\frac{\partial H_z}{\partial\rho}$	$\gamma = \frac{2\pi}{(\lambda_g)_{nl}}$
$E\phi = \frac{i\gamma}{4\pi^2 f_c^2\rho}\frac{\partial E_z}{\partial\phi}$	$(\alpha_w)_{nl} = \frac{R_s}{a(\mu/\epsilon)^{\frac{1}{2}}[1 - (f_c/f)^2]^{\frac{1}{2}}}$	$H\phi = \frac{i\gamma}{4\pi^2 f_c^2\rho}\frac{\partial H_z}{\partial\phi}$	$(\alpha_w)_{nl} = \frac{R_s}{a(\mu/\epsilon)[1 - (f_c/f)^2]^{\frac{1}{2}}} \times\left[\left(\frac{f_c}{f}\right)^2 + \frac{n^2}{(p'_{nl})^2 - n^2}\right]$

Notes:
1. a = radius of the waveguide
2. n, l are integers, for both *TE* and *TM* Wave, n can be zero but not l.
3. Dominant mode is the TM_{11} wave having the lowest cutoff frequency; TE_{01} mode is the only mode whose α_w decreases monotonically as frequencies increase.

TABLE 5b-2. FIELD CONFIGURATIONS FOR SEVERAL LOWER-ORDER MODES IN RECTANGULAR AND CIRCULAR GUIDES

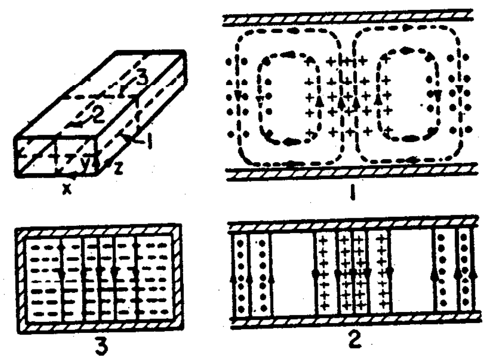
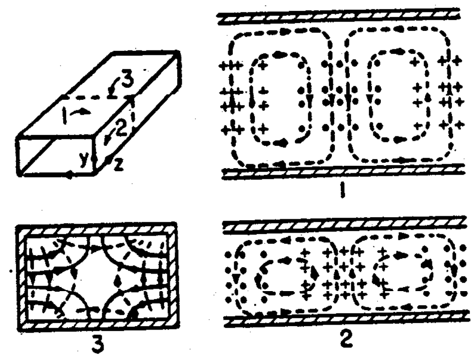
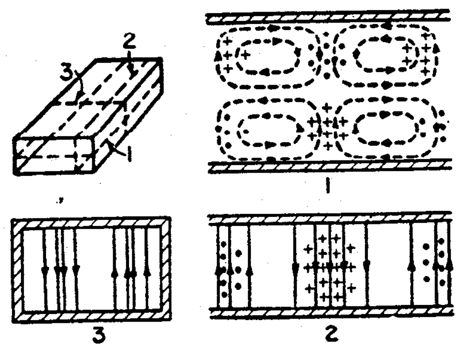
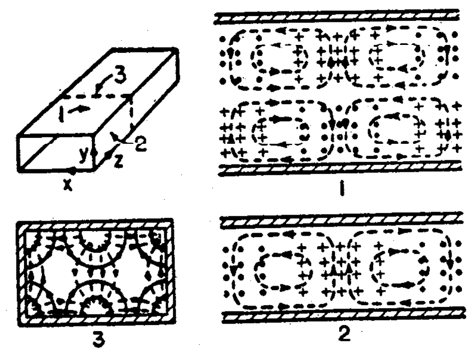
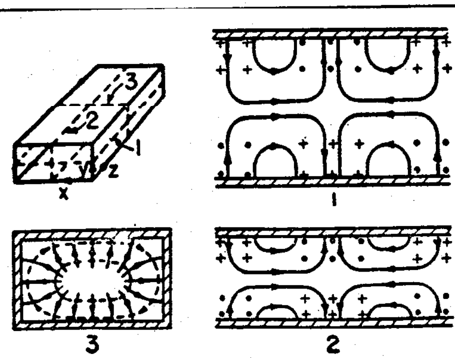
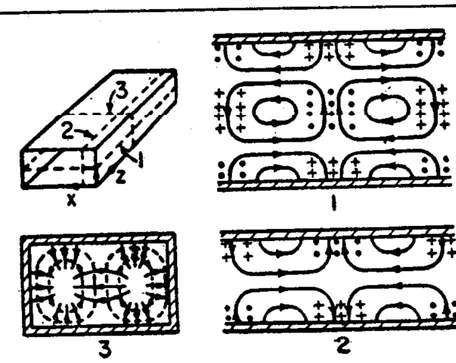
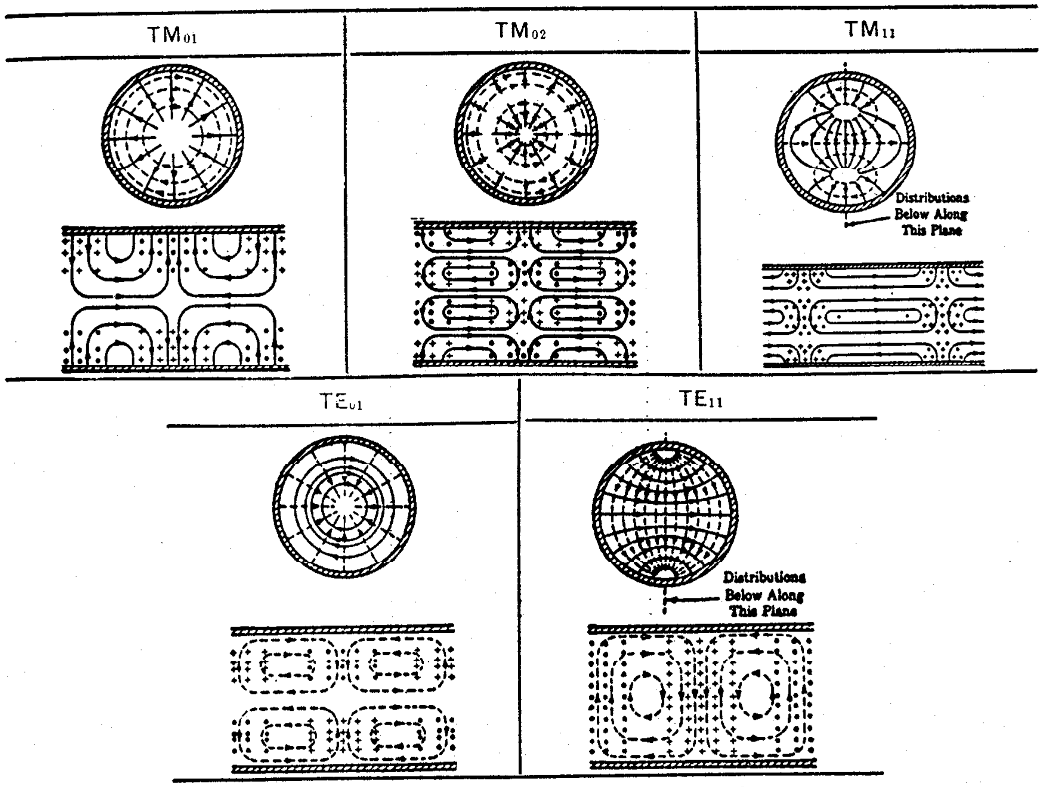
TE_{10}	TE_{11}
	
TE_{20}	TE_{21}
	
TM_{11}	TM_{21}
	

TABLE 5b-2. FIELD CONFIGURATIONS FOR SEVERAL LOWER-ORDER MODES IN RECTANGULAR AND CIRCULAR GUIDES (*Continued*)



Note 1. The solid dots represent vectors coming out of the paper, and the crosses represent vectors going into the paper.

Note 2. The solid lines represent electric lines of force, and the dotted lines represent magnetic lines of force.

with $E_z^{(n)} = 0$ on the boundary, then, for $n > 0$,

$$\Gamma_n^2 \leq \frac{\int_A (\nabla u)^2 \, dx \, dy}{\int_A u^2 \, dx \, dy} \tag{5b-137}$$

for a TE mode, if a trial function $v(x,y)$ which are not subjected to any boundary conditions on the boundary satisfies the conditions

$$\int_A vH_z^{(0)} \, dx \, dy = 0, \int vH_z^{(1)} \, dx \, dy = 0, \; . \; . \; . \; , \int vH_z^{(n-1)} \, dx \, dy = 0 \tag{5b-138}$$

where $H_z^{(0)}, H_z^{(1)}, \; . \; . \; . \; , H_z^{(n-1)}$ are the eigenfunctions for the equation

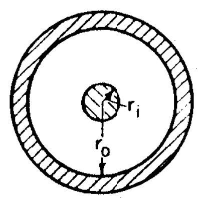
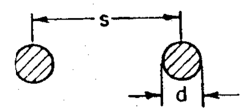
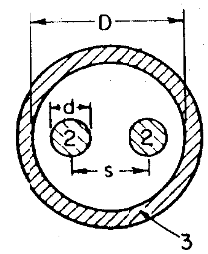
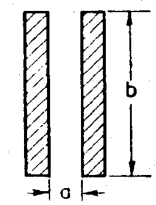
$$(\nabla^2 + \Gamma_n^2)H_z^{(n)} = 0$$

with $\partial H_z^{(n)}/\partial n = 0$ on the boundary then, for $n > 0$,

$$\Gamma_n^2 \leq \frac{\int_A (\nabla v)^2 \, dx \, dy}{\int_A v^2 \, dx \, dy} \tag{5b-139}$$

Both $u(x,y)$ and $v(x,y)$ must be continuous within the bound with sectionally continuous derivatives.

TABLE 5b-3. SOME CONSTANTS OF COAXIAL, PARALLEL-WIRE, SHIELDED PAIRS AND PARALLEL-BAR TRANSMISSION LINES*

			$p = \frac{s}{d}$, $q = \frac{s}{D}$ 	 Formulas for $a \ll b$
Capacitance C, farads/m	$\frac{2\pi\epsilon}{\ln\left(\frac{r_0}{r_i}\right)}$	$\frac{\pi\epsilon}{\cosh^{-1}\left(\frac{s}{d}\right)}$	....	$\frac{\epsilon b}{a}$
External inductance L, henrys/m	$\frac{\mu}{2\pi}\ln\left(\frac{r_0}{r_i}\right)$	$\frac{\mu}{\pi}\cosh^{-1}\left(\frac{s}{d}\right)$	....	$\mu\frac{a}{b}$
Conductance G, mhos/m	$\frac{2\pi\sigma}{\ln\left(\frac{r_0}{r_i}\right)} = \frac{2\pi\omega\epsilon_v\epsilon_r''}{\ln\left(\frac{r_0}{r_i}\right)}$	$\frac{\pi\sigma}{\cosh^{-1}\left(\frac{s}{d}\right)} = \frac{\pi\omega\epsilon_v\epsilon_r''}{\cosh^{-1}\left(\frac{s}{d}\right)}$	....	$\frac{\sigma b}{a} = \frac{\omega\epsilon_v\epsilon_r' b}{a}$
Resistance R, ohms/m	$\frac{R_s}{2\pi}\left(\frac{1}{r_0} + \frac{1}{r_i}\right)$	$\frac{2R_s}{\pi d}\left[\frac{s/d}{\sqrt{(s/d)^2 - 1}}\right]$	$\frac{2R_{s2}}{\pi d}\left[1 + \frac{1 + 2p^2}{4p^4}(1 - 4q^2)\right] + \frac{8R_{s3}}{\pi D} q^2 \left[1 + q^2 - \frac{1 + 4p^2}{8p^4}\right]$	$\frac{2R_s}{b}$
Internal inductance L_i, henrys/m (for high frequency)	← $\frac{R}{\omega}$ →			

Characteristic impedance at high frequency Z_0, ohms	$\frac{\eta}{2\pi}\ln\left(\frac{r_0}{r_i}\right)$	$\frac{\eta}{\pi}\cosh^{-1}\left(\frac{s}{d}\right)$	$\frac{\eta_1}{\pi}\left\{\ln\left[2p\left(\frac{1-q^2}{1+q^2}\right)\right] - \frac{1+4p^2}{16p^4}(1-4q^2)\right\}$	$\eta\frac{a}{b}$
Z_0 for air dielectric	$60\ln\left(\frac{r_0}{r_i}\right)$	$120\cosh^{-1}\left(\frac{s}{d}\right) \cong 120\ln\left(\frac{2s}{d}\right)$ if $s/d \gg 1$	$120\left\{\ln\left[2p\frac{(1-q^2)}{(1+q^2)}\right] - \frac{1+4p^2}{16p^4}(1-4q^2)\right\}$	$120\pi\frac{a}{b}$
Attenuation due to conductor α_c	← $\frac{R}{2Z_0}$ →			
Attenuation due to dielectric α_d	← $\frac{GZ_0}{2} = \frac{\sigma\eta}{2} = \frac{\pi\sqrt{\epsilon_r'\mu_r'}}{\lambda_0}\left(\frac{\epsilon''_r}{\epsilon'_r}\right)$ →			
Total attenuation, db/m	← $8.686(\alpha_c + \alpha_d)$ →			
Phase constant for low-loss lines β	← $\omega\sqrt{\mu\epsilon} = \frac{2\pi}{\lambda}$ →			

All units above are mks.
For the dielectric:

$\epsilon = \epsilon_r'\epsilon_v$ = dielectric constant, farads/m
$\mu = \mu_r'\mu_v$ = permeability, henrys/m
$\eta = \sqrt{\mu/\epsilon}$ ohms

ϵ_r'' = loss factor of dielectric $= \sigma/\omega\epsilon_v$
R_s = skin-effect surface resistivity of conductor, ohms
λ = wavelength in dielectric $= \lambda_0/\sqrt{\epsilon_r'\mu_r'}$

* Ramo and Whinnery, "Fields and Waves in Modern Radio," 2d ed., John Wiley & Sons, Inc., New York, 1953. Formulas for shielded pair obtained from Green, Leibe, and Curtis, *Bell System Tech. J.* **15**, 248–284 (April, 1936).

Conventional TEM Transmission Lines. For a two-conductor uniform line supporting the *TEM* waves, the differential equations for the voltage V and current I are

$$\frac{\partial V}{\partial z} = -L\frac{\partial I}{\partial t} - RI \tag{5b-140}$$

$$\frac{\partial I}{\partial z} = -C\frac{\partial V}{\partial t} - GV \tag{5b-141}$$

where L, C, R, and G are the inductance, capacitance, resistance, and conductance, respectively, all per unit length of the line.

If steady-state sinusoidal conditions of the form $e^{-i\omega t}$ are considered, then the equations become

$$\frac{\partial V}{\partial z} = -(R - i\omega L)I \tag{5b-142}$$

$$\frac{\partial I}{\partial z} = -(G - i\omega C)V \tag{5b-143}$$

Combining the above equations gives

$$\left(\frac{d^2}{dz^2} - \chi^2\right)\begin{Bmatrix} I \\ V \end{Bmatrix} = 0 \tag{5b-144}$$

where the propagation constant

$$\chi = \sqrt{(R - i\omega L)(G - i\omega C)} = \alpha + i\beta \tag{5b-145}$$

The solution for Eq. (5b-144) is

$$V = Ae^{-\chi z} + Be^{\chi z} \tag{5b-146}$$

$$I = \frac{1}{Z_0}(Ae^{-\chi z} - Be^{\chi z}) \tag{5b-147}$$

where $Z_0 = \sqrt{(R - i\omega L)/(G - i\omega C)}$ and is called the characteristic impedance. A and B are constants to be determined according to the input and termination conditions. Tables 5b-3 and 5b-4 summarize constants for some common lines and some important formulas for transmission lines.

Another kind of quasi-*TEM* microwave transmission line is the strip line[1] which basically consists of two (or more) parallel metallic strips of generally different width separated by a dielectric medium. This structure cannot support a *TEM* wave although the dominant mode closely resembles the *TEM* wave of a simplified microstrip with dielectric material uniformly filling the entire region. Under this *TEM* wave approximation, the problem is essentially one of finding the electrostatic potential $\Phi(x,y)$ which satisfies the Laplace's equation $\nabla^2\Phi = 0$ and the boundary conditions $\Phi = \Phi_1$ on surface C_1 and $\Phi = \Phi_2$ on surface C_2. $\mathbf{E}_{\text{transverse}} = \nabla\Phi e^{\pm ikz}$ and $\mathbf{H}_{\text{transverse}} = \sqrt{\frac{\epsilon}{\mu}}(\mathbf{e}_z \times \nabla\Phi)e^{\pm ikz}$ with $k = \omega\sqrt{\mu\epsilon}$. The characteristic impedance of the line is

$$Z_c = \sqrt{\frac{\mu}{\epsilon}}\frac{(\Phi_1 - \Phi_2)^2}{\int_A \nabla\Phi \cdot \nabla\Phi\, dA} = \frac{\sqrt{\epsilon\mu}}{C}$$

where C is the capacitance of the structure per unit length.

Surface Waveguides. Another family of waveguides which is capable of guiding electromagnetic waves is the open-boundary structures. These structures consist of dielectric-coated planes and wires, corrugated planes and wires, interface between

[1] R. E. Collin, "Field Theory of Guided Waves," chap. 4, McGraw-Hill Book Company, New York, 1960.

TABLE 5b-4. SEVERAL IMPORTANT FORMULAS FOR SOME COMMON TRANSMISSION LINES*

Quantity	General line	Ideal line	Approximate results for low-loss lines
Propagation constant $\gamma = \alpha - i\beta$.	$\sqrt{(R - i\omega L)(G - i\omega C)}$	$-i\omega\sqrt{LC}$	(See α and β below)
Phase constant β	$\mathrm{Im}\,(\gamma)$	$\omega\sqrt{LC} = \frac{\omega}{v} = \frac{2\pi}{\lambda}$	$\omega\sqrt{LC}\left(1 - \frac{RG}{4\omega^2 LC} + \frac{G^2}{8\omega^2 C^2} + \frac{R^2}{8\omega^2 L^2}\right)$
Attenuation constant α	$\mathrm{Re}\,(\gamma)$	0	$\frac{R}{2Z_0} + \frac{GZ_0}{2}$
Characteristic impedance Z_0	$\sqrt{\frac{R - i\omega L}{G - i\omega C}}$	$\sqrt{\frac{L}{C}}$	$\sqrt{\frac{L}{C}}\left[1 - i\left(\frac{G}{2\omega C} - \frac{R}{2\omega L}\right)\right]$
Input impedance Z_i	$Z_0\left(\frac{Z_L \cosh\gamma l - Z_0 \sinh\gamma l}{Z_0 \cosh\gamma l + Z_L \sinh\gamma l}\right)$	$Z_0\left(\frac{Z_L \cos\beta l - iZ_0 \sin\beta l}{Z_0 \cos\beta l - iZ_L \sin\beta l}\right)$	
Impedance of shorted line	$Z_0 \tanh\gamma l$	$-iZ_0 \tan\beta l$	$Z_0\left(\frac{\alpha l \cos\beta l - i \sin\beta l}{\cos\beta l - i\alpha l \sin\beta l}\right)$
Impedance of open line	$Z_0 \coth\gamma l$	$+iZ_0 \cot\beta l$	$Z_0\left(\frac{\cos\beta l - i\alpha l \sin\beta l}{\alpha l \cos\beta l - i \sin\beta l}\right)$
Impedance of quarter-wave line	$Z_0\left(\frac{Z_L \sinh\alpha l + Z_0 \cosh\alpha l}{Z_0 \sinh\alpha l + Z_L \cosh\alpha l}\right)$	$\frac{Z_0^2}{Z_L}$	$Z_0\left(\frac{Z_0 + Z_L\alpha l}{Z_L + Z_0\alpha l}\right)$
Impedance of half-wave line	$Z_0\left(\frac{Z_L \cosh\alpha l + Z_0 \sinh\alpha l}{Z_0 \cosh\alpha l + Z_L \sinh\alpha l}\right)$	Z_L	$Z_0\left(\frac{Z_L + Z_0\alpha l}{Z_0 + Z_L\alpha l}\right)$
Voltage along line $V(z)$	$V_i \cosh\gamma z - I_i Z_0 \sinh\gamma z$	$V_i \cos\beta z + iI_i Z_0 \sin\beta z$	
Current along line $I(z)$	$I_i \cosh\gamma z - \frac{V_i}{Z_0}\sinh\gamma z$	$I_i \cos\beta z + i\frac{V_i}{Z_0}\sin\beta z$	
Reflection coefficient K_R	$\frac{Z_L - Z_0}{Z_L + Z_0}$	$\frac{Z_L - Z_0}{Z_L + Z_0}$	
Standing-wave ratio	$\frac{1 + \lvert K_R\rvert}{1 - \lvert K_R\rvert}$	$\frac{1 + \lvert K_R\rvert}{1 - \lvert K_R\rvert}$	

R, L, G, C = distributed resistance, inductance, conductance, capacitance per unit length
l = length of line
Subscript i denotes input end quantities.
Subscript L denotes load end quantities.

z = distance along line from input end
λ = wavelength measured along line
v = phase velocity of line equals velocity of light in dielectric of line for an ideal line

* Ramo and Whinnery, "Fields and Waves in Modern Radio," 2d ed., John Wiley & Sons, Inc., New York, 1953.

two different media. Special features of surface-wave modes having the usual propagation constant $e^{i\gamma z}$ along the axis and the structure are given in the following:

1. The field is characterized by an exponential decay away from the surface of the structure.
2. In most cases in which ϵ/ϵ_0, $\mu/\mu_0 > 1$, the phase velocities of the propagating surface-wave modes are less than the velocity of light in vacuum.
3. Below the cutoff frequency, a mode simply does not exist. In other words unlike the bounded waveguide case no evanescent mode exists.
4. The finite number of discrete surface-wave modes does not represent a complete set of solutions. In addition to the eigenfunction solutions there exists solutions with a continuous spectrum. (This property is in direct contrast to the mode property in bounded waveguides.)
5. Only TE, TM, or HE modes may exist on a surface-wave structure.

Detailed formulas are given for the circular dielectric waveguide as a representative surface-wave structure. It is understood that all fields vary as $e^{i\gamma z - i\omega t}$. The dielectric rod of radius a, having ϵ_1 and μ_0 as its permittivity and permeability, is assumed to be embedded in another dielectric medium with $\epsilon = \epsilon_0$ and $\mu = \mu_0$. Furthermore $\epsilon_1 > \epsilon_0$.

FIELD COMPONENTS

1. HE_{nm} modes with $n \neq 0$:

$$E_z = A_n J_n(s_1 r) \cos n\phi \qquad r \leq a \tag{5b-148}$$

$$= B_n K_n(s_0 r) \cos n\phi \qquad r \geq a \tag{5b-149}$$

$$H_z = C_n J_n(s_1 r) \sin n\phi \qquad r \leq a \tag{5b-150}$$

$$= D_n K_n(s_0 r) \sin n\phi \qquad r \geq a \tag{5b-151}$$

2. TM_{om} modes:

$$\begin{aligned} E_z &= A_o J_0(s_1 r) && r \leq a \\ &= B_0 K_0(s_0 r) && r \geq a \\ H_z &= 0 && \text{for all } r \end{aligned}$$

3. TE_{om} modes:

$$\begin{aligned} E_z &= 0 && \text{for all } r \\ E_z &= C_0 J_0(s_1 r) && r \leq a \\ &= D_0 K_0(s_0 r) && r \geq a \end{aligned}$$

All other transverse field components may be found from Eqs. (5b-120) to (5b-123) with $\epsilon = \epsilon_1$, $\mu = \mu_0$ for $r \leq a$ and $\epsilon = \epsilon_0$, $\mu = \mu_0$ for $r \geq a$. A_n, B_n, C_n, D_n are amplitude coefficients. J_n and K_n are respectively the Bessel and modified Bessel functions.

PROPAGATION CONSTANT. The propagation constant γ is obtained by solving the following equations:

1. HE_{nm} modes ($n \neq 0$):

$$\left[\frac{(\epsilon_1/\epsilon_0)J_n'(s_1 a)}{s_1 a J_n(s_1 a)} + \frac{K_n'(s_0 a)}{s_0 a K_n(s_0 a)}\right]\left[\frac{J_n'(s_1 a)}{s_1 a J_n(s_1 a)} + \frac{K_n'(s_0 a)}{s_0 a K_n(s_0 a)}\right] = n^2\left(\frac{1}{s_0^2 a^2} + \frac{\epsilon_1/\epsilon_0}{s_1^2 a^2}\right)\left(\frac{1}{s_0^2 a^2} + \frac{1}{s_1^2 a^2}\right) \tag{5b-152}$$

$$s_0^2 a^2 + s_1^2 a^2 = \omega^2 \mu_0 \epsilon_0 a^2 \left(\frac{\epsilon_1}{\epsilon_0} - 1\right) \tag{5b-153}$$

$$\gamma^2 = \omega^2 \mu_0 \epsilon_0 + s_0^2 = \omega^2 \mu_0 \epsilon_1 - s_1^2 \tag{5b-154}$$

2. TM_{om} modes:

$$\frac{(\epsilon_1/\epsilon_0)J_0'(s_1 a)}{s_1 a J_0(s_1 a)} + \frac{K_0'(s_0 a)}{s_0 a K_0(s_0 a)} = 0 \tag{5b-155}$$

with Eqs. (5b-153) and (5b-154)

3. TE_{om} modes:

$$\frac{J_0'(s_1a)}{s_1aJ_0(s_1a)} + \frac{K_0'(s_0a)}{s_0aK_0(s_0a)} = 0 \qquad (5b\text{-}156)$$

with Eqs. (5b-153) and (5b-154).

Numerical solutions of the above equations show that only the HE_{11} mode processes no cutoff frequency. The propagation constants as a function of frequency for the three lowest-order modes are given by Fig. 5b-1.

ATTENUATION. If the dielectric ϵ_1 for the cylindrical rod is imperfect and has a small conductivity σ_d, then there exists an attenuation constant α_d for all field components (i.e., all field components vary as $e^{-\alpha_d z}e^{i\gamma z}e^{-i\omega t}$).

$$\alpha_d = 4.343\sigma_d \sqrt{\frac{\mu_0}{\epsilon_0}} \left| \frac{\int_{A_i} (\mathbf{E}\cdot\mathbf{E})\,dA}{\frac{\mu_0}{\epsilon_0}\int_{A_i+A_0} (\mathbf{E}_t \times \mathbf{H}_t^*)\cdot \mathbf{e}_z\,dA} \right| \qquad (5b\text{-}157a)$$

where α_d is in db/meter, $\mathbf{E}_t$ and $\mathbf{H}_t$ are the transverse fields, A_i is the cross-sectional area of the dielectric rod, and $A_i + A_0$ is the total cross-sectional area. Figures 5b-1 and 5b-2 show respectively the propagation constant γ and the attenuation α_d as a function of frequency for the three lowest-order modes.

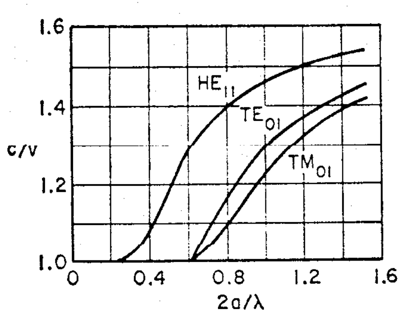


FIG. 5b-1. Velocity ratio c/v for polystyrene rod ($\epsilon_1 = 2.56$) embedded in free space. v is the phase velocity of the surface wave; c is the velocity of light in free space. a is the radius of the rod. λ is the free-space wavelength. [*From data obtained from W. Elsasser, J. Appl. Phys.* **20**, 1193 (1949).]

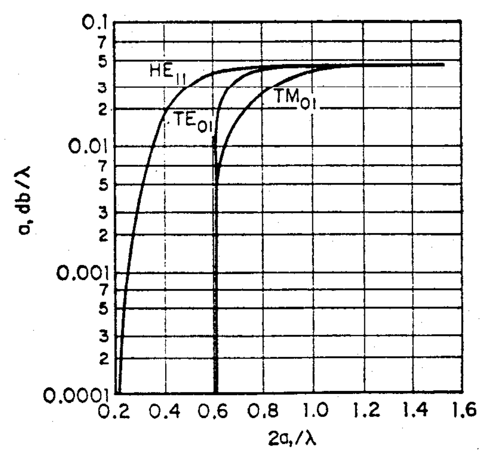


FIG. 5b-2. Attenuation for some surface wave modes along polystyrene rod of radius a ($\epsilon_1 = 2.56$ and $\tan\delta = 0.001$). Note that the attenuation α is in db/λ which is equal to 8.686 nepers/λ, where λ is the free-space wavelength. [*From data obtained from W. Elsasser, J. Appl. Phys.* **20**, 1193 (1949).]

If conductors are included as part of the surface-wave structures (such as the dielectric-coated wires), then in addition to the dielectric loss there is an attenuation constant α_c associated with the loss due to the finite conductivity of the conductors; i.e.,

$$\alpha_c = 4.343 \sqrt{\frac{\mu_0}{\epsilon_0}} \left| \frac{R_s/2 \oint_L |H_t|^2\,dl}{\frac{\mu_0}{\epsilon_0}\int_{A_i+A_0} (\mathbf{E}_t \times \mathbf{H}_t^*)\cdot \mathbf{e}_z\,dA} \right| \text{ db/m} \qquad (5b\text{-}157b)$$

where L is the cross-sectional curve around the conductor. In the case of the dielectric-coated wire, there are two dominate modes that have zero cutoff frequencies: the HE_{11} mode and the TM_{01} mode.

These formulas for the dielectric rod case are particularly useful in the study of fiber optics. Much more involved formulas for other types of surface-wave guides are also available; but for these the reader is referred to the literature.[1]

Inhomogeneously Filled Waveguides. Waveguides filled nonuniformly with homogeneous dielectrics or filled with inhomogeneous dielectrics offer many practical applications such as phase changers, matching transformers, etc. Previous formulas for homogeneously filled waveguides are not applicable for the present situation. Because of the complexity of the problem only several special cases have been treated. A procedure for deriving the electric and magnetic field components is given in the following for an inhomogeneously filled rectangular waveguide case[2]:

It is assumed that the nonuniformity is only in one of the three coordinate directions, say $\mathbf{e}_\xi$ where ξ may be x, y, or z. Then derive the electric and magnetic field components from the scalar potentials Ψ, Φ as follows:

$$\mathbf{E}_1 = \nabla \times (\mathbf{e}_\xi \Psi) \qquad (5b\text{-}158)$$

$$\mathbf{H}_1 = \frac{1}{i\omega\mu} \nabla \times \nabla \times (\mathbf{e}_\xi \Psi) \qquad (5b\text{-}159)$$

and

$$\mathbf{H}_2 = \nabla \times (\mathbf{e}_\xi \Phi) \qquad (5b\text{-}160)$$

$$\mathbf{E}_2 = \frac{i}{\omega\epsilon} \nabla \times \nabla \times (\mathbf{e}_\xi \Phi) \qquad (5b\text{-}161)$$

with $\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2$, $\mathbf{H} = \mathbf{H}_1 + \mathbf{H}_2$ in the general case. Substituting Eq. (5b-158) into Eq. (5b-58) gives a wave equation for Ψ and substituting Eq. (56-160) into Eq. (5b-59) gives a wave equation for Φ. Solving these differential equations for Φ and Ψ and satisfying the appropriate boundary conditions gives the solution of the problem. The above procedure is workable for rectangular waveguides filled nonuniformly with homogeneous dielectrics or filled with inhomogeneous dielectrics.

For a circular cylindrical waveguide filled with inhomogeneous dielectrics, the above procedure is, in general, not workable. This is because the resultant wave equation for Φ or Ψ is not separable. However, for special cases the procedure is still very useful. For example, when $\epsilon = \epsilon(z)$, $\mu = \mu_0$, Eqs. (5b-158) to (5b-161) may still be used to give the full set of solutions. When $\epsilon = \epsilon(r)$, $\mu = \mu_0$ in the cylindrical coordinates r, ϕ, z; only the circularly symmetric modes may be derived from Eqs. (5b-158) to (5b-161) with $\mathbf{e}_\xi = \mathbf{e}_r$.

It is noted that when a waveguide is filled by certain piecewise-homogeneous dielectrics (such as a circular waveguide filled with a concentric dielectric of different radius), the field components may still be derived for the region in which the dielectric is homogeneous and the complete solutions are obtained by matching the boundary conditions at the discontinuity.

Anisotropic Wave Propagation. A typical material having anisotropic electromagnetic property is the ferrite. In rectangular coordinates, the ferrite medium is characterized by the following relations:

$$\mathbf{B} = \boldsymbol{\mu} \cdot \mathbf{H} \qquad (5b\text{-}162)$$

$$\boldsymbol{\mu} = \begin{bmatrix} \mu_1 & -i\mu_2 & 0 \\ i\mu_2 & \mu_1 & 0 \\ 0 & 0 & \mu_3 \end{bmatrix} \qquad (5b\text{-}163)$$

$$\begin{aligned} \mu_1 &= \mu_0 \left(1 + \frac{\omega_c \omega_M}{\omega_c^2 - \omega^2}\right) & \omega_c &= \left|\frac{e}{m}\right| B_0 \\ \mu_2 &= \mu_0 \frac{\omega\omega_M}{\omega_c^2 - \omega^2} & \omega_M &= \left|\frac{e}{m}\right| \mu_0 M_0 \\ \mu_3 &= \mu_0 \end{aligned} \qquad (5b\text{-}164)$$

[1] G. Goubau, *J. Appl. Phys.* **21,** 119 (1950); C. Yeh, *J. Appl. Phys.* **33,** 3235 (1962).
[2] C. Yeh and K. F. Casey, *IEEE Trans. Microwave Theory and Tech.* **MTT-13,** 297 (1965).

with the applied d-c magnetic field B_0 in the z direction. M_0 is the internal magnetization.

FARADAY ROTATIONS. Solutions for plane-wave propagating in the ferrite medium in a direction parallel to the applied static magnetic field $B_0\mathbf{e}_z$ are

$$\mathbf{E}^{\pm} = (\mathbf{e}_x \pm i\mathbf{e}_y)A^{\pm}e^{i\gamma^{\pm}z}e^{-i\omega t} \tag{5b-165}$$

with

$$\gamma^{\pm} = \omega\sqrt{\epsilon(\mu_1 \mp \mu_2)} \tag{5b-166}$$

Equations (5b-165) and (5b-166) are derived directly from Maxwell's equations and Eqs. (5b-162) and (5b-163). It is recognized that $\mathbf{E}^+$ is the right-handed circularly polarized plane wave of amplitude A^+ and propagation constant γ^+, and $\mathbf{E}^-$ is the left-handed circularly polarized plane wave of amplitude A^- and propagation constant γ^-. A linearly polarized plane wave may be resolved into two counterrotating equal-amplitude circularly polarized waves, i.e., a linearly polarized wave $\mathbf{E} = \mathbf{E}^+ + \mathbf{E}^-$ with $A^+ = A^-$. Since the two circularly polarized component waves propagate at different velocities, the linearly polarized wave in the ferrite is rotated. When the wave is propagating in the $+z$ direction, the angle of rotation θ is given by

$$\begin{aligned}\theta = \tan^{-1}\frac{E_y}{E_x} &= \tan^{-1} i\,\frac{e^{i\gamma^-z} - e^{i\gamma^+z}}{e^{i\gamma^-z} + e^{i\gamma^+z}} \\ &= \frac{1}{2}(\gamma^+ - \gamma^-)z \\ &= \frac{1}{2}\omega\sqrt{\mu_0\epsilon}\,z\left(\sqrt{1 + \frac{\omega_M}{\omega_c + \omega}} - \sqrt{1 + \frac{\omega_M}{\omega_c - \omega}}\right)\end{aligned} \tag{5b-167}$$

Reversing the direction of propagation (i.e., replacing i by $-i$) does not change the sense of θ. Hence, regardless of whether the waves are traveling in the direction of the static magnetic field ($+z$) or in the opposite direction of the static magnetic field, their axes of polarization are rotated in the same sense with respect to the biasing magnetostatic field. This phenomenon is called the *Faraday rotation*. Reciprocity requires that the rotations be equal and opposite; thus the ferrite medium is nonreciprocal. Making use of the nonreciprocal nature of waves in ferrite medium, a number of very useful practical devices using ferrites have been invented: The ferrite gyrator, producing π phase shift in the $+z$ direction and zero phase shift in the $-z$ direction, the ferrite isolator, circulator, switch, etc.

FERRITE-LOADED WAVEGUIDES. Assuming that all field components in a waveguide containing anisotropic ferrites vary as $e^{i\gamma z - i\omega t}$, then the transverse components $\mathbf{E}_t$ and $\mathbf{H}_t$ can be derived from the longitudinal components E_z and H_z according to the following relations:

$$\begin{aligned}\mathbf{E}_t = \frac{1}{(\beta_1^2 - \gamma^2)^2 - \beta_2^4}\,\{&\nabla_t[(\beta_1^2 - \gamma^2)i\gamma E_z + \omega\gamma^2\mu_2 H_z] \\ &- i\mathbf{e}_z \times \nabla_t[\omega(\beta_1^2\mu_1 - \gamma^2\mu_1 + \beta_2^2\mu_2)H_z - i\gamma\beta_2^2 E_z]\}\end{aligned} \tag{5b-168}$$

$$\begin{aligned}\mathbf{H}_t = \frac{1}{(\beta_1^2 - \gamma^2) - \beta_2^4}\,\{&\nabla_t[\beta_2^2\omega\epsilon E_z + (\beta_1^2 - \gamma^2)i\gamma H_z] \\ &+ i\mathbf{e}_z \times \nabla_t[(\beta_1^2 - \gamma^2)\omega\epsilon E_z + \beta_2^2 i\gamma H_z]\}\end{aligned} \tag{5b-169}$$

with $\beta_1^2 = \omega^2\mu_1\epsilon$ and $\beta_2^2 = -\omega^2\mu_2\epsilon$. The longitudinal fields (E_z,H_z) satisfy

$$\nabla_t^2 E_z + a_1E_z + a_2H_z = 0 \tag{5b-170}$$

$$\nabla_t^2 H_z + a_3H_z + a_4E_z = 0 \tag{5b-171}$$

with

$$a_1 = (\beta_1^2 - \gamma^2) + \beta_2^2 \frac{\mu_2}{\mu_1} \tag{5b-172}$$

$$a_2 = \omega\mu_3 i\gamma \frac{\mu_2}{\mu_1} \tag{5b-173}$$

$$a_3 = (\beta_1^2 - \gamma^2) \frac{\mu_3}{\mu_1} \tag{5b-174}$$

$$a_4 = -i\omega\gamma\epsilon \frac{\mu_2}{\mu_1} \tag{5b-175}$$

According to Eqs. (5b-170) and (5b-171), a pure *TE, TM*, or *TEM* modes cannot exist in a waveguide filled with a ferrite, since if either of the longitudinal field E_z or H_z is zero, the entire longitudinal field vanishes and all transverse fields also vanish according to Eqs. (5b-168) and (5b-169).

Equations (5b-170) and (5b-171) may be decoupled by the introduction of functions Φ_1 and Φ_2 defined by

$$E_z = \Phi_1 + \Phi_2 \tag{5b-176}$$

$$H_z = g_1\Phi_1 + g_2\Phi_2 \tag{5b-177}$$

where

$$g_1 = \frac{p_1^2 - a_1}{a_2} = \frac{a_4}{p_1^2 - a_3} \tag{5b-178}$$

$$g_2 = \frac{p_2^2 - a_1}{a_2} = \frac{a_4}{p_2^2 - a_3} \tag{5b-179}$$

and p_1^2 and p_2^2 are the roots of the equation

$$p^4 - (a_1 + a_3)p^2 + a_1a_3 - a_2a_4 = 0 \tag{5b-180}$$

Φ_1 and Φ_2 satisfy the equations

$$(\nabla_t^2 + p_1^2)\Phi_1 = 0 \qquad (\nabla_t^2 + p_2^2) = 0 \tag{5b-181}$$

Rigorous theories of wave propagation in endless waveguides completely filled with a magnetized ferrite medium have been worked out for circular and rectangular waveguides. However, the solutions are too complicated to be included here.[1]

Periodic Structures. Any propagating mode in an empty perfectly conducting cylindrical tube has a phase velocity which is greater than the speed of light. From the need of electronic devices for the generation and amplification of microwaves and for the acceleration of charged elementary particles originated the demand for cylindrical waveguide structures in which the modes having a longitudinal component of the electric vector move with a phase velocity less than the velocity of light so as to enhance the energy exchange between the beam of particles and the wave fields. A practical slow-wave guide is the periodic structure.

Because of the periodicity of the structure, spatial harmonics of the modes must be taken into account. This is accomplished by the use of Floquet's theorem which states that for a given mode of propagation and at a given frequency, the wave functions at two points along the periodic guiding structure, separated by one spatial period of the structure, differ only by a complex constant. In other words, the general expression for the wave function, say E_z, should be of the form

$$E_z = \sum_{m=-\infty}^{\infty} A_m f_m(x,y) e^{i(\gamma_0 + 2m\pi/L)z} \tag{5b-182}$$

[1] H. Suhl and R. C. Walker, *Bell System Tech. J.* **33**, 579 (1954); P. S. Epstein, *Revs. Modern Phys.* **28**, 3 (1956).

where γ_0 is the propagation constant of the fundamental wave $m = 0$, and L is the period of the guide.

As a representative example, a periodic disk-load circular waveguide will be considered. The radii of the tube and the circular apertures are b and a, respectively. The spatial period of the structure is L and the thickness of each disk is w. Only the circularly symmetrical TM waves will be considered. In accordance with Floquet's theorem, appropriate representation for $E_z{}^{\text{I}}$ in region I ($r \leq a$) is

$$E_z{}^{\text{I}} = \sum_{m=-\infty}^{\infty} A_m I_0(\chi_m r) e^{i\gamma_m z} \tag{5b-183}$$

where

$$\gamma_m{}^2 = k^2 + \chi_m{}^2 \qquad \gamma_m = \gamma_0 + \frac{2\pi m}{L} \tag{5b-184}$$

with $m = 0, \pm 1, \ldots, k = \omega\sqrt{\mu\epsilon}$. I_0 is the modified Bessel function. Appropriate series representation for $E_z{}^{\text{II}}$ in region II ($a \leq r \leq b$) may also be assumed. However, upon matching the tangential E_z and H_ϕ across the surface $r = a$, an infinite determinant for the propagation constant results. For the case of closely spaced disks ($\gamma_0 w \ll 1$), the infinite determinant reduces to

$$\frac{1}{\chi_0 a}\frac{I_1(x_0 a)}{I_0(s_0 a)} = \frac{1}{ka}\frac{J_1(ka)N_0(kb) - N_1(ka)J_0(kb)}{J_0(ka)N_0(kb) - N_0(ka)J_0(kb)} \tag{5b-185}$$

A typical ω vs. γ_0 diagram is shown in Fig. 5b-3.

Formulas for propagating modes along helix (tape helix, sheath helix, or wire helix) are also available in the literature.[1] Problems concerning wave propagation in a waveguide filled longitudinally with a sinusoidally varying dielectric material have also been solved.[2]

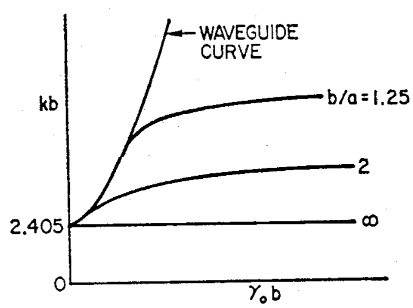


FIG. 5b-3. The ω-γ_0 diagram for the periodic disk-loaded circular waveguide with infinitesimal disk thickness. γ_0 is the propagation constant, b is the waveguide radius and a is the disk inner radius. [*From E. L. Chu* and *W. W. Hansen, J. Appl. Phys.* **18**, 996 (1947).]

Waveguide Junctions. Strictly speaking, the problem of waveguide junctions should be analyzed as a boundary-value problem. However, extreme difficulties are encountered owing to the complex geometry of the junction region.[3] The problem is therefore reformulated in terms of lumped-circuit representations for the junctions and transmission-line representations for the waveguides. For an arbitrary junction of m waveguide arms, assuming that only one mode may propagate in each waveguide arm, the power flowing into the nth arm is

$$\bar{P}_n = a_n a_n^* - b_n b_n^* \tag{5b-186}$$

where a_n is the incoming mode amplitude, and b_n is the reflected mode amplitude. The * represents the complex conjugate of the function. The voltage and current at the nth terminal are

$$v_n = \sum_m z_{nm} i_m \qquad i_n = \sum_m y_{nm} v_m \tag{5b-187}$$

[1] D. A. Watkins, "Topics in Electromagnetic Theory," John Wiley & Sons, Inc., New York, 1958; R. M. Bevensee, "Electromagnetic Slow Wave Systems," John Wiley & Sons, Inc., New York, 1964.

[2] C. Yeh and K. F. Casey, *IEEE Trans. Microwave Theory and Tech.* **MTT-13**, 297 (1965).

[3] G. L. Matthaei, L. Young, and E. M. T. Jones, "Microwave Filters, Impedance-matching Networks and Coupling Structures," McGraw-Hill Book Company, New York, 1964.

where z_{nm} and y_{nm} are respectively the impedance and admittance matrix. The mode amplitudes are related to i_n and z_{nm} by the relations

$$a_n = \sqrt{\frac{z_n}{8}} \sum_m \left(\frac{z_{nm}}{z_n} + \delta_{nm} \right) i_m \tag{5b-188}$$

$$b_n = \sqrt{\frac{z_n}{8}} \sum_m \left(\frac{z_{nm}}{z_n} - \delta_{nm} \right) i_m \tag{5b-189}$$

with δ_{nm} being the Kronecker delta. In matrix notation

$$\mathbf{A} = (\mathbf{Z}' + \mathbf{U})\mathbf{I}' \tag{5b-190}$$
$$\mathbf{B} = (\mathbf{Z}' - \mathbf{U})\mathbf{I}' \tag{5b-191}$$

where the elements of the matrices $\mathbf{Z}'$ and $\mathbf{I}'$ are respectively z_{nm}/z_n and $i_m \sqrt{z_n/8}$. $\mathbf{U}$ is the unit matrix. Combining Eqs. (5b-190) and (5b-191) gives

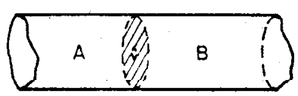
$$\mathbf{B} = \mathbf{SA} \tag{5b-192}$$

with $\mathbf{S} = (\mathbf{Z}' - \mathbf{U})(\mathbf{Z}' + \mathbf{U})^{-1}$. $\mathbf{S}$ is called the *scattering matrix.* So the junction problem is reduced to an evaluation of the scattering matrix. For a reciprocal lossless junction, the scattering matrix possesses the following properties:

$$\begin{aligned} \tilde{\mathbf{S}} &= \mathbf{S} \qquad \text{(symmetry property of S)} \\ \mathbf{S}^{-1} &= \tilde{\mathbf{S}}^{*} \qquad \text{(S is a unitary matrix)} \end{aligned} \tag{5b-193}$$

where $\tilde{\mathbf{S}}$ is the transpose of $\mathbf{S}$ and * indicates the complex conjugate.

Waveguide Discontinuities. A discontinuity in the structure of a waveguide results in the distortion of the nominal field distributions. It is assumed that only the dominant mode may propagate in this structure. Therefore, as far as the far-zone (away from the disturbance) dominant propagating mode is concerned, an equivalent circuit description of the disturbance will be adequate. Considering the discontinuity due to a post, an aperture or an abrupt change in the cross-sectional area of a waveguide, the disturbance may be represented by an admittance Y shunted across a uniform transmission line at the discontinuity. The shunt admittance is related to the reflection coefficient R of a dominant mode by the relation $R = Y/(2 - Y)$. For a time dependence of $e^{-i\omega t}$, and $Y = iB$, the admittance is inductive if B is positive and capacitive if B is negative.



(a)

FIG. 5b-4. A sketch of an iris in a waveguide.

The variational expression for the shunt admittance of an iris in a waveguide is (see Fig. 5b-4)

$$Y_d = \frac{\int_{\mathrm{Ap}} \int_{\mathrm{Ap}} \mathbf{E}_{nA}(x',y') \cdot \mathbf{G}(x',y',x,y) \cdot \mathbf{E}_{nA}(x,y)\, dx'\, dy'\, dx\, dy}{\left(\iint_{\mathrm{Ap}} \Phi_0 \cdot \mathbf{E}_{nA}\, dS \right)^2} \tag{5b-194}$$

where

$$\mathbf{G}\ (\textit{Green's function}) = \sum_{m,n} \sum_{p=A,B} Y_{mnp} \Phi_{mn}(x',y') \Phi_{mn}(x,y)$$

Y_{mnA}, Y_{mnB} = characteristic admittances for (m,n) modes in regions A and B
Φ_{mn} = transverse field vectors in the undisturbed guide
$\mathbf{E}_{nA}$ = normalized transverse aperture fields in region A
Ap = aperture area
Φ_0 = dominant mode transverse field vectors

The above expression for Y_d is variational in the sense that the variation of Y_d due to a small change of $\mathbf{E}_{nA}$ from its true value is vanishingly small.

For a rectangular waveguide of cross-sectional dimension $a \times b$, we have

$$\begin{aligned}\boldsymbol{\Phi}_{mn} &= A_{mn}\left[-\mathbf{e}_x\left(\frac{n}{b}\right)\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\right.\\ &\qquad\left.+\,\mathbf{e}_y\left(\frac{m}{a}\right)\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\right] \quad \text{for } TE \text{ modes}\\ &= A_{mn}\left[\mathbf{e}_x\left(\frac{m}{b}\right)\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\right.\\ &\qquad\left.+\,\mathbf{e}_y\left(\frac{n}{b}\right)\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\right] \quad \text{for } TM \text{ modes}\end{aligned} \tag{5b-195}$$

$$\begin{aligned}Y_{mn} &= \sqrt{\frac{\epsilon}{\mu}}\left[1-\left(\frac{k_{mn}}{k}\right)^2\right]^{\frac{1}{2}} \quad \text{for } TE \text{ modes}\\ &= \sqrt{\frac{\epsilon}{\mu}}\left[1-\left(\frac{k_{mn}}{k}\right)^2\right]^{-\frac{1}{2}} \quad \text{for } TM \text{ modes}\end{aligned} \tag{5b-196}$$

$$\gamma^2 = k^2 - k_{mn}{}^2 \qquad k = \omega\sqrt{\mu\epsilon} \tag{5b-197}$$

$$k_{mn}{}^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \tag{5b-198}$$

$$A_{mn}{}^2 = 2\left[\frac{n^2}{\epsilon_m}\left(\frac{a}{b}\right) + \frac{m^2}{\epsilon_n}\left(\frac{a}{b}\right)\right]^{-1} \tag{5b-199}$$

$$\epsilon_m = \begin{cases}1 & \text{for } m = 0\\ 2 & \text{for } m = 1, 2, \ldots\end{cases} \tag{5b-200}$$

For a circular waveguide of radius a, we have

$$\begin{aligned}\boldsymbol{\Phi}_{mn} &= A_{mn}\left[\mathbf{e}_r\frac{mJ_m(k_{mn}r)}{k_{mn}r}\sin m\phi\right.\\ &\qquad\left.+\,\mathbf{e}_\phi J'_m(k_{mn}r)\cos m\phi\right] \quad \text{for } TE \text{ modes}\\ &= B_{mn}\left[\mathbf{e}_r J'_m(k_{mn}r)\sin m\phi - \mathbf{e}_\phi\frac{mJ_m(k_{mn}r)}{k_{mn}r}\cos m\phi\right] \quad \text{for } TM \text{ modes}\end{aligned} \tag{5b-201}$$

Y_{mn} is given by Eq. (5b-196), and γ is given by Eq. (5b-197).

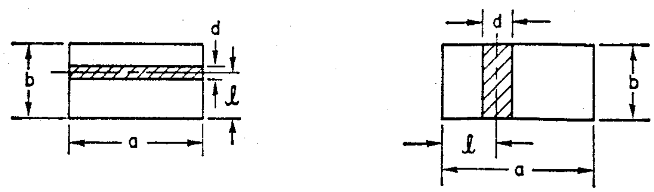
$$J'_m(k_{mn}a) = 0 \quad \text{for } TE \text{ modes} \tag{5b-202}$$

$$J_m(k_{mn}a) = 0 \quad \text{for } TM \text{ modes} \tag{5b-203}$$

$$A_{mn} = \left[\frac{\pi}{\epsilon_m}\left(a^2 - \frac{m^2}{k_{mn}{}^2}\right)J_m{}^2(k_{mn}a)\right]^{-\frac{1}{2}} \tag{5b-204}$$

$$B_{mn} = \left[\frac{\pi}{\epsilon_m}(aJ'_m(k_{mn}a))^2\right]^{-\frac{1}{2}} \tag{5b-205}$$

ϵ_m is given by Eq. (5b-200). Some representative examples are given in Fig. 5b-5.



$$\frac{Y_d}{Y_0} \approx -i\,\frac{4b}{\lambda_g}\,\ln\left(\csc\frac{\pi\ell}{b}\csc\frac{\pi d}{2b}\right) \qquad \text{For } \frac{b}{\lambda_g} < 0.75$$

$$\frac{Y_d}{Y_0} \approx i\left(\frac{\lambda_g}{a}\right)\left(\csc^2\frac{\pi\ell}{a}\csc^2\frac{\pi d}{2a} - 1\right) \qquad \text{For } \frac{\lambda_g}{a} > 1.5$$

FIG. 5b-5. *Left:* capacitive iris *Right:* inductive iris.