References

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5b-10. Cavity Resonators. Resonant cavities are used at high frequencies in place of lumped-circuit elements, primarily because they eliminate radiation and in general possess very low losses. Only eigenvalue solutions exist in a lossless cavity resonator completely enclosed by perfectly conducting walls. For a cavity filled with a homogeneous, isotropic dielectric, the pth eigenvector $\mathbf{E}_p$ satisfies

$$\begin{aligned} (\nabla^2 + k_p^2)\mathbf{E}_p &= 0 \qquad \text{(everywhere within the cavity)} \\ \mathbf{n} \times \mathbf{E}_p &= 0 \qquad \text{(on the enclosing wall)} \end{aligned} \tag{5b-206}$$

where $k_p = \omega_p \sqrt{\mu\epsilon}\ (p = 1, 2, 3, \ldots)$ are the eigenvalues. ω_p is the resonant frequency for the pth mode.

The Q_p of a resonator for the pth mode is defined as follows:

$$Q_p = \omega_p \frac{\text{total time-average energy stored}}{\text{time-average power dissipated}} \tag{5b-207}$$

$$= \frac{\Delta\omega}{\omega_p} \tag{5b-208}$$

where $\Delta\omega$ is the bandwidth of the resonance curve. Hence Q_p is a measure of the amount of power dissipated for the pth mode. For an enclosed cavity with slightly lossy walls,

$$Q_p = \frac{2}{\delta_s} \frac{\int_V \mathbf{H}_p \cdot \mathbf{H}_p^* \, dV}{\oint_A \mathbf{H}_p \cdot \mathbf{H}_p^* \, dA} \tag{5b-209}$$

where $\mathbf{H}_p$ is the magnetic field of the pth mode of the cavity without losses, and δ_s is the skin depth of the walls. A is the total surface enclosing the cavity region.

For a cavity composed of a uniform transmission line (which may support the *TE*, *TM*, *TEM*, or *HE* mode) with short-circuiting perfectly conducting ends, the Q_p of this cavity is related to the attenuation constant α_p of the transmission line by the relation[1]

$$Q_p = \frac{v_{\text{phase}}^p}{v_{\text{group}}^p} \frac{\gamma_p}{2\alpha_p} \tag{5b-210}$$

where v_{phase}^p, v_{group}^p and γ_p are respectively the phase velocity ω_p/γ_p, the group velocity $\partial\omega_p/\partial\gamma_p$, and the phase constant of the pth mode. If the end plates possess a very small loss, then the total Q_T of this cavity is

$$\frac{1}{Q_T} = \frac{1}{Q_{\text{end plates}}} + \frac{1}{Q_{\text{trans. line}}}$$

where $Q_{\text{end plates}}$ is calculated according to Eq. (5b-209) and $Q_{\text{trans. line}}$ can be calculated according to Eq. (5b-210).

[1] C. Yeh, *Proc. IRE* **50**, 2145 (1962).

Simple Resonators. The mode functions for a cylindrical waveguide of simple cross section closed at both ends by short-circuiting plates are[1] (with d = length of the cavity):

For TM_{mnl} modes

$$E_{zmnl} = A_{mn}\Phi_{mn} \cos \frac{l\pi z}{d} \tag{5b-211}$$

$$H_{zmnl} = 0 \tag{5b-212}$$

$$\mathbf{E}_{tmnl} = -\frac{l\pi}{d} \frac{A_{mn}}{\Gamma_{mn}^{(TM)}} \nabla_t\Phi_{mn} \sin \frac{l\pi z}{d} \tag{5b-213}$$

$$\mathbf{H}_{tmnl} = i\omega\epsilon \frac{A_{mn}}{\Gamma_{mn}^{(TM)}} (\mathbf{e}_z \times \nabla_t\Phi_{mn}) \cos \frac{l\pi z}{d} \tag{5b-214}$$

with $(\nabla_t^2 + \Gamma_{mn}^{(TM)^2})\Phi_{mn} = 0$ and $\Phi_{mn} = 0$ on the cylindrical wall. The resonant frequency

$$\omega_{mnl}^{TM} = \frac{1}{\sqrt{\mu\epsilon}} \left[\Gamma_{mn}^{(TM)^2} + \left(\frac{l\pi}{d}\right)^2 \right]^{\frac{1}{2}}$$

For TE_{mnl} modes

$$E_{zmnl} = 0 \tag{5b-215}$$

$$H_{zmnl} = B_{mn}\Psi_{mn} \sin \frac{l\pi z}{d} \tag{5b-216}$$

$$\mathbf{E}_{tmnl} = \frac{-i\omega\mu}{\Gamma_{mn}^{(TE)}} B_{mn}(\mathbf{e}_z \times \nabla_t\Psi_{mn}) \sin \frac{l\pi z}{d} \tag{5b-217}$$

$$\mathbf{H}_{tmnl} = \frac{l\pi}{d} \frac{1}{\Gamma_{mn}^{(TE)}} B_{mn}(\nabla_t\Psi_{mn}) \cos \frac{l\pi z}{d} \tag{5b-218}$$

with $(\nabla_t^2 + \Gamma_{mn}^{(TE)^2})\Psi_{mn} = 0$ and $\partial\Psi_{mn}/\partial n = 0$ along the cylindrical wall. The resonant frequency

$$\omega_{mnl}^{TE} = \frac{1}{\sqrt{\mu\epsilon}} \left[\Gamma_{mn}^{(TE)^2} + \left(\frac{l\pi}{d}\right)^2 \right]^{\frac{1}{2}}$$

For a rectangular resonator with cross sections $a \times b$ we have

$$\Phi_{mn} = \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b} \tag{5b-219}$$

$$\Psi_{mn} = \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \tag{5b-220}$$

with

$$\Gamma_{mn}^{(TM)} = \Gamma_{mn}^{(TE)} = \left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \right]^{\frac{1}{2}} \tag{5b-221}$$

For a circular cylindrical resonator of radius a, we have

$$\Phi_{mn} = J_m(\Gamma_{mn}^{(TM)} r) \begin{matrix}\cos \\ \sin\end{matrix}\, m\phi \tag{5b-222}$$

$$\Psi_{mn} = J_m(\Gamma_{mn}^{(TE)} r) \begin{matrix}\cos \\ \sin\end{matrix}\, m\phi \tag{5b-223}$$

where $\Gamma_{mn}^{(TM)}$ and $\Gamma_{mn}^{(TE)}$ satisfy the following equations:

$$J_m(\Gamma_{mn}^{(TM)} a) = 0 \tag{5b-224}$$

$$J'_m(\Gamma_{mn}^{(TE)} a) = 0 \tag{5b-225}$$

[1] Solutions are also available for resonators of more complex shapes, such as the ellipsoid-hyperbolid resonators [W. W. Hansen and R. D. Richtmeyer, *J. Appl. Phys.* **10,** 189 (1930)] and the reentrant cavities [D. C. Stinson, *Trans. IRE* **MTT-3,** 18 (1955)].

Solutions are also available for spherical cavity of radius a:

$$\mathbf{E}_{mnl}^{TE} = \nabla \times (\Phi_{mnl} r \mathbf{e}_r) \tag{5b-226}$$

$$\mathbf{H}_{mnl}^{TE} = -\frac{i}{\omega\mu} \nabla \times \nabla \times [\Phi_{mnl} r \mathbf{e}_r] \tag{5b-227}$$

$$\mathbf{H}_{mnl}^{TM} = \nabla \times [\Psi_{mnl} r \mathbf{e}_r] \tag{5b-228}$$

$$\mathbf{E}_{mnl}^{TM} = \frac{i}{\omega\epsilon} \nabla \times \nabla \times [\Psi_{mnl} r \mathbf{e}_r] \tag{5b-229}$$

$$\Phi_{mnl} = j_m(k_{mnl}^{(TE)} r) P_m{}^l (\cos\theta)_{\sin}^{\cos} l\phi \tag{5b-230}$$

$$\Psi_{mnl} = j_m(k_{mnl}^{(TM)} r) P_m{}^l (\cos\theta)_{\sin}^{\cos} l\phi \tag{5b-231}$$

where $j_m(x)$ is the spherical Bessel function. $k_{mnl}^{(TE)}$ and $k_{mnl}^{(TM)}$ satisfy

$$j_m(k_{mnl}^{(TE)} a) = 0, \; j'_m(k_{mnl}^{(TM)} a) = 0 \tag{5b-232}$$

with

$$\omega_{mnl}^{(TE),(TM)} = \frac{k_{mnl}^{(TE),(TM)}}{\sqrt{\mu\epsilon}}$$

Field configurations for a few lower-order modes are given in Fig. 5b-6.

Small Perturbation Formula. The resonant frequency shift of a cavity due to the presence of a small foreign body having a dielectric constant ϵ_1 and a permeability μ_1 is

$$-\frac{\delta\omega}{\omega_p} = \frac{\int_{V_1} (\epsilon_1 - \epsilon_0)\mathbf{E}_1 \cdot \mathbf{E}_0^* \, dV + \int_{V_1} (\mu_1 - \mu_0)\mathbf{H}_1 \cdot \mathbf{H}_0^* \, dV}{\epsilon_0 \int_V \mathbf{E}_0 \cdot \mathbf{E}_0^* \, dV + \mu_0 \int_V \mathbf{H}_0 \cdot \mathbf{H}_0^* \, dV} \tag{5b-233}$$

where $\mathbf{E}_1$, $\mathbf{H}_1$ denote the resulting field vectors within the volume V_1 of the foreign body, and $\mathbf{E}_0$, $\mathbf{H}_0$ denote the undisturbed field vectors. V is the volume of the cavity. ω_p is the resonant frequency of the unperturbed cavity.

The resonant frequency shift of a cavity due to a small wall deformation is

$$-\frac{\delta\omega}{\omega_p} = \frac{\int_{\Delta V} (\mu_0 \mathbf{H}_0 \cdot \mathbf{H}_0^* - \epsilon_0 \mathbf{E}_0 \cdot \mathbf{E}_0^*) \, dV}{\mu_0 \int_V (\mathbf{H}_0 \cdot \mathbf{H}_0^*) \, dV + \epsilon_0 \int_V (\mathbf{E}_0 \cdot \mathbf{E}_0^*) \, dV} \tag{5b-234}$$

where ΔV is the small change in cavity volume.

Open Resonators. For very high frequency waves (such as light waves) any enclosed metallic cavity of reasonable dimensions for machining would have to operate on a very high order mode. The resonances of the mode would be so closely grouped that the natural bandwidths of the oscillating modes could not be separated, and the use as a resonant system would be impractical. By removing the sides from a closed cavity, a large number of modes can be eliminated owing to energy loss by radiation from the open sides; only the low-loss modes which are essentially *TEM* modes will remain. Assuming that z is the axis of the open resonator, and x, y are the transverse directions, one may obtain, from Maxwell's equations, the simple beam solutions which are characterized by a direction of propagation (the z axis) and by a unique plane phase front perpendicular to this axis:[1]

$$E_{mn}(z) = E_0 \frac{w_0}{w} H_m\left(\sqrt{2}\,\frac{x}{w}\right) H_n\left(\sqrt{2}\,\frac{y}{w}\right) \exp\left(-\frac{x^2 + y^2}{w^2}\right) \tag{5b-235}$$

[1] G. D. Boyd and J. P. Gordon, *Bell System Tech. J.* **40**, 489 (1961); A. G. Fox and T. Li, *ibid.* 453.

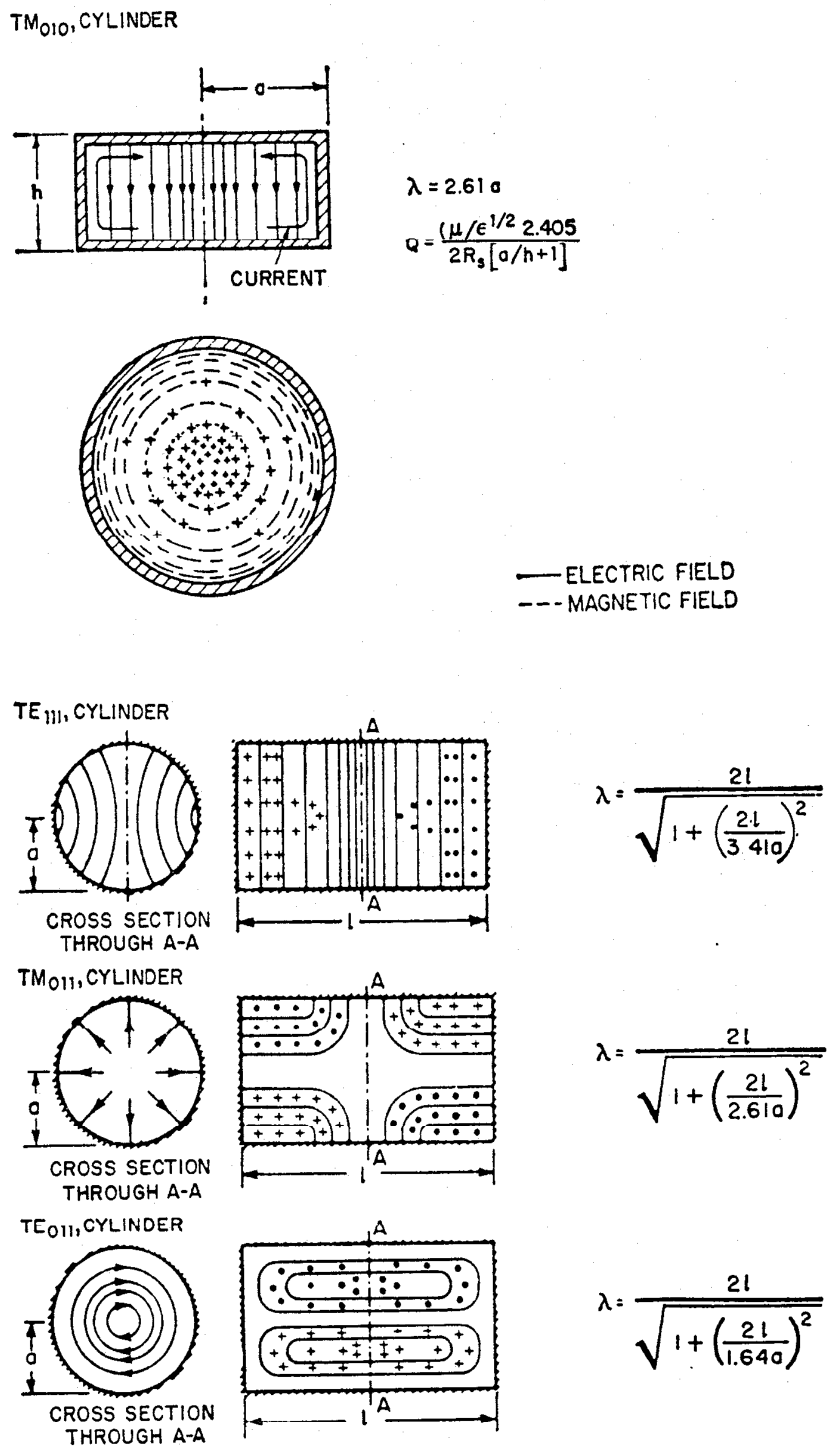

FIG. 5b-6. Field configurations for several selected modes. λ is the resonant wavelength.

where E_{mn} is a field component parallel to a wavefront for the (m,n) mode; H_m and H_n are Hermite polynomials of order m and n, respectively; $w_0 > \lambda$ is an arbitrary parameter with dimensions of length (w_0 may also be defined as the minimum spot size of the beam), $w(z) = w_0[1 + (z/z_0)^2]^{\frac{1}{2}}$, $z_0 = \pi w_0^2/\lambda$, and λ is the wavelength of a plane wave in the resonator medium. Possible positions of reflectors having radii of

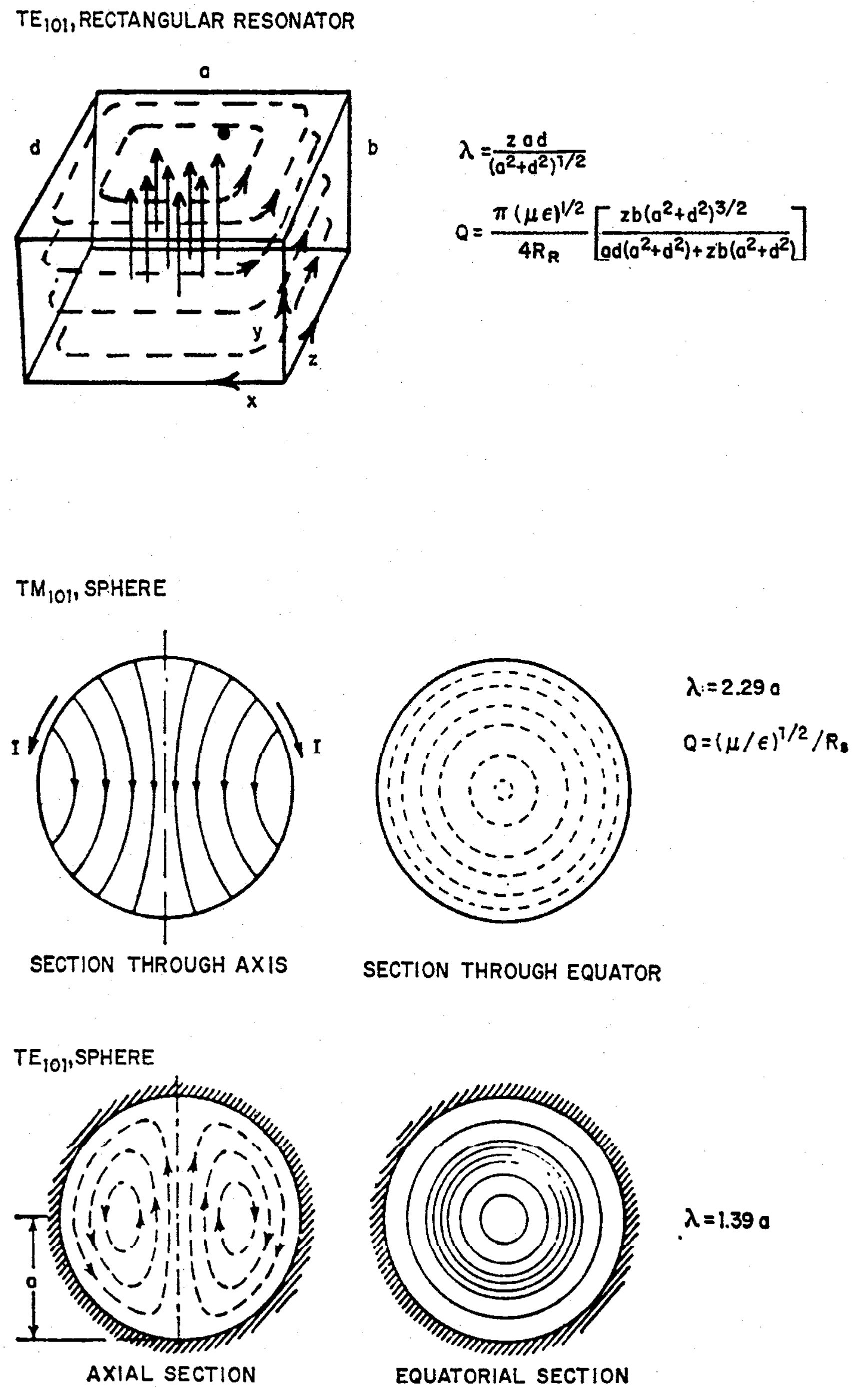

FIG. 5b-6 (*Continued*)

curvature R are given by the relation

$$R(z) = -w \frac{dz}{dw} = -\frac{1}{z}(z^2 + z_0^2) \qquad (5b\text{-}236)$$

The size of the reflector must be large enough to intercept substantially all the field for the mode of interest (say, $m = 0$, $n = 0$), so that energy loss due to diffraction may be acceptable. The modes with large m and n have fields extending farther out from the axis and so will suffer larger diffraction losses. In this way one can discriminate between the transverse modes and ensure that only a few will have low loss. As an example, let us design an optical resonator by using two reflectors having radii of curvature R_1 and R_2, and a mirror separation d. From Eq. (5b-236), we have

$$R_1 = -z_1 - \frac{z_0^2}{z_1} \qquad R_2 = -z_2 - \frac{z_0^2}{z_2}$$

with

$$z_2 - z_1 = d$$

Solving the above equations for z_0 gives

$$z_0 = \left[\frac{d(R_1 - d)(-R_2 - d)(R_1 - R_2 - d)}{(R_1 - R_2 - 2d)^2}\right]^{\frac{1}{2}}$$

which is the location of the minimum spot size $w_0 = (\lambda z_0/\pi)^{\frac{1}{2}}$.

The phase variation along the z axis for the (m,n) mode is

$$\beta z = -kz - (m + n + 1)\tan^{-1}\frac{z}{z_0} \qquad (5b\text{-}237)$$

where $k = 2\pi/\lambda$, and β is the propagation constant. The resonant condition requires

$$\beta d = q\pi \approx kd \qquad q = 1, 2, \ldots \qquad (5b\text{-}238)$$

where d is the minor separation. The frequency separation between longitudinal modes is $\Delta f = c/2d$; c = velocity of light in the resonator medium. Selected modal patterns are given in Fig. 5b-7.

The Q of an optical resonator is given by

$$Q = \frac{2\pi d}{\alpha\lambda} \qquad (5b\text{-}239)$$

where α is the fractional power loss per bounce from a reflector and is the sum of diffraction and reflection losses. The diffraction loss is small only if the Fresnel number $N = a_1a_2/\lambda d$, where a_1 and a_2 are radii of the mirrors, is much larger than unity.

General Considerations. DEGENERATE MODES. Modes with different field distributions but with the same resonant frequency.

EXCITATION OF CAVITY FIELDS. Excitation of cavity fields may be accomplished by the introduction of a conducting probe or antenna in the direction of the electric field lines, or by the introduction of a conducting loop with plane normal to the magnetic field lines, or by the introduction of a hole or iris between the cavity waveguide.[1] It is important to note that when the walls of the cavity have one or more apertures, the orthonormal sets $\mathbf{H}_p$ and $\mathbf{E}_p$, derived from the consideration of a completely enclosed cavity, are no longer adequate for an expansion of the cavity fields.[2] The electric vector **E** and the magnetic vector **H** of an electromagnetic field within a

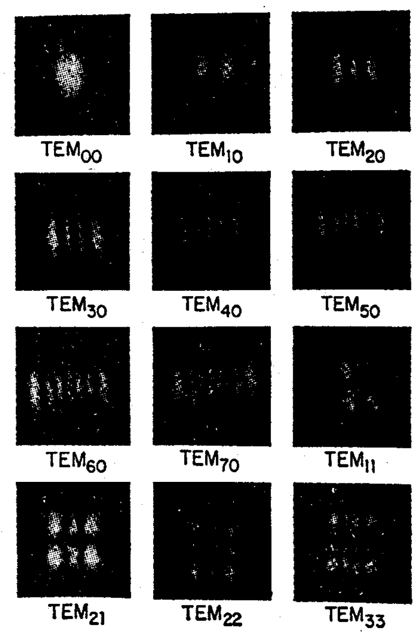

FIG. 5b-7. Modal patterns in optical resonators. [*From H. Kogelnik and W. W. Rigrod, Proc. IRE* **50**, 220 (1962).]

[1] Smythe, W. R., "Static and Dynamic Electricity," 3d ed., McGraw-Hill Book Company, New York, 1968.
[2] K. Kurokawa, *IRE Trans.* **MTT-6**, 178 (1958).

cavity coupled to an outside source by means of a waveguide must be derived according to the relations

$$\mathbf{E} = \epsilon \sum_{p=1}^{\infty} \mathbf{E}_p \int_V \mathbf{E} \cdot \mathbf{E}_p \, dV \tag{5b-240}$$

$$\mathbf{H} = \sum_{p=1}^{\infty} \mathbf{H}_p \left[\frac{-i\omega\epsilon\mu}{k_p^2 - k^2} \int_A (\mathbf{n} \times \mathbf{E}) \cdot \mathbf{H}_p \, dA \right] - \sum_{p=1}^{\infty} \mathbf{G}_p \left[\frac{i\omega\epsilon}{k^2} \int_A (\mathbf{n} \times \mathbf{E}) \cdot \mathbf{G}_p \, dA \right] \tag{5b-241}$$

where A consists of the perfectly conducting surface and the aperature surface, V is the volume of the cavity, $k^2 = \omega^2\mu\epsilon$, and

$$\left.\begin{aligned} \nabla^2\mathbf{E}_p + k_p^2\mathbf{E}_p &= 0 \\ \nabla \cdot \mathbf{E}_p &= 0 \end{aligned}\right\} \quad \text{in } V \tag{5b-242}$$

$$\mathbf{n} \times \mathbf{E}_p = 0 \qquad \text{on } A$$

$$\left.\begin{aligned} \nabla^2\mathbf{H}_p + k_p^2\mathbf{H}_p &= 0 \\ \nabla \cdot \mathbf{H}_p &= 0 \end{aligned}\right\} \quad \text{in } V \tag{5b-243}$$

$$\mathbf{n} \times (\nabla \times \mathbf{H}_p) = 0 \qquad \text{on } A$$

$$\left.\begin{aligned} \nabla^2\mathbf{G}_p + g_p^2\mathbf{G}_p &= 0 \\ \nabla \cdot \mathbf{G}_p &= 0 \end{aligned}\right\} \quad \text{in } V \tag{5b-244}$$

$$\mathbf{n} \times (\nabla \times \mathbf{G}_p) = 0 \qquad \mathbf{n} \cdot \mathbf{G}_p = 0 \qquad \text{on } A$$

Hence $\mathbf{G}_p$ is derivable from scalar potential as follows:

$$\mathbf{G} = \nabla v_p \tag{5b-245}$$

References

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5b-11. Radiation. Solutions of radiation problems must satisfy not only Maxwell's equations and the appropriate boundary conditions but also Sommerfeld's radiation condition.

Radiation Field from Known Current Distributions. Given a distribution of electric and magnetic currents, specified by the density functions $\mathbf{J}(\mathbf{r})$ and $\mathbf{J}_m(\mathbf{r})$ occupying a finite region of space. Formal expressions for the electric vector $\mathbf{E}$ and the magnetic vector $\mathbf{H}$ in an unbounded space are given earlier by Eqs. (5b-81) through (5b-84). Consider a reference frame with its origin in the vicinity of the sources; let $\mathbf{r}$ be the coordinates of the observation point, and $\mathbf{r}'$ be the coordinates of the source point. In the far-zone region (i.e., $r \gg r'$ and $kr \gg 1$), the radiated fields which are purely transverse to the direction of propagation are

$$E_\theta = i\omega\mu \frac{e^{ikr}}{4\pi r} F_\theta(\theta,\phi) = \left(\frac{\mu}{\epsilon}\right)^{\frac{1}{2}} H_\phi \tag{5b-246}$$

$$E_\phi = i\omega\mu \frac{e^{ikr}}{4\pi r} F_\varphi(\theta,\phi) = -\left(\frac{\mu}{\epsilon}\right)^{\frac{1}{2}} H_\theta \tag{5b-247}$$

with

$$F_\theta(\theta,\phi) = \int_{V_0} \left[\mathbf{J}(\mathbf{r}') \cdot \mathbf{e}_\theta + \left(\frac{\epsilon}{\mu}\right)^{\frac{1}{2}} \mathbf{J}_m(\mathbf{r}') \cdot \mathbf{e}_\phi \right] e^{-ik\mathbf{e}_r\cdot\mathbf{r}'}\, dV' \qquad (5b\text{-}248)$$

$$F_\phi(\theta,\phi) = \int_{V_0} \left[\mathbf{J}(\mathbf{r}') \cdot \mathbf{e}_\phi - \left(\frac{\epsilon}{\mu}\right)^{\frac{1}{2}} \mathbf{J}_m(\mathbf{r}') \cdot \mathbf{e}_\theta \right] e^{-ik\mathbf{e}_r\cdot\mathbf{r}'}\, dV' \qquad (5b\text{-}249)$$

where r, θ, ϕ are the spherical coordinates of the observation point and $\mathbf{e}_r$, $\mathbf{e}_\theta$, $\mathbf{e}_\phi$ are the corresponding unit vectors. The Poynting's vector (the time-average intensity of power flow) is

$$\mathbf{S} = \tfrac{1}{2}\mathrm{Re}\,(\mathbf{E} \times \mathbf{H}^*) = \frac{1}{8\lambda^2 r^2}\left(\frac{\mu}{\epsilon}\right)^{\frac{1}{2}} \{|F_\theta|^2 + |F_\phi|^2\}\mathbf{e}_r \qquad (5b\text{-}250)$$

and the time-average power per unit solid angle is

$$p(\theta,\phi) = r^2|\mathbf{S}| = \frac{1}{8\lambda^2}\left(\frac{\mu}{\epsilon}\right)^{\frac{1}{2}} \{|F_\theta|^2 + |F_\phi|^2\} \qquad (5b\text{-}251)$$

The total time-average radiated power is

$$P = \int p(\theta,\phi)\, d\Omega = \int_0^{2\pi}\int_0^{\pi} p(\theta,\phi) \sin\theta\, d\theta\, d\phi \qquad (5b\text{-}252)$$

The directivity characteristics of the radiating system are expressed by the gain function which is the ratio of the power radiated per unit solid angle in a direction (θ,ϕ) to average power radiated per unit solid angle. It is also referred to as the gain function with respect to an isotropic radiator radiating the same total power. Thus,

$$G(\theta,\phi) = \frac{p(\theta,\phi)}{\frac{1}{4\pi}\int p(\theta,\phi)\, d\Omega} = \frac{4\pi(|F_\theta|^2 + |F_\phi|^2)}{\int_0^{2\pi}\int_0^{\pi} (|F_\theta|^2 + |F_\phi|^2) \sin\theta\, d\theta\, d\phi} \qquad (5b\text{-}253)$$

The absolute gain is the maximum value of the gain function, and directivity = $10 \log_{10}[G(\theta,\phi)]_{\max}$ db.

THE ELECTRIC DIPOLE. The fields of an oscillating electric dipole of moment $\mathbf{p} = p_0\mathbf{e}_z = -(Id/i\omega)\mathbf{e}_z$, where $\mathbf{e}_z$ is a unit vector in the z direction, I is the uniform current, and d is the length of the structure, are

$$E_r = \frac{1}{2\pi\epsilon}\left(\frac{1}{r^3} - \frac{ik}{r^2}\right)\cos\theta\, p_0 e^{ikr}e^{-i\omega t} \qquad (5b\text{-}254)$$

$$E_\theta = \frac{1}{4\pi\epsilon}\left(\frac{1}{r^3} - \frac{ik}{r^2} - \frac{k^2}{r}\right)\sin\theta\, p_0 e^{ikr}e^{-i\omega t} \qquad (5b\text{-}255)$$

$$H_\phi = -\frac{i\omega}{4\pi}\left(\frac{1}{r^2} - \frac{ik}{r}\right)\sin\theta\, p_0 e^{ikr}e^{-i\omega t} \qquad (5b\text{-}256)$$

The dipole is located at the origin of the spherical coordinates. The far-zone Poynting vector is

$$\mathbf{S} = \frac{\omega k^3}{32\pi^2\epsilon}|p_0|^2\frac{\sin^2\theta}{r^2}\mathbf{e}_r \qquad (5b\text{-}257)$$

and the gain function is

$$G(\theta,\phi) = \tfrac{3}{2}\sin^2\theta \qquad (5b\text{-}258)$$

THE MAGNETIC DIPOLE. The fields of an oscillating magnetic dipole of moment $\mathbf{m} = m_0\mathbf{e}_z = IA\mathbf{e}_z$, where $\mathbf{e}_z$ is a unit vector in the z direction, I is the uniform current, and A is the area of the small current loops, are

$$E_\phi = \frac{k^2}{4\pi}\left(\frac{\mu}{\epsilon}\right)^{\frac{1}{2}}\left(\frac{1}{r}+\frac{i}{kr^2}\right)\sin\theta\, m_0 e^{ikr}e^{-i\omega t} \qquad (5b\text{-}259)$$

$$H_r = \frac{1}{2\pi}\left(\frac{1}{r^3}-\frac{ik}{r^2}\right)\cos\theta\, m_0 e^{ikr}e^{-i\omega t} \qquad (5b\text{-}260)$$

$$H_\theta = \frac{1}{4\pi}\left(\frac{1}{r^3}-\frac{ik}{r^2}-\frac{k^2}{r}\right)\sin\theta\, m_0 e^{ikr}e^{-i\omega t} \qquad (5b\text{-}261)$$

$$A \ll \lambda^2$$

The dipole is located at the origin of the coordinates, and $\mathbf{e}_z$ is normal to the area A. The gain function is the same as that of an electric dipole.

THE LINEAR THIN-WIRE ANTENNA. The far-zone fields of a thin-wire center-driven antenna with the current distribution

$$\mathbf{J}(\mathbf{r}) = \mathbf{e}_z I_0\, \delta(x)\, \delta(y) \sin k(l - |z|) \qquad (5b\text{-}262)$$

where I_0 is the current amplitude, $2l$ is the length of the wire, and the wire is centered at the origin, are

$$E_\theta = -i\sqrt{\frac{\mu}{\epsilon}}\, I_0 \frac{\cos(kl\cos\theta) - \cos kl}{\sin\theta}\cdot\frac{e^{ikr}e^{-i\omega t}}{2\pi r} \qquad (5b\text{-}263)$$

$$H_\phi = \sqrt{\frac{\epsilon}{\mu}}\, E_\theta \qquad (5b\text{-}264)$$

The far-zone Poynting vector is

$$\mathbf{S} = \mathbf{e}_r \sqrt{\frac{\mu}{\epsilon}}\frac{I_0^2}{8\pi^2 r^2}\left[\frac{\cos(kl\cos\theta) - \cos kl}{\sin\theta}\right]^2 \qquad (5b\text{-}265)$$

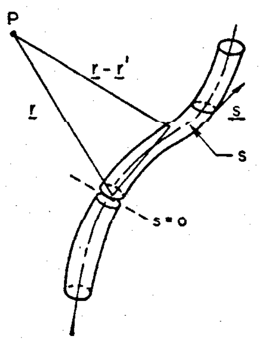

FIG. 5b-8. Coordinates of a curved wire.

Here, the length l does not have to be much less than the wavelength λ.

Integral Equation of Thin-wire Antennas. Equations (5b-81) through (5b-84) show that when the current distribution is known, the radiation fields may be found by integration. Strictly speaking, the current distribution must be found by solving the boundary-value problem which is usually quite difficult except for some special cases.[1] For a thin curved wire antenna excited by a generator which produces an electric field $E_s^i(s)$ across a gap centered at $s = 0$, the current along the wire satisfies the following integral equation.[2] (Symmetric wire antenna)

$$\int_{L_s} J(s')\pi(s,s')\, ds' = c'\cos ks + \frac{i}{\sqrt{\mu/\epsilon}}\int_0^s E_\xi^i(\xi)\sin k(s-\xi)\, d\xi \qquad (5b\text{-}266)$$

$$\pi(s,s') = G(s,s')\mathbf{s}\cdot\mathbf{s}' - \int_0^s\left[\frac{\partial G(\xi,s')}{\partial\xi}\,\boldsymbol{\xi}\cdot\mathbf{s}' + \frac{\partial G(\xi,s')}{\partial s'} + G(\xi,s')\frac{\partial(\boldsymbol{\xi}\cdot\mathbf{s}')}{\partial\xi}\right]\cos k(s-\xi)\, d\xi \qquad (5b\text{-}267)$$

$$G(s,s') = \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} \qquad (5b\text{-}268)$$

where $\int_{L_s} ds'$ will represent the surface integral over the wire, s is the arc length measured from the center of the gap, and $\mathbf{s}$, $\mathbf{s}'$, $\boldsymbol{\xi}'$ are all unit vectors along the wire (see Fig. 5b-8). c' is determined by the condition that the current vanishes at both ends of the antenna. Equation (5b-266) may be solved numerically by reducing the integral equation to a finite set of algebraic equations.[2]

[1] S. A. Schelkunoff, "Advanced Antenna Theory," John Wiley & Sons, Inc., New York, 1952.

[2] K. K. Mei, *IEEE Trans.* **AP-13**, 374 (1965).

Knowing the exact current distribution, one may also find the input impedance at the gap. For example, for a delta gap source at $s = 0$—i.e., the last integral in Eq. (5b-266) becomes $[iV_0/(2\sqrt{\mu/\epsilon})] \sin ks$, where V_0 is the voltage across the gap—the input impedance is $Z_{\text{in}} = V_0/I(0)$, where $I(0)$ is the current at $s = 0$.

Radiation Field from Apertures. Given a surface S enclosing the sources and the values of **E** and **H** over the entire surface S. The field at a point P outside the region of the surface is given by

$$\mathbf{E}_p(\mathbf{r}) = \int_S \{i\omega\mu[\mathbf{n}' \times \mathbf{H}(\mathbf{r}')]G(\mathbf{r},\mathbf{r}') + [\mathbf{n}' \times \mathbf{E}(\mathbf{r}')] \times \nabla' G(\mathbf{r},\mathbf{r}') + [\mathbf{n}' \cdot \mathbf{E}(\mathbf{r}')]\nabla' G(\mathbf{r},\mathbf{r}')\}\, dS' \qquad (5b\text{-}269)$$

$$\mathbf{H}_p(\mathbf{r}) = \int_S \{-i\omega\epsilon[\mathbf{n}' \times \mathbf{E}(\mathbf{r}')]G(\mathbf{r},\mathbf{r}') + [\mathbf{n}' \times \mathbf{H}(\mathbf{r}')] \times \nabla' G(\mathbf{r},\mathbf{r}') + [\mathbf{n}' \cdot \mathbf{H}(\mathbf{r}')]\nabla' G(\mathbf{r},\mathbf{r}')\}\, dS' \qquad (5b\text{-}270)$$

$$G(\mathbf{r},\mathbf{r}') = \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r} - \mathbf{r}'|} \qquad (5b\text{-}271)$$

where $\mathbf{n}'$ is the unit vector normal to S directed outward from the region of the sources, $|\mathbf{r} - \mathbf{r}'|$ is the distance from dS' to P; the gradient operator ∇' is with respect to the primed coordinates on S. Another form of Eqs. (5b-269) and (5b-270) is

$$\mathbf{E}_p(\mathbf{r}) = -\int_S \left[G(\mathbf{r},\mathbf{r}') \frac{\partial \mathbf{E}(\mathbf{r}')}{\partial n'} - \mathbf{E}(\mathbf{r}') \frac{\partial G(\mathbf{r},\mathbf{r}')}{\partial n'} \right] dS' \qquad (5b\text{-}272)$$

$$\mathbf{H}_p(\mathbf{r}) = -\int_S \left[G(\mathbf{r},\mathbf{r}') \frac{\partial \mathbf{H}(\mathbf{r}')}{\partial n'} - \mathbf{H}(\mathbf{r}') \frac{\partial G(\mathbf{r},\mathbf{r}')}{\partial n'} \right] dS' \qquad (5b\text{-}273)$$

where $G(\mathbf{r},\mathbf{r}')$ is given by Eq. (5b-271). Equations (5b-269) and (5b-270) or Eqs. (5b-272) and (5b-273) may be regarded as an analytical formulation of the Huygens-Fresnel principle which states that each point on a given wavefront can be regarded as a secondary source which gives rise to a spherical wavelet; the wave at a field point is to be obtained by superposition of these elementary wavelets, with due regard to their phase differences when they reach the point in question.

Since the values of **E** and **H** over the entire surface S are not known for most antenna problems, it is therefore desirable to provide modified expressions for $\mathbf{E}_p(\mathbf{r})$ and $\mathbf{H}_p(\mathbf{r})$. For very high frequencies, we have

$$\mathbf{E}_p(\mathbf{r}) \approx \frac{i}{\omega\epsilon} \int_A \{k^2[\mathbf{n}' \times \mathbf{H}_a(\mathbf{r}')]G(\mathbf{r},\mathbf{r}') + [\mathbf{n}' \times \mathbf{H}_a(\mathbf{r}')] \cdot \nabla'[\nabla' G(\mathbf{r},\mathbf{r}')] - i\omega\epsilon[\mathbf{n}' \times \mathbf{E}_a(\mathbf{r}')] \times \nabla' G(\mathbf{r},\mathbf{r}')\}\, dS' \qquad (5b\text{-}274)$$

$$\mathbf{H}_p(\mathbf{r}) \approx \frac{-i}{\omega\mu} \int_A \{k^2[\mathbf{n}' \times \mathbf{E}_a(\mathbf{r}')]G(\mathbf{r},\mathbf{r}') + [\mathbf{n}' \times \mathbf{E}_a(\mathbf{r}')] \cdot \nabla'[\nabla' G(\mathbf{r},\mathbf{r}')] + i\omega\mu[\mathbf{n}' \times \mathbf{H}_a(\mathbf{r}')] \times \nabla' G(\mathbf{r},\mathbf{r}')\}\, dS' \qquad A \gg \lambda^2 \qquad (5b\text{-}275)$$

where $\mathbf{E}_a(\mathbf{r}')$ and $\mathbf{H}_a(\mathbf{r}')$ are the fields over the aperture. It is noted that in the present case the integration is carried out over an open surface A in contrast with that of the previous equations, i.e., Eqs. (5b-269) to (5b-273). Equations (5b-274) to (5b-275) may be used for the computation of fields for reflectors, lenses, and horns under the approximations that the field over the aperture is related in the most simple way possible to the primary sources—in the cases of lenses and reflectors, by the use of geometrical optics; in the case of horns, by considering the field distribution which would exist over the aperture plane of the horn extended to infinity. In the far zone ($r \gg r'$, $kr \gg 1$),

$$\mathbf{E}_p(\mathbf{r}) \approx \frac{ik}{4\pi r} e^{ikr}\, \mathbf{e}_r \times \int_A \left\{\mathbf{n}' \times \mathbf{E}_a(\mathbf{r}') - \left(\frac{\mu}{\epsilon}\right)^{\frac{1}{2}} \mathbf{e}_r \times [\mathbf{n}' \times \mathbf{H}_a(\mathbf{r}')]\right\} e^{ik\mathbf{e}_r \cdot \mathbf{r}'}\, dS' \qquad (5b\text{-}276a)$$

$$\mathbf{H}_p(\mathbf{r}) \approx \sqrt{\frac{\epsilon}{\mu}}\, [\mathbf{e}_r \times \mathbf{E}_p(\mathbf{r})] \qquad A \gg \lambda^2 \qquad (5b\text{-}276b)$$

When the aperture field is obtained by the simple considerations stated above, there is an elementary relation between the tangential components of the electric and magnetic vectors over the aperture:

$$\mathbf{H}_a = \eta(\mathbf{s}' \times \mathbf{E}_a) \tag{5b-277}$$

and

$$\mathbf{E}_p^{(\text{far zone})}(\mathbf{r}) \simeq \frac{ike^{ikr}}{4\pi r} \mathbf{e}_r \times \int_A \Big(\mathbf{n}' \times \mathbf{E}_a(\mathbf{r}') - \eta \left(\frac{\mu}{\epsilon}\right)^{\frac{1}{2}} \{\mathbf{e}_r \cdot [\mathbf{s}' \times \mathbf{E}_a(\mathbf{r}')]\mathbf{n}' - [\mathbf{s}' \times \mathbf{E}_a(\mathbf{r}')](\mathbf{n}' \cdot \mathbf{e}_r)\}\Big) e^{ik\mathbf{e}_r \cdot \mathbf{r}'} \, dS' \tag{5b-278a}$$

$$\mathbf{H}_p^{(\text{far zone})}(\mathbf{r}) \simeq \sqrt{\frac{\epsilon}{\mu}} \, [\mathbf{e}_r \times \mathbf{E}_p^{(\text{far zone})}(\mathbf{r})] \tag{5b-278b}$$

where $\mathbf{s}'$ is a unit vector along a ray through the aperture; $\eta = (\epsilon/\mu)^{\frac{1}{2}}$ for lenses and reflectors in free space; and for a horn, $\eta = (\Gamma_{mn}/\omega\mu)(1 - R)/(1 + R)$ for TE modes and $[\omega\epsilon/\Gamma_{mn}](1 - R)/(1 + R)$ for TM modes; with R being the reflection coefficient of the mode in the horn, and Γ_{mn} the eigenvalues of modes in an infinite horn.

Another approximate formula for the computation of radiated fields from a perfectly conducting reflector at very high frequencies is also available. The formula is based on the knowledge of the induced current distribution over the reflector; the induced current is obtained on the basis of geometrical optics. Let $(\mathbf{E}_i, \mathbf{H}_i)$ be the incident field, $\mathbf{s}_0$ a unit vector in the direction of the incident ray, and $\mathbf{n}$ a unit vector normal to the surface S at the point of incidence. The induced surface current density is

$$\mathbf{J} = 2(\mathbf{n} \times \mathbf{H}_i) = 2\sqrt{\frac{\epsilon}{\mu}} \, [\mathbf{n} \times (\mathbf{s}_0 \times \mathbf{E}_i)] \tag{5b-279a}$$

The induced surface charge density is

$$\rho = 2\epsilon(\mathbf{n} \cdot \mathbf{E}_i) \tag{5b-279b}$$

The radiated fields are

$$\mathbf{E}(\mathbf{r}) = \frac{2i}{\omega\epsilon} \int_S \{[\mathbf{n}' \times \mathbf{H}_i(\mathbf{r}')] \cdot \nabla'[\nabla' G(\mathbf{r},\mathbf{r}')] + k^2[\mathbf{n}' \times \mathbf{H}_i(\mathbf{r}')]G(\mathbf{r},\mathbf{r}')\} \, dS' \tag{5b-280a}$$

$$\mathbf{H}(\mathbf{r}) = 2 \int_S \{[\mathbf{n}' \times \mathbf{H}_i(\mathbf{r}')] \times \nabla' G(\mathbf{r},\mathbf{r}')\} \, dS' \tag{5b-280b}$$

where S is the illuminated surface of the reflector and

$$G(\mathbf{r},\mathbf{r}') = \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r} - \mathbf{r}'|}$$

The far-zone radiated fields are

$$\mathbf{E}(\mathbf{r}) \simeq \frac{i\omega\mu}{2\pi r} e^{ikr} \int_S (\mathbf{n}' \times \mathbf{H}_i(\mathbf{r}') - \{[\mathbf{n}' \times \mathbf{H}_i(\mathbf{r}')] \cdot \mathbf{e}_r\}\mathbf{e}_r)e^{-ik\mathbf{r}' \cdot \mathbf{e}_r} \, dS \tag{5b-281a}$$

$$\mathbf{H}(\mathbf{r}) \simeq \left(\frac{\epsilon}{\mu}\right)^{\frac{1}{2}} [\mathbf{e}_r \times \mathbf{E}(\mathbf{r})] \tag{5b-281b}$$

Linear Arrays of Antennas. A great variety of radiation patterns can be realized by arranging in space a set of antennas operating at the same frequency. The linear array has been used quite successfully to synthesize certain desired radiation patterns.[1] A linear array is assumed to consist of n antennas with centers at the points x_p ($p = 0, 1, \ldots, n - 1$) on the x axis. Each antenna is independently fed. Under the

[1] See the article by R. S. Elliott in "Microwave Scanning Antennas," edited by R. C. Hansen, Academic Press, Inc., New York, 1966.

approximation that the antennas do not interact with one another, the Poynting vector for the radiated field is

$$\mathbf{S} = \mathbf{e}_r \sqrt{\frac{\mu}{\epsilon}} \frac{1}{8\pi^2 r^2} |G(\theta,\phi)A(\theta,\phi)|^2 \tag{5b-282}$$

where $G(\theta,\phi)$ is the radiation pattern of each individual antenna, and $A(\theta,\phi)$ is called the array factor. The radiation pattern of the entire array is

$$u(\theta,\phi) = |G(\theta,\phi)A(\theta,\phi)| = G(\theta,\phi)|A(\theta,\phi)| \tag{5b-283}$$

For an array made up of center-fed half-wave dipoles ($kl = \pi/2$, $2l$ = length of the dipole) oriented parallel to the z axis,

$$G(\theta,\phi) = \frac{\cos[(\pi/2)\cos\theta]}{\sin\theta} \tag{5b-284a}$$

$$A(\theta,\phi) = \sum_{p=0}^{n-1} A_p e^{-ikx_p \sin\theta\cos\phi} \tag{5b-284b}$$

where A_p denotes the complex magnitude of the current. By the appropriate choice of A_p and x_p, many desired radiation patterns may be synthesized.

EQUALLY SPACED LINEAR ARRAY ($x_p = pd$; d is the uniform spacing). The array factor for an equally spaced linear array is

$$A(\theta,\phi) = \sum_{p=0}^{n-1} a_p \xi^p \tag{5b-285}$$

where $\xi = e^{i\alpha}$, $\alpha = -kd \sin\theta\cos\phi - \gamma$, and $A_p = a_p e^{-ip\gamma}$. $e^{-ip\gamma}$ is the progressive phasing γ of the array currents, and a_p is the magnitude of the currents.

1. Uniform Array (a_p = constant). The array factor of the uniform array is

$$A(\theta,\phi) = \sum_{p=0}^{n-1} \xi^p = \frac{\xi^n - 1}{\xi - 1} \tag{5b-286}$$

Broadside Array ($\gamma = 0$, $kd < 2\pi$). Radiation is cast principally in the broadside direction; $\psi = \pi/2$. (ψ is the angle between the x axis and the line of observation, and $\cos\psi = \sin\theta\cos\phi$.)

End-fire Array ($kd = -\gamma$ or $+\gamma$). Radiation is cast principally in the direction of the line of sources.

Hansen-Woodyard Unilateral End-fire Array [$\gamma = -(kd + \pi/n)$ or $\gamma = +(kd + \pi/2)$]. Radiation is cast principally in the direction $\psi = \pi$ when $\gamma = +(kd + \pi/n)$, and in the direction $\psi = 0$ when $\gamma = -(kd + \pi/n)$.

Phase Array [n and kd ($<\pi$) are fixed]. By varying γ from 0 to kd, the major lobe rotates from the broadside direction to the end-fire direction.

2. Nonuniform Array ($a_p \neq$ constant). The array factor of a nonuniform array is given by Eq. (5b-285).

Binomial Array. a_p are chosen as the binomial coefficient: $a_p = (n-1)!/[(n-1-p)!p!]$. When $\gamma = 0$, $kd = \pi$, the binomial array yields a broadside pattern without side lobes.

Dolph-Chebyshev Array (n even, $d \geq \lambda/2$). By matching the polynomial $A(\theta,\phi)$ in Eq. (5b-285) to a Chebyshev polynomial, one may obtain an array of a given number of elements which gives the lowest side lobes for a prescribed antenna gain, or highest gain for a prescribed side-lobe level.

UNEQUALLY SPACED LINEAR ARRAYS. Although Eq. (5b-284) for unequally spaced array is considerably more difficult to handle than Eq. (5b-285) for equally spaced array, with the use of a computer numerical results can be obtained in a straightforward manner. An unequally spaced array is generally more "broadband" than an equally spaced array.

Radio-astronomical Antennas. Consideration must be given to the case where the incident wave from cosmic sources is partially polarized and polychromatic.[1] Assuming that the radio-astronomical antenna is conjugate-matched to the load, the power absorbed by the load is

$$P_{abs} = \frac{1}{2}\sqrt{\frac{\epsilon}{\mu}}\, A(\theta,\phi)\, Tr(p^{rad}p^{rad*}) \cdot (\widetilde{\langle E^{inc}E^{inc*}\rangle}) \tag{5b-287}$$

where $A(\theta,\phi)$ is the effective area of the receiving antenna, i.e., $A(\theta,\phi) = (\lambda^2/4\pi)G(\theta,\phi)$, $G(\theta,\phi)$ is the gain function of antenna in transmission. p^{rad} is the field polarization vector, .

$$p^{rad} = \frac{\mathbf{E}^{rad}}{\sqrt{\mathbf{E}^{rad}\cdot\mathbf{E}^{rad*}}} \tag{5b-288}$$

where $\mathbf{E}^{rad}$ is the electric vector of the far-zone field radiated by the antenna in transmission. $\widetilde{\langle E^{inc}E^{inc*}\rangle}$ is the transpose of $\langle E^{inc}E^{inc}\rangle$ and, $\langle W\rangle$ means

$$\langle W\rangle = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} W\,dt \tag{5b-289}$$

For example, if the incident polychromatic wave is narrow-band and has the form

$$E^{inc}(\mathbf{r},t) = [\mathbf{e}_\theta E_\theta(t) + \mathbf{e}_\phi E_\varphi(t)]e^{-ikr}e^{-i\omega t} \tag{5b-290}$$

where $E_\theta(t)$ and $E_\phi(t)$ are slowly varying functions of time, and ω is a mean frequency, then the time-average power absorbed by the conjugate-matched load is

$$P_{abs} = \frac{1}{2}(1-m)A(\theta,\phi)\langle S^{inc}(\theta,\phi)\rangle + mA(\theta,\phi)\langle S^{inc}(\theta,\phi)\rangle\cos^2\frac{\gamma}{2} \tag{5b-291}$$

with

$$\cos\gamma = \cos 2\chi'\cos 2\chi\cos(2\psi' - 2\psi)\sin 2\chi'\sin 2\chi \tag{5b-292}$$

$$\langle S^{inc}(\theta,\phi)\rangle = \frac{1}{2}\sqrt{\frac{\epsilon}{\mu}}\,(\langle E_\theta E_\theta^*\rangle + \langle E_\phi E_\phi^*\rangle) \tag{5b-293}$$

γ is the angle between the point $(2\psi, -2\chi)$ describing the polarization ellipse of the incident wave and the point $(2\psi', 2\chi')$ describing the polarization ellipse of the radiated wave, and m is the degree of polarization which is the ratio of the power density of the polarized part to the total power density.[2]

A way to measure the degree of coherence $|\gamma|$ of an incoming polychromatic signal by the use of a correlation interferometer which requires no phase-preserving link has been suggested by Brown and Twiss.[3] The correlation interferometer (which consists of two identical antennas) measures the correlation coefficient $|\gamma|$ which is defined as

$$|\gamma| = \left\{\frac{\langle[M_1^2 - \langle M_1^2\rangle][M_2^2 - \langle M_2^2\rangle]\rangle}{\sigma(M_1^2)\sigma(M_2^2)}\right\}^{\frac{1}{2}} \tag{5b-294}$$

where

$$\sigma^2(M_1^2) = \langle(M_1^2 - \langle M_1^2\rangle)^2\rangle \tag{5b-295}$$

$$\sigma^2(M_2^2) = \langle(M_2^2 - \langle M_2^2\rangle)^2\rangle \tag{5b-296}$$

$$M_1^2 = V_1V_1^* \qquad M_2^2 = V_2V_2^* \tag{5b-297}$$

[1] H. C. Ko, *Proc. IRE* **49**, 1446 (1961).
[2] M. Born and E. Wolf, "Principles of Optics," 2d ed., Pergamon Press, New York, 1964.
[3] R. H. Brown and R. Q. Twiss, *Phil. Mag.* **45**, 663 (1954).

$V_1(t)V_1^*(t)$ is the power output of one antenna operating singly and $V_2(t)V_2^*(t)$ is the power output of the other antenna operating singly. The correlation interferometer of Brown and Twiss is an interferometer that measures $|\gamma(M_1^2,M_2^2)|$, while the conventional interferometer measures $\gamma(V_1,V_2)$. Hence, no phase-preserving link is necessary in the measurement of $|\gamma(M_1^2,M_2^2)|$, the antennas can be separated greatly, and thus high resolving powers can be realized. If the source of the polychromatic signal is a rectangular distribution of width $2w$, t'e correlation coefficient $|\gamma|$ is related to the width by the equation

$$|\gamma| = \left|\frac{\sin klw}{klw}\right| \tag{5b-298}$$

where l is the separation of the interferometer, $k = \omega/c$, and ω is a mean frequency.

Lorentz Reciprocity Theorem. Let $(\mathbf{E}_a,\mathbf{H}_a)$ be the fields generated by sources $(\mathbf{J}_a,\mathbf{J}_{ma})$, and $(\mathbf{E}_b,\mathbf{H}_b)$ be the fields generated by sources $(\mathbf{J}_b,\mathbf{J}_{mb})$, operating at the same frequency. Then, Lorentz reciprocity theorem states that

$$\int_{\text{all space}} (\mathbf{E}_a \cdot \mathbf{J}_b - \mathbf{H}_a \cdot \mathbf{J}_{mb})\, dV = \int_{\text{all space}} (\mathbf{E}_b \cdot \mathbf{J}_a - \mathbf{H}_b \cdot \mathbf{J}_{ma})\, dV \tag{5b-299}$$

With regard to antennas, the above theorem means that the receiving pattern of any antenna constructed of linear isotropic matter is identical to its transmitting pattern.

In general, reciprocity does not hold for an anisotropic medium. However, for the special case of an anisotropic plasma or ferrite, the concept of reciprocity can be generalized. This is based on the fact that the dielectric tensor of a magnetically biased plasma or the permeability tensor of a ferrite is symmetrical under a reversal of the biasing magnetostatic field: i.e., $\epsilon(\mathbf{B}_0) = \tilde{\epsilon}(-\mathbf{B}_0)$ or $\mu(\mathbf{B}_0) = \tilde{\mu}(-\mathbf{B}_0)$ where the tilde indicates the transpose dyadic. The reciprocity theorem then becomes

$$\int_{\text{all space}} (\mathbf{E}_b(-B_0) \cdot \mathbf{J}_a - \mathbf{H}_b(-B_0) \cdot \mathbf{J}_{ma})\, dV = \int_{\text{all space}} (\mathbf{E}_a(B_0) \cdot \mathbf{J}_b - \mathbf{H}_a(B_0) \cdot \mathbf{J}_{mb})\, dV \tag{5b-300}$$

Elementary Relations Concerning Antennas. Consider a transmitting antenna and a receiving antenna separated by a large distance r. The power absorbed by the receiving antenna is

$$P_r = P\frac{G_tG_r\lambda^2}{16\pi^2r^2} \tag{5b-301}$$

where P is the total power transmitted by the transmitting antenna, G_t and G_r are the respective gain functions of the two antennas for the direction of transmission, and λ is the wavelength of the radiated wave.

Now if an antenna is used for transmission as well as reception, such as for radar application, the power absorbed by the receiver from the scattered wave is

$$P_r = P\frac{\sigma\lambda^2G_t^2}{(4\pi)^3r^4} \tag{5b-302}$$

where r is the distance from the antenna to the scatterer, and σ is the scattering cross section of the scatterer. The scattering cross section is defined as the actual cross section of a sphere that in the same position as the scatterer would scatter back to the receiver the same amount of energy as is returned by the scatterer.

Radiation from Charged Particles. Radiation results when a charged particle accelerates or decelerates (Bremsstrahlung), when a charged particle moves along a curved path at a constant velocity (cyclotron radiation), when a charged particle moves at a constant velocity which is faster than the phase velocity of light in the medium (Čerenkov radiation), when a charged particle moves at a uniform velocity along an uneven surface (Smith-Purcell radiation), or when a charged particle moves through two media with different electrical properties (transition radiation).[1]

POINT CHARGE IN ARBITRARY MOTION IN FREE SPACE. The fields are:

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0 s^3}\left\{\mathbf{r}_u\left(1 - \frac{u^2}{c^2}\right) + \frac{1}{c^2}[\mathbf{r} \times (\mathbf{r}_u \times \dot{\mathbf{u}})]\right\} \tag{5b-303}$$

$$\mathbf{B} = \frac{1}{rc}\mathbf{r} \times \mathbf{E} \tag{5b-304}$$

with

$$s = r - \frac{\mathbf{r} \cdot \mathbf{u}}{c} \tag{5b-305}$$

$$\mathbf{r}_u = \mathbf{r} - \frac{r\mathbf{u}}{c} \tag{5b-306}$$

$$\mathbf{u} = -\frac{d\mathbf{r}}{dt'} \qquad \dot{\mathbf{u}} = \frac{d\mathbf{u}}{dt'} \tag{5b-307}$$

where $\mathbf{r}$ is the retarded radius vector which is the radius vector from the retarded position of the particle to the field point, $\mathbf{u}$ and $\dot{\mathbf{u}}$ are respectively the velocity vector and the acceleration vector of the particle at the retarded position, t' is the time of emission, and c is the velocity of light in vacuum. q is the charge of the particle. The second term in Eq. (5b-303) represents the radiated field. ϵ_0 is the free-space permittivity. The directional rate of radiation is

$$-\frac{dU}{dt'}\,d\Omega = \frac{q^2 r}{16\pi^2\epsilon_0 s^5 c^3}\,|\mathbf{r} \times (\mathbf{r}_u \times \dot{\mathbf{u}})|^2\,d\Omega \tag{5b-308}$$

and the total rate of radiation is

$$-\frac{dU}{dt'} = \frac{q^2}{6\pi\epsilon_0 c^3}\,\frac{|\dot{\mathbf{u}}|^2 - |\mathbf{u} \times \dot{\mathbf{u}}|^2/c^2}{(1 - u^2/c^2)^3} \tag{5b-309}$$

$-dU/dt'$ is also the rate of energy loss by the particle. Two useful special cases are listed in the following:

$\mathbf{u} \| \dot{\mathbf{u}}$ (Linear Motion)

$$-\frac{dU}{dt'}\,d\Omega = \frac{|\dot{\mathbf{u}}|^2}{c^3}\left(\frac{q^2}{16\pi^2\epsilon_0}\right)\frac{\sin^2\theta}{(1 - (u/c)\cos\theta)^5}\,d\Omega \tag{5b-310}$$

$$-\frac{dU}{dt'} = \frac{q^2|\dot{\mathbf{u}}|^2}{6\pi\epsilon_0 c^3(1 - u^2/c^2)^3} \tag{5b-311}$$

where θ is the angle between $\mathbf{u}$ and $\mathbf{r}$.

$\dot{\mathbf{u}} \perp \mathbf{u}$ (Circular Motion)

$$-\frac{dU}{dt'}\,d\Omega = \frac{q^2|\dot{\mathbf{u}}|^2}{16\pi^2\epsilon_0 c^3}\,\frac{(1 - u^2/c^2)\cos^2\alpha + (u/c - \sin\alpha\cos\phi)^2}{[1 - (u/c)\sin\alpha\cos\phi]^5}\,d\Omega \tag{5b-312}$$

$$-\frac{dU}{dt'} = \frac{q^2|\dot{\mathbf{u}}|^2}{6\pi\epsilon_0 c^3}\,\frac{1}{(1 - u^2/c^2)^2} \tag{5b-313}$$

where $\sin\alpha = \cos\theta/\cos\phi$, θ is the angle between $\mathbf{u}$ and $\mathbf{r}$, $\phi = \omega_0 t'$, $|\mathbf{u}| = a\omega_0$, and $|\dot{\mathbf{u}}| = a\omega_0^2$. The charge is assumed to be moving in a circle of radius a with a constant angular velocity ω_0.

ČERENKOV RADIATION. Čerenkov radiation occurs when a charged particle is moving in a material medium at a uniform speed u which is faster than the phase velocity of light in the medium. If n is the index of refraction of the medium,

[1] J. V. Jelley, "Čerenkov Radiation," Pergamon Press, New York, 1958.

Čerenkov radiation occurs (when $nu > c$, $n > 1$) at a cone angle of $\theta = \cos^{-1}(c/nu)$ with respect to the direction of motion. The field components are singular in that direction. c is the speed of light in vacuum. Energy radiated per unit length of path per frequency interval $(dU/dl)\,d\omega$ is

$$\frac{dU}{dl}\,d\omega = \frac{q^2}{4\pi\epsilon_0 c^2}\left(1 - \frac{c^2}{n^2u^2}\right)\omega\,d\omega \tag{5b-314}$$

and the total radiation rate is

$$\frac{dU}{dt'} = \frac{q^2u}{4\pi\epsilon_0 c^2}\int\left(1 - \frac{c^2}{n^2u^2}\right)\omega\,d\omega \tag{5b-315}$$

where the integration is carried out over ranges of ω where $c^2/n^2u^2 < 1$.

TRANSITION RADIATION. A burst of radiation occurs when a charged particle, moving at constant speed u, passes through the boundary between two media having different optical properties. Unlike Čerenkov radiation, this transition radiation will occur at any velocity of the particle, though its intensity increases with the energy. Assuming that a charged particle enters normally into a half space of refractive index n from vacuum, the energy radiated per frequency interval per unit solid angle u for $u \ll c$ is

$$U\,d\Omega\,d\omega = \frac{q^2u^2}{4\pi^3c^3\epsilon_0}\sin^2\theta\cos^2\theta\left(\frac{n^2-1}{n^2\cos\theta + \sqrt{n^2-\sin^2\theta}}\right)^2 d\Omega\,d\omega \tag{5b-316}$$

where θ is the angle between the outward unit normal from the dielectric half space and the line connecting the observation point with the point that the charge particle enters into the dielectric half space. If the half space is a perfect conductor, we have

$$U\,d\Omega\,d\omega = \frac{q^2u^2}{4\pi^3c^3\epsilon_0}\sin^2\theta\,d\Omega\,d\omega \qquad u \ll c \tag{5b-317}$$

The total energy spectral density per unit frequency interval for the perfectly conducting half-space case is

$$U\,d\omega = \frac{q^2u^2}{3\pi^2c^3\epsilon_0}\,d\omega \qquad u \ll c \tag{5b-318}$$

SMITH-PURCELL RADIATION. Radiation occurs when a charged particle moves at a uniform velocity u along an uneven surface. Assuming that the uneven surface is a sinusoidal diffraction grating of period d and amplitude a, and the medium above the grating is vacuum, the power radiated per unit solid angle (for $u \ll c$) is

$$P\,d\Omega = \frac{2q^2a^2\pi^2u^4}{\epsilon_0c^3d^4}\,\frac{\{[1-(u/c)\cos\theta]^2 - [1-(u/c)^2]\sin^2\theta\cos^2\phi\}}{[1-(u/c)\cos\theta]^5}\,d\Omega$$

and the total power radiated is $16a^2q^2\pi^3u^4/3d^4\epsilon_0c^3$. θ is the angle between the axis of the grating and the line connecting the point of observation with the retarted position of the charged particle, and ϕ is the azimuthal angle.

References

1. Silver, S.: "Microwave Antenna Theory and Design," Boston Technical Lithographers, Boston, 1963.
2. Van Bladel, J.: "Electromagnetic Fields," McGraw-Hill Book Company, New York, 1964.
3. Harrington, R. F.: "Time-harmonic Electromagnetic Fields," McGraw-Hill Book Company, New York, 1961.
4. Panofsky, W. K. H. and M. Phillips: "Classical Electricity and Magnetism," 2d ed., Addison-Wesley Publishing Company, Reading, Mass., 1962.

5b-12. Scattering and Diffraction. Scattering occurs when a propagating wave is interrupted by an obstacle. In an unbounded region, the scattered fields must satisfy the appropriate boundary conditions as well as the radiation condition.

Scattering Cross Sections. The fundamental problem in diffraction is the determination of the total field in amplitude, phase, and polarization. However, in many cases it is not required to know the total field in complete detail at all points; it is often sufficient to know such quantities as the total scattered power, the total power absorbed by the obstacle, or the amplitude of the electric field in a specified direction and at a great distance from the obstacle.

The fields of a plane wave propagating in a direction **u** are given by

$$\mathbf{E}_i = \mathbf{A}e^{i\mathbf{k}\cdot\mathbf{r}} \tag{5b-319}$$

$$\mathbf{H}_i = \left(\frac{\epsilon}{\mu}\right)^{\frac{1}{2}} (\mathbf{u}_3 \times \mathbf{A})e^{i\mathbf{k}\cdot\mathbf{r}} \tag{5b-320}$$

where $\mathbf{k} = k\mathbf{u}_3 = \omega\sqrt{\mu\epsilon}\,\mathbf{u}_3$, and **A** may be $(A_1\mathbf{u}_1 + A_2\mathbf{u}_2)$. $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ form an orthogonal set of unit vectors. At large distances, the scattered fields $(\mathbf{E}_{sc},\mathbf{H}_{sc})$ resulting from this incident plane wave are

$$\mathbf{E}_{sc} = \frac{e^{ikr}}{r}\mathbf{F} \tag{5b-321}$$

$$\mathbf{H}_{sc} = \frac{e^{ikr}}{r}\left(\frac{\epsilon}{\mu}\right)^{\frac{1}{2}} (\mathbf{e}_r \times \mathbf{F}) \tag{5b-322}$$

where **F** is a complex vector which is transverse to the unit vector $\mathbf{e}_r$ in the radial direction. For example, in the spherical coordinates r, θ, ϕ,

$$\mathbf{F} = F_\theta(\theta,\phi)\mathbf{e}_\theta + F_\phi(\theta,\phi)\mathbf{e}_\phi \tag{5b-323}$$

The time-averaged power scattered by the obstacle is

$$P_{sc} = \frac{1}{2}\,\mathrm{Re}\left[\int_S (\mathbf{E}_{sc} \times \mathbf{H}_{sc}^*)\cdot d\mathbf{S}\right] = \frac{1}{2}\left(\frac{\epsilon}{\mu}\right)^{\frac{1}{2}} \int_{\substack{\text{unit}\\\text{sphere}}} |\mathbf{F}|^2\,d\Omega \tag{5b-324}$$

where S is the surface of the scatterer, and Ω is the solid angle. If P_{abs} is the time averaged power dissipated in the scatterer, the following relationship can be shown

$$P_{abs} + P_{sc} = -\frac{2\pi}{\omega\mu}\,\mathrm{Im}\,[\mathbf{A}^*\cdot\mathbf{F}(\mathbf{u}_3)] \tag{5b-325}$$

where $\mathbf{F}(\mathbf{u}_3)$ is the radiation vector of the scattered wave in the direction of incidence (i.e., in the forward direction). The time-averaged incident power per unit area is

$$P_i = \frac{1}{2}\left(\frac{\epsilon}{\mu}\right)^{\frac{1}{2}} |\mathbf{A}|^2 \tag{5b-326}$$

The following quantities are defined to characterize the reradiating, absorbing or transmitting properties of a three-dimensional obstacle in an incident plane-wave field:

TOTAL SCATTERING CROSS SECTION

$$\sigma_{sc} = \frac{P_{sc}}{P_i} = \frac{\int_\Omega |\mathbf{F}|^2\,d\Omega}{|\mathbf{A}|^2} \tag{5b-327}$$

ABSORPTION CROSS SECTION

$$\sigma_a = \frac{P_{abs}}{P_i} \tag{5b-328}$$

EXTINCTION CROSS SECTION

$$\sigma_{\text{ext}} = \frac{P_{sc} + P_{\text{abs}}}{P_i} = \sigma_a + \sigma_{sc}$$
$$= -\frac{4\pi}{k} \operatorname{Im}\left[\frac{\mathbf{A}^* \cdot \mathbf{F}(\mathbf{u}_3)}{|\mathbf{A}|^2}\right] \tag{5b-329}$$

BISTATIC CROSS SECTION

$$\sigma_{\text{bistatic}} = \frac{4\pi|\mathbf{F}(\mathbf{u}')|^2}{|\mathbf{A}|^2} \tag{5b-330}$$

$\mathbf{u}'$ is the observation direction. The power scattered per unit solid angle in the $\mathbf{u}'$ direction is

$$\frac{dP_{sc}}{d\Omega} = \sigma_{\text{bistatic}} \frac{P_i}{4\pi} \tag{5b-331}$$

MONOSTATIC (RADAR) CROSS SECTION OR BACKSCATTERING CROSS SECTION

$$\sigma_{\text{mono}} = \frac{4\pi|\mathbf{F}(-\mathbf{u}_3)|^2}{|\mathbf{A}|^2} \tag{5b-332}$$

TRANSMISSION CROSS SECTION. A plane wave polarized in the $\mathbf{u}_p$ direction and propagating in the $\mathbf{u}$ direction is incident on a metallic screen provided with an aperture S. $\mathbf{F}(\mathbf{u})$ is the radiation vector of the transmitted wave with respect to the forward direction $\mathbf{u}$. The transmission cross section is

$$\sigma_t = \frac{2\pi}{k} \operatorname{Im}\,[\mathbf{u}_p \cdot \mathbf{F}(\mathbf{u})] \tag{5b-333}$$

Similar expressions to those given above are also available for two-dimensional scatterers. The far-zone scattered field due to an incident $\mathbf{E}$ wave whose electric vector is polarized in the z direction, which is parallel to the axis of the two-dimensional scatterer, is

$$E_z^{sc} \simeq \frac{e^{ikr}}{r^{\frac{1}{2}}} F(\phi)$$

where the cylindrical coordinates r, ϕ, z have been used. The scattering cross section, the extinction cross section, and the bistatic cross section are respectively

$$\sigma_{sc} = \frac{1}{|A|^2} \int_0^{2\pi} |F(\phi)|^2 \, d\phi$$
$$\sigma_{\text{ext}} = -\frac{4}{k} \operatorname{Re}\left[\left(\frac{\pi k}{-2i}\right)^{\frac{1}{2}} \frac{F(\mathbf{u})}{|A|}\right]$$
$$\sigma_{\text{bistatic}} = \frac{2\pi|F(\mathbf{u}')|^2}{|A|^2}$$

where $\mathbf{u}$ is the direction of incidence (i.e., the forward direction), $\mathbf{u}'$ is the observation direction, and A is the amplitude of the incident E wave. The formulas derived for an E wave can also be applied to an H wave provided that we replace E_z^{sc} by H_z^{sc}, $F(\phi)$ by $F(\phi)/(\mu/\epsilon)^{\frac{1}{2}}$, and A by $A/(\mu/\epsilon)^{\frac{1}{2}}$ in the above equations.

Integral Formulations. Scattering from objects of arbitrary shapes may be formulated in terms of integral equations. This formulation eliminates the necessity of separating the vector wave equations. Although in general the resultant integral equation is difficult to solve formally, with the help of high-speed computors numerical solutions may be readily obtained.[1]

[1] R. F. Harrington, "Field Computation by Moment Methods," The Macmillan Company, New York, 1968.

The surface current density on a perfectly conducting three-dimensional scatterer satisfies the following integral equation:[1]

$$\mathbf{J}_s(\mathbf{r}_0) - 2\int_S \mathbf{n}(\mathbf{r}_0) \times [\mathbf{J}_s(\mathbf{r}') \times \nabla' G(\mathbf{r}_0,\mathbf{r}')]\, dS' = 2\mathbf{n}(\mathbf{r}_0) \times \mathbf{H}_i(\mathbf{r}_0) \qquad (5b\text{-}334)$$

or

$$\mathbf{n}(\mathbf{r}_0) \times \mathbf{E}_i(\mathbf{r}_0) = -i\omega\mu \int_S \mathbf{n}(\mathbf{r}_0) \times \mathbf{J}_s(\mathbf{r}')G(\mathbf{r}',\mathbf{r}_0)\, dS' - \frac{1}{i\omega\epsilon}\int_S [\nabla' \cdot \mathbf{J}_s(\mathbf{r}')]\mathbf{n}(\mathbf{r}_0) \times \nabla' G(\mathbf{r}',\mathbf{r}_0)\, dS' \qquad (5b\text{-}335)$$

where $\mathbf{E}_i(\mathbf{r}_0)$ and $\mathbf{H}_i(\mathbf{r}_0)$ are the known incident electric and magnetic fields at $\mathbf{r}_0$ on the scatterer, $\mathbf{n}(\mathbf{r}_0)$ is the unit outward normal on the scatterer at $\mathbf{r}_0$, $G(\mathbf{r}_0,\mathbf{r}') = e^{ik|\mathbf{r}_0-\mathbf{r}'|}/4\pi|\mathbf{r}_0 - \mathbf{r}'|$, and S is the surface of the scatterer. The scattered field may then be found from the relation

$$\mathbf{E}^{sc}(\mathbf{r}) = i\omega\mu \int_S \left(u + \frac{1}{k^2}\nabla'\nabla'\right) G(\mathbf{r},\mathbf{r}') \cdot \mathbf{J}_s(\mathbf{r}')\, dS' \qquad (5b\text{-}336)$$

or

$$\mathbf{H}^{sc}(\mathbf{r}) = \int_S \mathbf{J}_s(\mathbf{r}') \times \nabla' G(\mathbf{r},\mathbf{r}')\, dS' \qquad (5b\text{-}337)$$

in which $G(\mathbf{r},\mathbf{r}') = e^{ik|\mathbf{r}-\mathbf{r}'|}/4\pi|\mathbf{r} - \mathbf{r}'|$. $\mathbf{J}_s$ is obtained from Eq. (5b-334), and u is the unit dyadic.

The surface current density on a perfectly conducting two-dimensional cylindrical scatterer satisfies the following integral equation

$$E_z^i(\mathbf{r}_0) - i\omega\mu \int_l J_z(\mathbf{r}')G_c(\mathbf{r}_0,\mathbf{r}')\, dl' = 0 \qquad (5b\text{-}338)$$

for an incident E wave, or

$$\frac{1}{2}J_t(\mathbf{r}_0) + \int_l J_t(\mathbf{r}')\frac{\partial}{\partial \mathbf{n}(\mathbf{r}')}G_c(\mathbf{r}_0,\mathbf{r}')\, dl' = -H_z^i(\mathbf{r}_0) \qquad (5b\text{-}339)$$

for an incident H wave. J_z is the current density along the axis of the cylindrical scatterer; J_t is the current density tangent to the boundary of the cylindrical scatterer but normal to the z axis. $\mathbf{n}$ is still the unit outward normal on the scatterer. $G_c(\mathbf{r},\mathbf{r}')$ is the two-dimensional Green's function:

$$G_c(\mathbf{r},\mathbf{r}') = -\frac{i}{4}H_0^{(1)}(k|\mathbf{r} - \mathbf{r}'|) \qquad (5b\text{-}340)$$

where $H_0^{(1)}(p)$ is the Hankel function of the first kind of order zero and argument p. l is the cross-sectional bounding curve of the scatterer. The scattered field may then be found from the relation

$$E_z^{sc}(\mathbf{r}) = -i\omega\mu \int_l J_z(\mathbf{r}')G_c(\mathbf{r},\mathbf{r}')\, dl' \qquad (5b\text{-}341)$$

or

$$H_z^{sc}(\mathbf{r}) = \int_l J_t(\mathbf{r}')\frac{\partial}{\partial \mathbf{n}(\mathbf{r}')}G_c(\mathbf{r},\mathbf{r}')\, dl' \qquad (5b\text{-}342)$$

Formulation of the problem of the scattering by dielectric obstacles in terms of integral equations is also possible. However, the results are too involved to be included here. The reader is referred to the literature.[2]

[1] A. W. Maue, *Z. Physik.* **126**, 601 (1949).
[2] P. C. Waterman, Scattering by Dielectric Obstacles, *Mitre Corp. Rept.* MTP-84, July, 1968.

Rayleigh Scattering (Low-frequency Scattering). Rather simple formulas for the scattered fields in the far zone are available when the wavelength of the incident wave is much greater than the largest linear dimension of the scatterer.[1]

DIELECTRIC SCATTERER. For three-dimensional dielectric scatterers,

$$\mathbf{E}^{sc} \simeq -\frac{k^2}{4\pi\epsilon}\mathbf{e}_r \times (\mathbf{e}_r \times \mathbf{p}^e)\frac{e^{ikr}}{r} \tag{5b-343}$$

$$\mathbf{H}^{sc} \simeq \frac{\omega k}{4\pi}\mathbf{e}_r \times \mathbf{p}^e \frac{e^{ikr}}{r} \tag{5b-344}$$

where $\mathbf{p}^e$ is the induced electric dipole moment which is orientated in the same direction as the electric vector of the incident field. $k = \omega\sqrt{\mu\epsilon}$ is the free-space wave number and $\mathbf{e}_r$ is a unit vector in the r direction. The induced electric dipole moment is the same as that for the static value for the dielectric sphere immersed in a static electric field orientated in the same direction as the incident electric vector. For a perfect dielectric sphere of radius a and a dielectric constant ϵ_1

$$\mathbf{p}^e = 4\pi\epsilon a^3 \frac{\epsilon_1/\epsilon - 1}{\epsilon_1/\epsilon + 2}\mathbf{e} \tag{5b-345}$$

$\mathbf{e}$ is a unit vector in the same direction as the electric vector of the incident field. The total scattering cross section is

$$\sigma_{sc} \simeq \frac{8}{3}\left(\frac{\epsilon_1/\epsilon - 1}{\epsilon_1/\epsilon + 2}\right)^2 (ka)^4 \pi a^2 \tag{5b-346}$$

The magnitude of the scattered field is

$$|F| = \left|\frac{\epsilon_1/\epsilon - 1}{\epsilon_1/\epsilon + 2} k^2 a^3 \sin\theta_s\right| \tag{5b-347}$$

where θ_s is the angle between the axis of the induced dipole and the point of observation.

For two-dimensional dielectric scatterers:

$$\left.\begin{aligned} E_z^{sc} &\sim \frac{ik^2}{4}\left(\frac{-2i}{\pi k}\right)^{\frac{1}{2}} \frac{e^{ikr}}{r^{\frac{1}{2}}} E_z^i \left(\frac{\epsilon_1}{\epsilon} - 1\right) S \\ \sigma_{sc}^E &= \left(\frac{\epsilon_1}{\epsilon} - 1\right)^2 \frac{k^3 S^2}{4} \end{aligned}\right\} \text{incident } E \text{ waves}$$

$$\sigma_{sc}^H = \frac{k^3}{8\epsilon\mu}\frac{|\mathbf{p}_e|^2}{|H_z^i|^2} \text{ (incident } H \text{ wave)}$$

where S is the cross-section area of the cylinder, ϵ_1 is the dielectric constant of the cylinder, and $\mathbf{p}_e$ is the induced electric dipole moment. E_z^i and H_z^i are the magnitudes of the incident waves.

PERFECTLY CONDUCTING SCATTERER. The scattered wave for a small perfectly conducting obstacle is due not only to an induced electric dipole of moment $\mathbf{p}^e$ but also to an induced magnetic dipole of moment $\mathbf{p}^m$. For the case of a perfectly conducting sphere of radius a, the induced electric dipole moment is $4\pi\epsilon a^3\mathbf{e}_x$, and the induced magnetic dipole moment is $(-2\pi\omega\mu\epsilon a^3/k)\mathbf{e}_y$. The incident electric vector is polarized in the $\mathbf{e}_x$ direction. The far-zone scattered electric fields are

$$E_\theta^{sc} \simeq \frac{e^{ikr}}{r} k^2 a^3 \cos\phi(\cos\theta - \tfrac{1}{2}) \tag{5b-348}$$

$$E_\phi^{sc} \simeq \frac{e^{ikr}}{r} k^2 a^3 \sin\phi(\tfrac{1}{2}\cos\theta - 1) \tag{5b-349}$$

[1] A. F. Stevenson, *J. Appl. Phys.* **24**, 1134, 1143 (1953).

The backscattering cross section is

$$\sigma_{\text{mono}} = 9\pi k^4 a^6 \tag{5b-350}$$

For two-dimensional perfectly conducting obstacles,

$$\sigma_{sc}{}^{E} = \frac{\pi^2}{k(\log kL)^2} \quad \text{(incident } E \text{ wave)}$$
$$\sigma_{sc}{}^{H} = \frac{1}{8} k^3 \left(2S^2 + \frac{|\mathbf{p}_e|^2}{\mu\epsilon|H^i|^2}\right) \quad \text{(incident } H \text{ wave)}$$

where L is the length of the contour, S is the cross-sectional area of the cylinder, and $\mathbf{p}_e$ is the induced electric dipole moment. For example, for a circular cylinder of radius a, $L = a$, $S = \pi a^2$, and $|\mathbf{p}_e| = 2\pi a^2 \sqrt{\mu\epsilon}\,|H^i|$.

Rayleigh-Gans Scattering or Born Approximation. Under the assumption that $|\epsilon_1/\epsilon - 1| \ll 1$, the scattered field by such a dielectric scatterer may be approximated by the following formula:

$$\mathbf{E}^{sc}(\mathbf{r}) = (\nabla^2 + k^2) \int_V \left(\frac{\epsilon_1}{\epsilon} - 1\right) \mathbf{E}_0 G(\mathbf{r},\mathbf{r}')\, dV' \tag{5b-351}$$

where $G(\mathbf{r},\mathbf{r}') = e^{ik|\mathbf{r}-\mathbf{r}'|}/4\pi|\mathbf{r} - \mathbf{r}'|$, V is the volume of the scatterer, and $\mathbf{E}_0$ is the incident field. In the far zone of the scatterer and in the direction of the unit vector $\mathbf{e}$,

$$\mathbf{E}^{sc}(\mathbf{r}) \sim \frac{k^2 e^{ikr}}{4\pi r} \int_V [\mathbf{E}_0 - (\mathbf{E}_0 \cdot \mathbf{e})\mathbf{e}] \left(\frac{\epsilon_1}{\epsilon} - 1\right) e^{-ik\mathbf{e}\cdot\mathbf{r}'}\, dV' \tag{5b-352}$$

For a dielectric sphere of radius a and $k^2a^3\,|\epsilon_1/\epsilon - 1| \ll 1$,

$$\mathbf{E}^{sc} \sim \frac{ka^2 e^{ikr}}{r} [\mathbf{E}_0 - (\mathbf{E}_0 \cdot \mathbf{e})\mathbf{e}] \frac{\epsilon_1/\epsilon - 1}{2 \sin \frac{1}{2}\theta} j_1(2ka \sin \tfrac{1}{2}\theta) \tag{5b-353}$$

where j_1 is the spherical Bessel's function of order 1. The total scattering cross section is

$$\sigma_{sc} = \frac{\pi a^2}{4} \left(\frac{\epsilon_1}{\epsilon} - 1\right)^2 \left\{\frac{5}{2} - \frac{\sin 4ka}{ka} + \frac{7}{16k^2a^2} (\cos 4ka - 1) + 2k^2a^2 + \left(\frac{1}{2k^2a^2} - 2\right) [\gamma + \ln (4ka) - \mathrm{Ci}(4ka)]\right\} \tag{5b-354}$$

where $\gamma = 0.5772 \ldots$ is the Euler's constant, and Ci is the cosine integral.

High-frequency Scattering. If the wavelength of an incident wave is much smaller than the smallest dimension of the scatterer, several approximation techniques for finding the scattered fields are available.

GEOMETRIC OPTICS APPROACH. Assume that a linearly polarized incident electric field in the direction $\mathbf{e}_p$ which is given by

$$\mathbf{E}^i = \mathbf{E}_0(\mathbf{e}_i) \frac{e^{ikR}}{R} \tag{5b-355}$$

impinges upon a perfectly conducting body (see Fig. 5b-9). The far-zone reflected electric field at the observation point is

$$\mathbf{E}^{rc} = D^{\frac{1}{2}} \{\mathbf{n}[\mathbf{n} \cdot \mathbf{E}^i] + \mathbf{n} \times [\mathbf{n} \times \mathbf{E}^i]\} \frac{R}{r} e^{ikr} \tag{5b-356}$$

where

$$D = \frac{R_1 R_2 \cos \theta}{(4R^2 + R_1R_2) \cos \theta + 2R(R_1 \sin^2 \theta_1 + R_2 \sin^2 \theta_2)} \tag{5b-357}$$

and θ_1 and θ_2 are the angles between the incident ray and the directions of the principal radii of curvature R_1 and R_2. θ is the angle between the incident ray and **n**. **n** is the outward unit normal on the surface of the scatterer at the point where the incident ray intersects the scatterer. Equation (5b-356) is valid only if $r \gg R_1$ or R_2 and if diffraction effects are not important.

A slightly better approximation for the scattered field can be obtained by assuming that the scattered fields are due to the induced current density in the illuminated region and the induced line distribution of charge along the bounding curve between the illuminated and the shadow regions. The induced current density is found according to the geometric optics method. The scattered fields can then be obtained according to Eqs. (5b-280*a*) and (5b-280*b*).

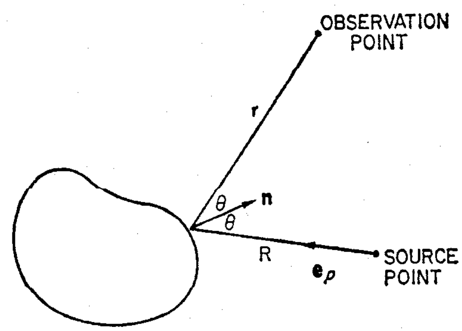

FIG. 5b-9. Reflection from a conducting obstacle.

A useful expression for the high-frequency radar cross section of perfectly conducting convex scatterer is also available:[1]

$$\sigma_{rad} = \sigma_{mono} = \pi R_1 R_2 \qquad (5b\text{-}358)$$

where R_1 and R_2 are the principal radii of curvature at the point at which the incident ray is perpendicular to the surface.

GEOMETRICAL THEORY OF DIFFRACTION. An extension of geometrical optics to account for diffraction phenomena has been proposed by Keller.[2] The main feature of the theory is the introduction of diffracted rays in addition to the usual rays of geometrical optics. These diffracted rays are produced by incident rays which hit edges, corners, or vertices of boundary surfaces, or which graze such surfaces. Some of these rays penetrate into the shadow regions and account for the existence of fields there. The initial value of the field on a diffracted ray is obtained by multiplying the field on the incident ray by a diffraction coefficient which takes different values for edge diffraction, vertex diffraction, etc. The value of the field along the diffracted ray is then obtained from its value at the diffraction point by the ordinary laws of geometrical optics. Several specific examples are given in the following.

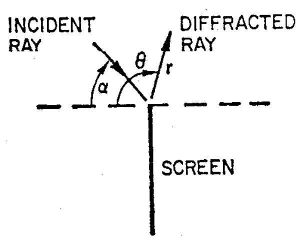

FIG. 5b-10. Diffracted rays from a straight conducting edge.

Fields Diffracted by Straight Edges. Let u_e be the field on a ray diffracted from an edge which is a straight line and the incident rays all lie in planes normal to the edge (see Fig. 5b-10). The diffracted field is

$$u_e = D u_i r^{-\frac{1}{2}} e^{ikr} \qquad (5b\text{-}359)$$

where D is the diffraction coefficient:

$$D = -\frac{e^{i\pi/4}}{2(2\pi k)^{\frac{1}{2}} \sin\beta} \left[\sec \tfrac{1}{2}(\theta - \alpha) \pm \csc \tfrac{1}{2}(\theta + \alpha)\right] \qquad (5b\text{-}360)$$

β is the angle between the incident ray and the edge, which is $\pi/2$ in the present normal incidence case. r is the distance from the edge. The angles between the incident

[1] R. G. Kouyoumjian, *Proc. IEEE* **53**, 864 (1965). High-frequency scattering by conducting ellipsoid has been treated by J. E. Burke and V. Twersky, *J. Acoust. Soc. Am.* **38**, 589 (1965).

[2] J. B. Keller, *J. Opt. Soc. Am.* **52**, 116 (1962).

and diffracted rays and the normal to the screen are θ and α, respectively. The upper sign applies when the boundary condition on the half-plane is $u = 0$ (i.e., $u_e = E_d$ where the incident E field is parallel to the edge), while the lower sign applies if it is $\partial u/\partial n = 0$.[1] (i.e., $u_e = H_d$ where the incident H field is parallel to the edge.) Equation (5b-359) is still valid for obliquely incident waves provided that θ and α are defined as above after the rays are first projected into the plane normal to the edge.

Fields Diffracted by Curved Edges. The diffracted field for a curved edge is

$$u_e = Du_i \left[r \left(1 + \frac{r}{\rho_1} \right) \right]^{-\frac{1}{2}} e^{ikr} \tag{5b-361}$$

where ρ_1 is the distance from the edge to the caustic of the diffracted rays, measured negatively in the direction of propagation. When the edge is a plane curve, ρ_1 is given by the relation

$$\frac{1}{\rho_1} = -\frac{\dot{\beta}}{\sin \beta} - \frac{\cos \delta}{\rho \sin^2 \beta} \tag{5b-362}$$

$\rho \geq 0$ denotes the radius of curvature of the edge, β is the angle between the incident ray and the (positive) tangent to the edge, $\dot{\beta}$ is the derivative of β with respect to arc length s along the edge, and δ is the angle between the diffracted ray and the normal to the edge.

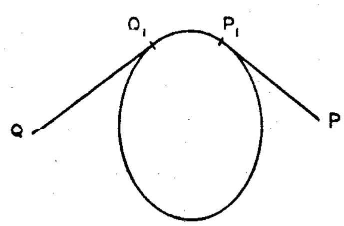

FIG. 5b-11. Surface diffracted rays.

Fields of Vertex-diffracted Ray. The diffracted field from a vertex is

$$u = Cu_i \frac{e^{ikr}}{r} \tag{5b-363}$$

where C is the vertex diffraction coefficient which has been evaluated only for a circular cone.

Fields of Surface-diffracted Rays. The diffracted rays are produced by incident rays which are tangent to the surface of the impenetrable body (see Fig. 5b-11). Each tangent ray splits at the point of tangency. One part continues along the path of the incident ray; another part travels along the surface of the body as a surface ray. This surface ray is a geodesic or shortest path on the body surface. Thus a single grazing incident ray gives rise to infinitely many diffracted rays.

The diffracted field is

$$u_d(P) = A_i(Q_1) \exp \{ik[\phi_i(Q_1) + t + s]\} \left[\frac{d\sigma(Q_1)}{d\sigma(P_1)} \right]^{\frac{1}{2}} \left[\frac{\rho_1}{s(\rho_1 + s)} \right]^{\frac{1}{2}} \cdot \sum_m D_m(P_1) D_m(Q_1) \exp \left[-\int_0^t \alpha_m(\tau)\, d\tau \right] \tag{5b-364}$$

where $A_i(Q_1)$ and $\phi_i(Q_1)$ are the amplitude and phase of the incident field at Q_1, t is the distance along the diffracted ray from Q_1 to P_1, s is the distance from P_1 to P, ρ_1 is the principal radius of curvature of the diffracted wavefront on the body, and $d\sigma(Q_1)/d\sigma(P_1)$ is the ratio of the width of a narrow strip of diffracted rays at Q_1 to that at P_1 on the surface of the body. The diffraction coefficients $D_m(P_1)$ and $D_m(Q_1)$ and the decay exponents α_m are obtained from a canonical problem with the appropriate boundary conditions. u_d corresponds to E_d with $u = 0$ on the cylindrical body when the incident E field is parallel to the axis of the cylinder while u_d corresponds to H_d with $\partial u/\partial n = 0$ on the cylindrical body when the incident H field is parallel to the

[1] For the special case of $\alpha = \pi/2$ with the boundary condition $\partial u/\partial n = 0$ on the screen, Eqs. (5b-359) and (5b-360) are not applicable. The revised form is

$$u_e = D' \frac{\partial u_i}{\partial n} r^{-\frac{1}{2}} e^{ikr} \qquad D' = -\frac{1}{ik} \left[\frac{\partial}{\partial \alpha} D(\theta,\alpha) \right] \Big|_{\alpha=\pi/2}$$

axis of the cylinder. Equation (5b-364) is not applicable without modification in the determination of the fields near the diffracting surface or near the shadow boundary. Application of Eq. (5b-364) to the problems of diffraction of waves by circular cylinders, spheres, parabolic cylinders, elliptic cylinders, etc., has been carried out successfully by Keller and his coworkers.[1]

Babinet's Principle. Consider three cases of a given source (1) radiating in free space, (2) radiating in the presence of an electrically conducting screen, and (3) radiating in the presence of a magnetically conducting screen. The electric and magnetic screens are said to be complementary if the two screens superimposed cover the entire $y = 0$ plane with no overlapping. Let the fields $y > 0$ be designated $(\mathbf{E}^i,\mathbf{H}^i)$, $(\mathbf{E}^e,\mathbf{H}^e)$, and $(\mathbf{E}^m,\mathbf{H}^m)$ for the cases 1, 2, and 3, respectively. Then Babinet's principle for complementary screens states that

$$\mathbf{E}^e + \mathbf{E}^m = \mathbf{E}^i \qquad \mathbf{H}^e + \mathbf{H}^m = \mathbf{H}^i \tag{5b-365}$$

The above Babinet's principle allows replacement of the aperture problem with an equivalent "disk" problem. Consider a plane metallic obstacle (disk) at $y = 0$ immersed in an incident wave $(\mathbf{E}^i = \mathbf{E}_0, \mathbf{H}^i = \mathbf{H}_0)$. The scattered fields are $(\mathbf{E}^{sc},\mathbf{H}^{sc})$. If one assumes that a wave $[\mathbf{E}^i = -\sqrt{(\mu/\epsilon)}\,\mathbf{E}_0,\ \mathbf{H}^i = \sqrt{(\epsilon/\mu)}\,\mathbf{H}_0]$ impinges on a metallic screen at $y = 0$ with an aperture of the same shape as the disk, the scattered fields on the shadow side of the aperture is $\mathbf{E}^{sc}_{\text{screen}} = \sqrt{(\mu/\epsilon)}\,\mathbf{H}^{sc}$, $\mathbf{H}^{sc}_{\text{screen}} = -\sqrt{(\epsilon/\mu)}\,\mathbf{E}^{sc}$ where $(\mathbf{E}^{sc},\mathbf{H}^{sc})$ are the scattered fields on the $y > 0$ side of the disk.

Diffraction by Simple Objects. DIFFRACTION BY SPHERE. A plane wave in an infinite, homogeneous medium (ϵ,μ), whose electric vector is linearly polarized in the x direction, is incident upon a sphere of radius a and constitutive parameters ϵ_1, μ_1 from the negative z axis. The incident wave $(\mathbf{E}_i,\mathbf{H}_i)$, the penetrated wave $(\mathbf{E}_p,\mathbf{H}_p)$, and the scattered wave $(\mathbf{E}_{sc},\mathbf{H}_{sc})$ are respectively

$$\mathbf{E}_i = \mathbf{e}_x E_0 e^{ikz} = E_0[\nabla \times \nabla \times (v_i r\mathbf{e}_r) + i\omega\mu\nabla \times (w_i r\mathbf{e}_r)] \tag{5b-366a}$$

$$\mathbf{H}_i = \mathbf{e}_y \frac{k}{\mu\omega} E_0 e^{ikz} = E_0[-i\omega\epsilon\nabla \times (v_i r\mathbf{e}_r) + \nabla \times \nabla \times (w_i r\mathbf{e}_r)] \tag{5b-366b}$$

$$\mathbf{E}_p = E_0[\nabla \times \nabla \times (v_p r\mathbf{e}_r) + i\omega\mu_1\nabla \times (w_p r\mathbf{e}_r)] \tag{5b-367a}$$

$$\mathbf{H}_p = E_0[-i\omega\epsilon_1\nabla \times (v_p r\mathbf{e}_r) + \nabla \times \nabla \times (w_p r\mathbf{e}_r)] \tag{5b-367b}$$

$$\mathbf{E}_{sc} = E_0[\nabla \times \nabla \times (v_s r\mathbf{e}_r) + i\omega\mu\nabla \times (w_s r\mathbf{e}_r)] \tag{5b-368a}$$

$$\mathbf{H}_{sc} = E_0[-i\omega\epsilon\nabla \times (v_s r\mathbf{e}_r) + \nabla \times \nabla \times (w_s r\mathbf{e}_r)] \tag{5b-369b}$$

with

$$v_i = \frac{-i}{k}\cos\phi \sum_{n=1}^{\infty} \frac{(i)^n 2n+1}{n(n+1)} j_n(kr)P_n{}^1(\cos\theta) \tag{5b-370a}$$

$$w_i = \frac{-i}{\omega\mu}\sin\phi \sum_{n=1}^{\infty} \frac{(i)^n 2n+1}{n(n+1)} j_n(kr)P_n{}^1(\cos\theta) \tag{5b-370b}$$

$$v_p = \frac{-i}{k_1}\cos\phi \sum_{n=1}^{\infty} c_n \frac{(i)^n 2n+1}{n(n+1)} j_n(k_1 r)P_n{}^1(\cos\theta) \tag{5b-371}$$

$$w_p = \frac{-i}{\omega\mu_1}\sin\phi \sum_{n=1}^{\infty} d_n \frac{(i)^n 2n+1}{n(n+1)} j_n(k_1 r)P_n{}^1(\cos\theta) \tag{5b-372}$$

$$v_s = \frac{-i}{k}\cos\phi \sum_{n=1}^{\infty} a_n \frac{(i)^n 2n+1}{n(n+1)} h_n{}^{(1)}(kr)P_n{}^1(\cos\theta) \tag{5b-373}$$

[1] B. R. Levy and J. B. Keller, *Communs. Pure Appl. Math.* **12**, 159 (1959).

$$w_s = \frac{-i}{\omega\mu} \sin\phi \sum_{n=1}^{\infty} b_n \frac{(i)^n 2n+1}{n(n+1)} h_n^{(1)}(kr) P_n^1(\cos\theta) \tag{5b-374}$$

$$a_n = \frac{(\epsilon_1/\epsilon) j_n(k_1a)[kaj_n(ka)]' - j_n(ka)[k_1aj_n(k_1a)]'}{(\epsilon_1/\epsilon) j_n(k_1a)[kah_n^{(1)}(ka)]' - h_n^{(1)}(ka)[k_1aj_n(k_1a)]'} \tag{5b-375a}$$

$$b_n = \frac{(\mu_1/\mu) j_n(k_1a)[kaj_n(ka)]' - j_n(ka)[k_1aj_n(k_1a)]'}{(\mu_1/\mu) j_n(k_1a)[kah_n^{(1)}(ka)]' - h_n^{(1)}(ka)[k_1aj_n(k_1a)]'} \tag{5b-375b}$$

$$c_n = \frac{(\mu_1\epsilon_1/\mu\epsilon)^{\frac{1}{2}}}{[k_1aj_n(k_1a)]'} \{[kaj_n(ka)]' - a_n[kah_n^{(1)}(ka)]'\} \tag{5b-375c}$$

$$d_n = \frac{1}{j_n(k_1a)} \{j_n(ka) - b_n h_n^{(1)}(ka)\} \tag{5b-375d}$$

The prime indicates the derivative of the function with respect to its argument, $k_1 = \omega\sqrt{\mu_1\epsilon_1}$, and $k = \omega\sqrt{\mu\epsilon}$. j_n and $h_n^{(1)}$ are respectively the Bessel and Hankel functions. P_n^1 is the associated Legendre function. For a perfectly conducting sphere, $a_n = [kaj_n(ka)]'/[kah_n^{(1)}(ka)]'$, $b_n = j_n(ka)/h_n^{(1)}(ka)$, $c_n = 0$, $d_n = 0$.

Far-zone Scattered Electric Field

$$\mathbf{E}_{sc}^{\text{far zone}} = \frac{e^{ikr}}{r} (F_\theta \mathbf{e}_\theta + F_\varphi \mathbf{e}_\phi) \tag{5b-376}$$

with

$$F_\theta = \frac{i}{k} \cos\phi \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)} \left[a_n \frac{d}{d\theta} P_n^1(\cos\theta) + b_n \frac{P_n^1(\cos\theta)}{\sin\theta} \right] \tag{5b-377}$$

$$F_\phi = -\frac{i}{k} \sin\phi \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)} \left[a_n \frac{P_n^1(\cos\theta)}{\sin\theta} + b_n \frac{d}{d\theta} P_n^1(\cos\theta) \right] \tag{5b-378}$$

Total Scattering Cross Section

$$\sigma_{sc} = \frac{2\pi}{k^2} \sum_{n=1}^{\infty} (2n+1)(|a_n|^2 + |b_n|^2) \tag{5b-379}$$

Extinction Cross Section

$$\sigma_{ext} = \frac{2\pi}{k^2} \text{Re} \left[\sum_{n=1}^{\infty} (2n+1)(a_n + b_n) \right] \tag{5b-380}$$

Radar Cross Section

$$\sigma_{rad} = \frac{\pi}{k^2} \left| \sum_{n=1}^{\infty} (2n+1)(-1)^n (a_n - b_n) \right|^2 \tag{5b-381}$$

Several typical curves for σ_{sc}, σ_{ext}, and σ_{rad} are given in Figs. 5b-12 and 5b-13.

High- and Low-frequency Limits

$$\sigma_{sc}^{(\text{conducting sphere})} \underset{ka\to\infty}{\simeq} 2\pi a^2[1 + 0.06595661(ka)^{-\frac{4}{3}} + 0.7797489(ka)^{-\frac{4}{3}} - 2.8713350(ka)^{-2} - 0.3385447(ka)^{-\frac{8}{3}} + 0.058460(ka)^{-\frac{10}{3}} + \cdots] \tag{5b-382}$$

$$\sigma_{rad}^{(\text{conducting sphere})} \underset{ka\to\infty}{\simeq} \pi a^2 \tag{5b-383}$$

$$\sigma_{sc}^{(\text{conducting sphere})} \underset{ka\to 0}{\simeq} \frac{10\pi}{3} k^4a^6 \left[1 + \frac{6}{25}(ka)^2 \right] \tag{5b-384}$$

$$\sigma_{rad}^{(\text{conducting sphere})} \underset{ka\to 0}{\simeq} 9\pi k^4a^6 \tag{5b-385}$$

$$\sigma_{sc}^{(\text{dielectric sphere})} \underset{ka\to 0}{\simeq} \frac{8\pi k^4a^6}{3} \left[\left(\frac{\epsilon_1/\epsilon - 1}{2 + \epsilon_1/\epsilon} \right)^2 + \left(\frac{\mu_1/\mu - 1}{2 + \mu_1/\mu} \right)^2 \right]$$

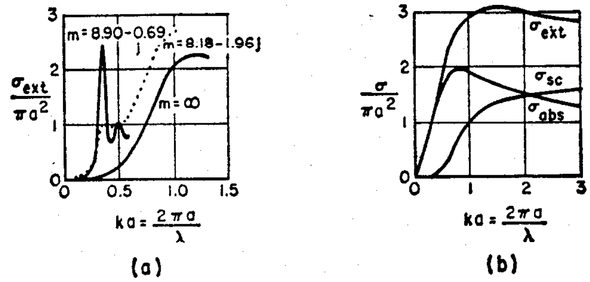

FIG. 5b-12. Typical cross sections for a sphere: (*a*) Typical extinction cross sections for various values of the index of refraction m [$= (\epsilon_1/\epsilon)^{\frac{1}{2}}$, where ϵ_1 is the complex dielectric constant of the sphere and ϵ is the free-space permittivity]. (*b*) Cross sections for an iron sphere. (At $\lambda = 0.42 \times 10^{-6}$ meter, the index of refraction for the iron sphere is $1.27 - j1.37$). a is the radius of the sphere, and λ is the free-space wavelength. (*From H. C. van der Hulst, "Light Scattering by Small Particles," John Wiley & Sons, Inc., New York,* 1957.)

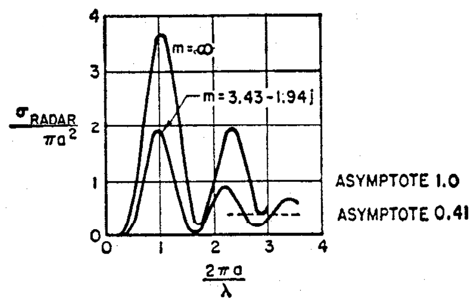

FIG. 5b-13. Typical radar cross section of a sphere with complex index of refraction, m. a is the radius of the sphere, and λ is the free-space wavelength. (*From H. C. van der Hulst, "Light Scattering by Small Particles," John Wiley & Sons, Inc., New York,* 1957.)

DIFFRACTION BY CIRCULAR CYLINDERS. A plane wave in an infinite, homogeneous medium (ϵ,μ) is incident upon a circular cylinder of radius a and constitutive parameters ϵ_1, μ_1 from the negative x axis. The axis of the cylinder is parallel to the z axis.[1] For an incident E wave, the incident wave $(\mathbf{E}_i{}^E,\mathbf{H}_i{}^E)$, the penetrated wave $(\mathbf{E}_p{}^E,\mathbf{H}_p{}^E)$ and the scattered wave $(\mathbf{E}_{sc}{}^E,\mathbf{H}_{sc}{}^E)$ are respectively

$$\mathbf{E}_i{}^E = E_0 e^{ikx}\mathbf{e}_z = E_0 \sum_{n=-\infty}^{\infty} (i)^n J_n(kr) e^{in\phi}\mathbf{e}_z \qquad (5b\text{-}386a)$$

$$\mathbf{H}_i{}^E = \frac{1}{i\omega\mu}(\nabla \times \mathbf{E}_i{}^E) \qquad (5b\text{-}386b)$$

$$\mathbf{E}_p{}^E = E_0 \sum_{n=-\infty}^{\infty} (i)^n b_n{}^E J_n(k_1 r) e^{in\phi}\mathbf{e}_z \qquad (5b\text{-}387a)$$

$$\mathbf{H}_p{}^E = \frac{1}{i\omega\mu_1}(\nabla \times \mathbf{E}_p{}^E) \qquad (5b\text{-}387b)$$

$$\mathbf{E}_{sc}{}^E = E_0 \sum_{n=-\infty}^{\infty} (i)^n a_n{}^E H_n{}^{(1)}(kr) e^{in\phi}\mathbf{e}_z \qquad (5b\text{-}388a)$$

$$\mathbf{H}_{sc}{}^E = \frac{1}{i\omega\mu}(\nabla \times \mathbf{E}_{sc}{}^E) \qquad (5b\text{-}388b)$$

[1] The problem of the scattering by an elliptical dielectric cylinder has been treated [C. Yeh, *J. Math. Phys.* **4**, 65 (1963)]. Solution for the scattering by parabolic dielectric cylinder has also been obtained [C. Yeh, *J. Opt. Soc. Am.* **57**, 195 (1967)].

with

$$a_n{}^E = \frac{(\epsilon_1\mu/\mu_1\epsilon)^{\frac{1}{2}} J_n'(k_1a)J_n(ka) - J_n(k_1a)J_n'(ka)}{H_n{}^{(1)\prime}(ka)J_n(k_1a) - (\epsilon_1\mu/\mu_1\epsilon)^{\frac{1}{2}} H_n{}^{(1)}(ka)J_n'(k_1a)} \tag{5b-389a}$$

$$b_n{}^E = \frac{H_n{}^{(1)\prime}(ka)J_n(ka) - J_n'(ka)H_n{}^{(1)}(ka)}{H_n{}^{(1)\prime}(ka)J_n(k_1a) - (\epsilon_1\mu/\mu_1\epsilon)^{\frac{1}{2}} H_n{}^{(1)}(ka)J_n'(k_1a)} \tag{5b-389b}$$

For an incident H wave, the incident wave $(\mathbf{E}_i{}^H,\mathbf{H}_i{}^H)$, the penetrated wave $(\mathbf{E}_p{}^H,\mathbf{H}_p{}^H)$ and the scattered wave $(\mathbf{E}_{sc}{}^H,\mathbf{H}_{sc}{}^H)$ are respectively

$$\mathbf{E}_i{}^H = \frac{-1}{i\omega\epsilon}(\nabla \times \mathbf{H}_i{}^H) \tag{5b-390a}$$

$$\mathbf{H}_i{}^H = H_0 e^{ikx}\mathbf{e}_z = H_0 \sum_{n=-\infty}^{\infty} (i)^n J_n(kr) e^{in\phi}\mathbf{e}_z \tag{5b-390b}$$

$$\mathbf{E}_p{}^H = \frac{-1}{i\omega\epsilon_1}(\nabla \times \mathbf{H}_p{}^H) \tag{5b-391a}$$

$$\mathbf{H}_p{}^H = H_0 \sum_{n=-\infty}^{\infty} (i)^n b_n{}^H J_n(k_1r) e^{in\phi}\mathbf{e}_z \tag{5b-391b}$$

$$\mathbf{E}_{sc}{}^H = \frac{-1}{i\omega\epsilon}(\nabla \times \mathbf{H}_{sc}{}^H) \tag{5b-392a}$$

$$\mathbf{H}_{sc}{}^H = H_0 \sum_{n=-\infty}^{\infty} (i)^n a_n{}^H H_n{}^{(1)}(kr) e^{in\phi}\mathbf{e}_z \tag{5b-392b}$$

with

$$a_n{}^H = \frac{(\epsilon\mu_1/\mu\epsilon_1)^{\frac{1}{2}} J_n'(k_1a)J_n(ka) - J_n(k_1a)J_n'(ka)}{H_n{}^{(1)\prime}(ka)J_n(k_1a) - (\epsilon\mu_1/\mu\epsilon_1)^{\frac{1}{2}} H_n{}^{(1)}(ka)J_n'(k_1a)} \tag{5b-393a}$$

$$b_n{}^H = \frac{H_n{}^{(1)\prime}(ka)J_n(ka) - J_n'(ka)H_n{}^{(1)}(ka)}{H_n{}^{(1)\prime}(ka)J_n(k_1a) - (\epsilon\mu_1/\mu\epsilon_1)^{\frac{1}{2}} H_n{}^{(1)}(ka)J_n'(k_1a)} \tag{5b-393b}$$

where the prime signifies the derivative of the function with respect to its argument, $k_1 = \omega\sqrt{\mu_1\epsilon_1}$ and $k = \omega\sqrt{\mu\epsilon}$. J_n and $H_n{}^{(1)}$ are respectively the Bessel and Hankel functions. For a perfectly conducting circular cylinder $a_n{}^E = -J_n(ka)/H_n{}^{(1)}(ka)$, $b_n{}^E = 0$ and $a_n{}^H = -J_n'(ka)/H_n{}^{(1)\prime}(ka)$, $b_n{}^H = 0$.

Far-zone Scattered Field

$$\mathbf{E}_{sc}{}^E \text{ (far zone)} \simeq \frac{e^{ikr}}{\sqrt{r}}\mathbf{e}_z F_z{}^E \tag{5b-394}$$

$$\mathbf{H}_{sc}{}^H \text{ (far zone)} \simeq \frac{e^{ikr}}{\sqrt{r}}\mathbf{e}_z F_z{}^H \tag{5b-395}$$

with

$$F_z{}^{E,H} = \left(\frac{-2i}{\pi k}\right)^{\frac{1}{2}} \sum_{n=-\infty}^{\infty} a_n{}^{E,H} e^{in\phi} \tag{5b-396}$$

Total Scattering Cross Section

$$\sigma_{sc}{}^{E,H} = -\frac{4}{k}\,\mathrm{Re} \sum_{n=-\infty}^{\infty} a_n{}^{E,H} \tag{5b-397}$$

Radar Cross Section

$$\sigma_{\text{rad}}^{E,H} = \frac{4}{k}\left|\sum_{n=0}^{\infty} d_n(-1)^n a_n{}^{E,H}\right|^2 \tag{5b-398}$$

with $d_0 = 1$, and $d_n = 2$ for $n \neq 0$.

High- and Low-frequency Limits

$$\sigma_{sc}{}^{E}\ \text{(conducting cylinder)} \underset{ka\to\infty}{\simeq} 4a[1 + 0.49807659(ka)^{-2/3} - 0.01117656(ka)^{-4/3} + \cdots] \qquad \text{(5b-399)}$$

$$\sigma_{sc}{}^{H}\ \text{(conducting cylinder)} \underset{ka\to\infty}{\simeq} 4a[1 - 0.43211998(ka)^{-2/3} - 0.21371236(ka)^{-4/3} + \cdots] \qquad \text{(5b-400)}$$

$$\sigma_{sc}{}^{E}\ \text{(conducting cylinder)} \underset{ka\to 0}{\simeq} \frac{\pi^2}{k\,(\log ka)^2} \qquad \text{(5b-401)}$$

$$\sigma_{sc}{}^{H}\ \text{(conducting cylinder)} \underset{ka\to 0}{\simeq} \frac{3\pi^2}{4}\,a(ka)^3 \qquad \text{(5b-402)}$$

A typical scattering cross section of a circular cylinder is given in Fig. 5b-14.

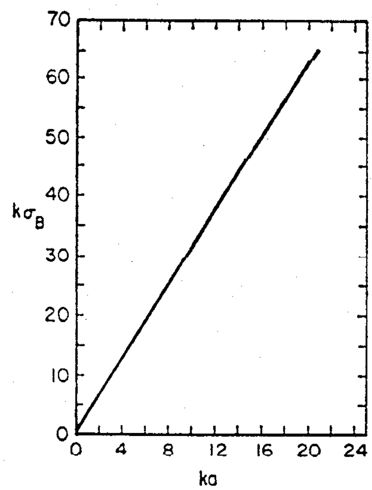

FIG. 5b-14. Backscattering cross section σ_B for a conducting cylinder of radius a. $k = \omega\sqrt{\mu\epsilon}$.

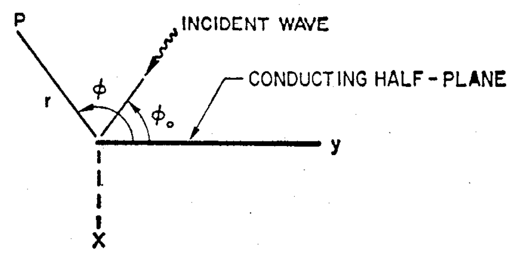

FIG. 5b-15. Diffraction by a semi-infinite conducting half plane.

DIFFRACTION BY A PERFECTLY CONDUCTING HALF PLANE. A perfectly conducting thin plane is located at the plane $x = 0$ and extends from $y = 0$ to $y = +\infty$ (see Fig. 5b-15). The solution for an incident plane wave

$$\left.\begin{matrix} E_z{}^i \\ H_z{}^i \end{matrix}\right\} = \left\{\begin{matrix} E_0 \\ H_0 \end{matrix}\right\} e^{-ikr\cos(\phi-\phi_0)} \qquad \text{(5b-403)}$$

is

$$\left.\begin{matrix} E_z{}^i + E_z{}^{sc} \\ H_z{}^i + H_z{}^{sc} \end{matrix}\right\} = \left\{\begin{matrix} E_0 \\ H_0 \end{matrix}\right\} e^{-ikr\cos(\phi-\phi_0)}\frac{1-i}{2}\int_{-\infty}^{a} e^{i\pi\tau^2/2}\,d\tau \mp e^{-ikr\cos(\phi-\phi_0)}\frac{1-i}{2}\int_{-\infty}^{b} e^{i\pi\tau^2/2}\,d\tau \qquad \text{(5b-404)}$$

with

$$a = 2\left(\frac{kr}{\pi}\right)^{1/2}\cos\left(\frac{\phi-\phi_0}{2}\right)$$

$$b = 2\left(\frac{kr}{\pi}\right)^{1/2}\cos\left(\frac{\phi+\phi_0}{2}\right)$$

Numerical results may be obtained with the help of the tabulated values for the Fresnel integral which is $F(w) = \int_0^w e^{i(\pi/2)\tau^2}\,d\tau$.

DIFFRACTION BY AN APERTURE IN AN INFINITE CONDUCTING SCREEN. The total scattering cross section σ_{sc} of a strip is related to the transmission coefficient t of a slit by the relation

$$t^{H,E} = \frac{\sigma_{sc}{}^{E,H}}{2A} \tag{5b-405}$$

where A is the cross-sectional area of the aperture.[1]

Transmission through a Slit of Width δ

$$t^H = \frac{\sigma_{sc}{}^E}{2\delta} \underset{k\delta\to 0}{\simeq} \frac{\pi^2/2k\delta}{[\gamma + \log (k\delta/8)]^2 + \pi^2/4}\left[1 + \frac{(k\delta)^2}{16} + \cdots\right] \tag{5b-405a}$$

$$t^E = \frac{\sigma_{sc}{}^H}{2\delta} \underset{k\delta\to 0}{\simeq} \frac{\pi^2(k\delta)^3}{256}\left\{1 + \frac{5(k\delta)^2}{64}\left[1 - \frac{8}{5}\left(\gamma + \log\frac{k\delta}{8}\right)\right]\right\} \tag{5b-406}$$

$$t^H = \frac{\sigma_{sc}{}^E}{2\delta} \underset{k\delta\to \infty}{\simeq} \left\{1 - \frac{\sin (k\delta - \pi/4)}{(2\pi)^{\frac{1}{2}}(k\delta)^{\frac{3}{2}}} + \frac{27 \sin (k\delta + \pi/4)}{8(2\pi)^{\frac{1}{2}}(k\delta)^{\frac{5}{2}}} + \frac{\sin 2(k\delta - \pi/4)}{8\pi(k\delta)^4} + \cdots\right\} \tag{5b-407}$$

$$t^E = \frac{\sigma_{sc}{}^H}{2\delta} \underset{k\delta\to \infty}{\simeq} \left\{1 - 2\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\frac{\cos (k\delta - \pi/4)}{(k\delta)^{\frac{3}{2}}} + \frac{2}{\pi}\frac{\cos 2k\delta}{(k\delta)^2} - \frac{1}{\pi}\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\frac{\cos (3k\delta + \pi/4) - (7\pi/4)\cos (k\delta + \pi/4)}{(k\delta)^{\frac{5}{2}}} - \frac{1}{\pi^2}\frac{\sin 4k\delta - (5\pi/2)\sin 2k\delta}{(k\delta)^3} + \cdots\right\} \tag{5b-408}$$

with $\gamma = 0.5772$.

Transmission through a Circular Aperture of Radius a

$$t = \frac{\sigma_{sc}}{2\pi a^2} \underset{ka\to 0}{=} \frac{64}{27\pi^2}(ka)^4[1 + \tfrac{22}{25}(ka)^2 + 0.3979\,(ka)^4 + \cdots] \tag{5b-409}$$

$$t = \frac{\sigma_{sc}}{2\pi a^2} \underset{ka\to \infty}{=} 1 - \frac{1}{\sqrt{\pi}}\frac{1}{(ka)^{\frac{3}{2}}}\sin\left(2ka - \frac{\pi}{4}\right) + \frac{1}{(ka)^2}\left[\frac{3}{4} + \frac{1}{2\pi}\sin 2\left(2ka - \frac{\pi}{4}\right)\right] - \cdots \tag{5b-410}$$

A typical transmission coefficient of a circular aperture is given in Fig. 5b-16.

Holography. Holography may be described as a method for recording and reconstructing the amplitude and phase information of a propagating field in a given plane.[2] Strictly speaking, rigorous electromagnetic theory of diffraction and polarization is required for an exact treatment of optical holography. Since the electromagnetic field under consideration is almost completely linearly polarized (i.e., only a small fraction of the energy is in the cross-polarization component of the field) and the wavelength of the field is much smaller than the smallest characteristic length of the scattering objects, a scalar physical optics description of the field is therefore adequate.

THE RECORDING PROCESS. The magnitude and the phase of a scattered wavefront can be recorded photographically by superposing a coherent reference wave on the field striking the photographic plate. One of the techniques for carrying out this

[1] H. Levine and J. Schwinger, *Communs. Pure Appl. Math.* **3**, 355 (1950).

[2] D. Gabor, *Proc. Roy. Soc.* (*London*), ser. A, **197**, 454 (1949); ser. B, **64**, 449 (1951); E. N. Leith and J. Upatnieks, *J. Opt. Soc. Am.* **52**, 1123 (1962); G. W. Stroke, "An Introduction to Coherent Optics and Holography," Academic Press, Inc., New York, 1966.

superposition is illustrated in Fig. 5b-17 wherein a plane wave illuminates a region containing the scattering object and a triangular prism. The scattering object diffracts the incident radiation to generate a field with magnitude $A(x)$ and phase $\phi(x)$ at the recording photographic plate, while the prism turns the incident plane wave through a small angle θ to give a field with a uniform magnitude A_0 and a linear phase variation αx where $\alpha = 2\pi \sin \theta/\lambda \underset{\theta \text{ small}}{\simeq} 2\pi\theta/\lambda$ with λ = wavelength. The total field at the recording plate is

$$u_{\text{total}} = A_0 e^{-i\alpha x} + A(x)e^{i\phi(x)} \qquad (5\text{b-}411)$$

and the intensity to which the emulsion is sensitive is

$$I(x) = |u_{\text{total}}|^2 = A_0^2 + A^2(x) + 2A_0A(x) \cos [\alpha x + \phi(x)] \qquad (5\text{b-}412)$$

Note that the intensity recorded by the photographic plate contains information concerning not only $A(x)$, the amplitude of the scattered wave, but also $\phi(x)$, the phase of the scattered wave.

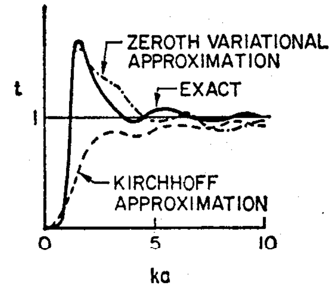

FIG. 5b-16. Transmission coefficient of a circular aperture of radius a. [*From C. Huang, R. D. Kodis, and H. Levine, J. Appl. Phys.* **26**, 151 (1955).]

THE RECONSTRUCTION PROCESS. Let us first consider the transmission characteristics of the recording photographic plate. The transmittance $T(x)$ of the resultant photographic plate, provided that the linear range of the Hurter-Driffield curve is used, is

$$T(x) \sim [I(x)]^{-\gamma/2} = \{A_0^2 + A^2(x) + 2A_0A(x) \cos [\alpha x + \phi(x)]\}^{-\gamma/2}$$
$$\sim 2A_0^2 - \gamma A^2(x) - \gamma A_0A(x)e^{i\phi(x)+i\alpha x} - \gamma A_0A(x)e^{-i\phi(x)-i\alpha x} \qquad (5\text{b-}413)$$

where γ is the slope of the Hurter-Driffield curve. It has been assumed that the intensity of the reference wave is much greater than that of the radiation scattered

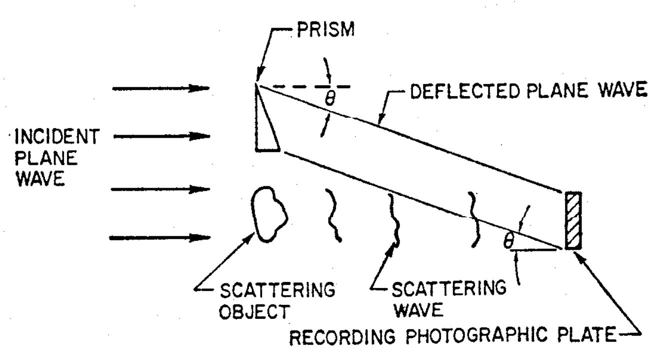

FIG. 5b-17. Schematic arrangement to illustrate recording of hologram.

by the object, so that the approximation made in dropping the higher-orders terms of the binomial expansion is justified. Note that neither the sign nor the exact magnitude of γ is of any consequence in the recording process; i.e., making a contact print of the photograph (hologram), which is equivalent to changing the sign of γ, serves only to shift the phase of the nonconstant portion of the transmittance an inconsequential 180°, whereas changing slightly the magnitude of γ serves only to enhance or to suppress the magnitude of this same portion of the transmittance.

To reconstruct the original wavefront it is only necessary to illuminate the hologram with a plane incident wave, as shown in Fig. 5b-18. As the plane wave passes through the photographic plate, it is multiplied by the transmittance $T(x)$, thereby producing four distinct components of radiation corresponding to four terms of Eq. (5b-413). The first term, being a constant, attenuates the parallel beam uniformly, but otherwise does not alter it. The second term also attenuates the beam, but not uniformly, so that the plane wave suffers some diffraction as it passes through the hologram. Recall that a common triangular prism shifts the phase of an incident ray by an amount proportional to its thickness at the point of incidence, a positive phase shift deflecting the ray upward and a negative one deflecting it downward. In the case of the third term in Eq. (5b-413), it represents an upward deflected beam multiplied by the scattered wave $A(x)e^{i\phi(x)}$; hence it is a reconstruction of the scattered wavefront. The fourth term represents a downward beam multiplied by the complex conjugate of the scattered wave. Hence, a copy of the scattered wavefront is con-

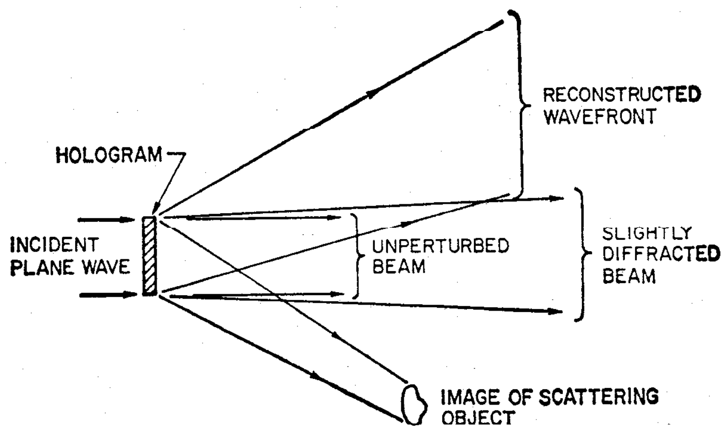

FIG. 5b-18. The reconstruction process—image formation from a hologram for the case of plane-wave illumination.

structed except that it travels backward in time. Consequently, a three-dimensional image of the scattering object is constructed.

Magnification. Magnification or demagnification of the image may be accomplished if one uses an incident wave with wavelength λ for making the hologram and uses an incident wave with wavelength λ' in the reconstruction of the image. The formula for linear magnification M is

$$M = \frac{\lambda'}{\lambda}\frac{q'}{q} \qquad (5b\text{-}414)$$

where q is the distance of the original object from the hologram, and q' is the distance of the hologram from the final image plane.

Resolution. The ultimate resolution of the conventional Fresnel-transform projection wavefront-reconstruction technique described above is approximately one-half that of the recording media. However, higher resolutions may be obtained by the use of Fourier-transform holography.

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5b-13. Waves in Plasma. Three basic features characterize plasmas and distinguish them from ordinary solids, liquids, or gases. The first feature is that at least some or all of the particles in a plasma are charged although the plasma as a whole is electrically neutral. The second feature is that Debye shielding effect must be present in plasmas. The third feature is that the product $\omega\tau$ must be large in order that plasma effects may be important. (ω = frequency of the wave in plasma, τ = the average time an electron travels between collisions with neutral molecules, or lattice ions, or impurities, etc.)

Basic Equations. The basic equations governing the waves in plasmas are the Boltzmann equation and Maxwell's equations:

$$\frac{\partial f_\alpha}{\partial t} + \mathbf{v}\cdot\frac{\partial f_\alpha}{\partial \mathbf{r}} + \mathbf{a}\cdot\frac{\partial f_\alpha}{\partial \mathbf{v}} = \left(\frac{\partial f_\alpha}{\partial t}\right)_c \tag{5b-415}$$

$$\begin{aligned} \nabla\times\mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \qquad \mathbf{D} = \epsilon_0\mathbf{E} \\ \nabla\times\mathbf{H} &= \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \qquad \mathbf{B} = \mu_0\mathbf{H} \end{aligned} \tag{5b-416}$$

$$\begin{aligned} \nabla\cdot\mathbf{D} &= \rho_c \\ \nabla\cdot\mathbf{B} &= 0 \\ \rho_c &= \sum_\alpha q_\alpha \iiint f_\alpha\, dv_x\, dv_y\, dv_z \\ \mathbf{J} &= \sum_\alpha q_\alpha \iiint \mathbf{v} f_\alpha\, dv_x\, dv_y\, dv_z \end{aligned} \tag{5b-417}$$

where $f_\alpha(\mathbf{x},\mathbf{v},t)$ is the distribution function for particles of type α, and $\mathbf{a}$ is the acceleration due to external forces, which for an electromagnetic field would be the Lorentz accleration $\mathbf{a} = (q_\alpha/m_\alpha)(\mathbf{E} + \mathbf{v}\times\mathbf{B})$. $(\partial f_\alpha/\partial t)_c$ is the time rate of change due to collisions. $\mathbf{E}$, $\mathbf{H}$, $\mathbf{B}$, $\mathbf{D}$ are the electromagnetic field vectors. ρ_c and $\mathbf{J}$ are respectively the charged density and the vector current density. q_α and m_α are respectively the charge and mass for particles of type α. $\mathbf{v}$ and $\mathbf{r}$ are the velocity and position vectors.

When collisions are neglected, we may set $(\partial f_\alpha/\partial t)_c = 0$ in Eq. (5b-415). This equation is called the collisionless Boltzmann equation or the Boltzmann-Vlasov equation.

HYDRODYNAMIC-CONTINUUM MODEL. Taking the appropriate moments of Eq. (5b-415) and making the assumption that (1) the mass density ρ_α for each species is unchanged, (2) the Lorentz force per unit mass for each species is $\langle\mathbf{a}\rangle_\alpha = (q_\alpha/m_\alpha)(\mathbf{E} + \mathbf{u}_\alpha\times\mathbf{B})$, (3) viscous effects are negligible, i.e., the pressure is a scalar quantity, and (4) the flow-velocity difference among the various gas species is small and each gas has a maxwellian velocity distribution, one obtains the following equations for the hydrodynamic-continuum model:

$$\frac{\partial \rho_\alpha}{\partial t} + \nabla\cdot\rho_\alpha\mathbf{u}_\alpha = 0 \qquad \text{(mass conservation)} \tag{5b-418}$$

$$\frac{\partial \mathbf{u}_\alpha}{\partial t} + \mathbf{u}_\alpha\cdot\nabla\mathbf{u}_\alpha = \frac{q_\alpha}{m_\alpha}(\mathbf{E} + \mathbf{u}_\alpha\times\mathbf{B}) - \frac{\nabla p_\alpha}{\rho_\alpha} - \sum_\beta \nu_\alpha(\mathbf{u}_{\alpha\beta} - \mathbf{u}_\beta) \qquad \text{(momentum conservation)} \tag{5b-419}$$

$$\nabla p_\alpha = U_\alpha^2\nabla\rho_\alpha \qquad \text{(energy conservation)} \tag{5b-420}$$

with $\rho_\alpha = m_\alpha n_\alpha$, $\rho_c = \sum_\alpha q_\alpha n_\alpha$, and $\mathbf{J} = \sum_\alpha q_\alpha n_\alpha \mathbf{u}_\alpha$. The subscript α refers to particles of type α. ρ_α, m_α, n_α, ρ_c, $\mathbf{J}$, $\mathbf{u}_\alpha$, and p_α are respectively the mass density, mass, number density, charge density, current density, average velocity vector, and scalar pressure. U_α is the adiabatic or the isothermal sound speed, depending on the problem at hand. $\nu_{\alpha\beta}$ is the collision frequency for momentum transfer for particles of type α with those of type β.

Equations (5b-418) to (5b-420), together with Maxwell's equations (5b-416) provide a complete set of equations for the hydrodynamic model.

LINEARIZED MAGNETOHYDRODYNAMIC (MHD) MODEL. A set of linearized mhd equations may be obtained if we replace the above set of individual-species equations (5b-418) to (5b-420) by a set of equations for the gas as a whole:

$$\frac{\partial \rho}{\partial t} + \rho_0 \nabla \cdot \mathbf{u} = 0 \tag{5b-421}$$

$$\rho_0 \frac{\partial \mathbf{u}}{\partial t} = \mathbf{J} \times \mathbf{B}_0 - \nabla p \tag{5b-422}$$

$$\nabla p = U_s^2 \nabla \rho \tag{5b-423}$$

$$\mathbf{J} = \sigma_0(\mathbf{E} + \mathbf{u} \times \mathbf{B}_0) \tag{5b-424}$$

where $\mathbf{B}_0$ is the applied magnetostatic field, σ_0 is the conductivity of the gas, U_s is the isothermal or adiabatic sound speed for the gas, ρ_0 is the equilibrium mass density of the gas. ρ, $\mathbf{u}$, p, $\mathbf{E}$, and $\mathbf{B}$ are all infinitesimal disturbances. A simplified Ohm's law [Eq. (5b-424) has been assumed. Equations (5b-421) to (5b-424), together with Maxwell's equations (5b-416)—with the assumption that the displacement vector term $\partial \mathbf{D}/\partial t$ is negligible—provide a complete set of equations for the linearized mhd model.

MAGNETOIONIC MODEL (COLD PLASMA MODEL). If we further assumed that the thermovelocity of electrons or ions is zero, (i.e., the term $\nabla p_\alpha/\rho_\alpha$ in Eq. (5b-419) is zero, and the inertial term $\mathbf{u}_\alpha \cdot \nabla \mathbf{u}_\alpha$ is omitted, then

$$\frac{\partial \mathbf{u}_\alpha}{\partial t} = \frac{q_\alpha}{m_\alpha}(\mathbf{E} + \mathbf{u}_\alpha \times \mathbf{B}) - \mathbf{u}_\alpha \sum_\beta \nu_{\alpha\beta} \tag{5b-425}$$

$$\mathbf{J} = \sum_\alpha q_\alpha n_\alpha \mathbf{u}_\alpha \tag{5b-426}$$

Equations (5b-425) and (5b-426), together with Maxwell's equations (5b-416), provide a complete set of equations for the cold plasma model.

Waves in Cold Plasmas. The linearized equations[1] (with harmonic time dependence $e^{-i\omega t}$) for waves in cold (electron)[2] plasmas are

$$\nabla \times \mathbf{B} = -i\omega\mu_0 \boldsymbol{\varepsilon} \cdot \mathbf{E} \tag{5b-427}$$

$$\nabla \times \mathbf{E} = i\omega \mathbf{B} \tag{5b-428}$$

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \epsilon_{xx} & -i\epsilon_{xy} & 0 \\ i\epsilon_{xy} & \epsilon_{yy} & 0 \\ 0 & 0 & \epsilon_{zz} \end{bmatrix} \tag{5b-429}$$

[1] The linearization procedures are justified if the phase velocities of the waves under consideration are much greater than the average electron velocity.

[2] In an electron plasma, only the motion of electrons is important. The ions and the neutrons are assumed to be stationary. For very low frequency waves the motion of ions may be important. In that case the components of the dielectric tensor must be modified. See E. Astrom, *Arkiv Fysik.* **2**, 443 (1950).

$$\epsilon_{xx} = \epsilon_0 \left\{ 1 - \frac{\omega_p{}^2(\omega + i\nu)}{\omega[(\omega + i\nu)^2 - \omega_c{}^2]} \right\}$$

$$\epsilon_{xy} = \epsilon_0 \left[\frac{\omega_p{}^2\omega_c}{\omega(\omega + i\nu + \omega_c)(\omega + i\nu - \omega_c)} \right]$$

$$\epsilon_{zz} = \epsilon_0 \left[1 - \frac{\omega_p{}^2}{\omega(\omega + i\nu)} \right]$$

with $\omega_p = (n_e e^2/m_e\epsilon_0)^{\frac{1}{2}}$ and $\omega_c = -(e/m_e)B_0$. ω_p, ω_c, and ν are respectively the plasma frequency for electrons, and the gyro frequency and collision frequency of electrons with all other heavy particles. $\mathbf{B}_0 = B_0\mathbf{e}_z$ is the applied static magnetic field. **E** and **B** are the complex amplitudes of the electromagnetic fields.

For a plane wave propagating in the **n** direction, the electric vector has the form

$$\mathbf{E} = \mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r}} \qquad \text{(5b-430)}$$

where $\mathbf{E}_0$ is a constant vector, **r** is the position vector, $\mathbf{k} = \mathbf{n}\omega/v_{\text{ph}}$ is the vector wave number, and v_{ph} is the phase velocity of the wave. The dispersion relation for the phase velocity, called the Appleton-Hartree equation, is obtained by substituting Eq. (5b-430) into Eqs. (5b-427) and (5b-428):

$$\Phi = U - \frac{Y^2 \sin^2 \gamma}{2(U - X)} \pm \left[\frac{Y^4 \sin^4 \gamma}{4(U - X)^2} + Y^2 \cos^2 \gamma \right]^{\frac{1}{2}} \qquad \text{(5b-431)}$$

$$k^2c^2 = \omega^2 - \frac{\omega_p{}^2}{\Phi} \qquad \text{(5b-432)}$$

$$U = 1 + i\frac{\nu}{\omega} \qquad X = \frac{\omega_p{}^2}{\omega^2} \qquad Y^2 = \frac{\omega_c{}^2}{\omega^2}$$

$$c = \frac{1}{\sqrt{\mu_0\epsilon_0}}$$

where γ is the angle between the direction of propagation and the direction of the static magnetic field $\mathbf{e}_z \cdot \mathbf{n}$ is in the yz plane. A sketch of the phase velocity vs. frequency for waves traveling in an arbitrary direction relative to $\mathbf{B}_0$ is given in Fig. 5b-19.

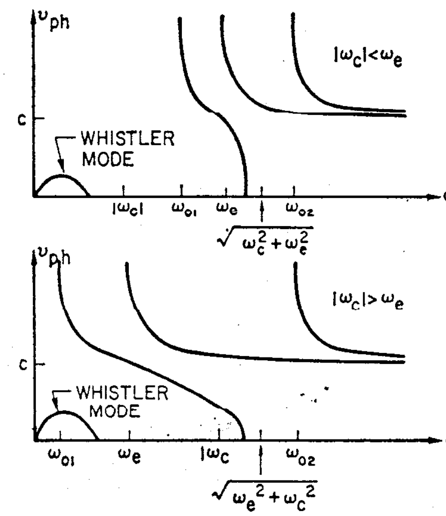

FIG. 5b-19. Phase velocity vs. frequency for waves traveling in an arbitrary direction relative to $\mathbf{B}_0$, the applied magnetic field, in an electron plasma. The above results are obtained according to the cold plasma model. $|\omega_c| = eB_0/m_e$, $\omega_e = (e^2n_0/m_e\epsilon_0)^{\frac{1}{2}}$, $\omega_{01} = [-|\omega_c| + (\omega_c{}^2 + 4\omega_e{}^2)^{\frac{1}{2}}]/2$, $\omega_{02} = \omega_{01} + |\omega_c|$, n_0 = number density of electrons, m_e = mass of electrons, and c = velocity of light in vacuum.

A great deal of work on wave propagation in plasma filled guide[1] and on the scattering of waves by a plasma column[2] has also been carried out.

Alfvén Wave. Alfvén wave exists in a plasma at very low frequencies when the plasma can be adequately represented by the linearized mhd model. Assuming that ρ, **u**, p, **E**, and **B** in Eqs. (5b-416) and (5b-421) to (5b-424) are all proportional to exp $i(kx-\omega t)$ $\sigma_0 = \infty$; and the applied static magnetic field $\mathbf{B}_0$ lies in the xy plane and makes an angle γ with the positive x axis; one may obtain the following set of equations:

$$u_x(\omega^2 - k^2V_a{}^2 \sin^2 \gamma - k^2U^2) + u_yk^2V_a{}^2 \sin \gamma \cos \gamma = 0 \qquad \text{(5b-433)}$$

$$u_xk^2V_a{}^2 \sin \gamma \cos \gamma + u_y(\omega^2 - k^2V_a{}^2 \cos^2 \gamma) = 0 \qquad \text{(5b-434)}$$

$$u_z(\omega^2 - k^2V_a{}^2 \cos^2 \gamma) = 0 \qquad \text{(5b-435)}$$

[1] See, for example, A. W. Trivelpiece and R. W. Gould, *J. Appl. Phys.* **30,** 1784 (1959)
[2] See, for example, C. Yeh and W. V. T. Rusch, *J. Appl. Phys.* **36,** 2302 (1965).

where $V_a = B_0/(\mu_0\rho_0)^{\frac{1}{2}}$ is called the Alfvén velocity. According to Eq. (5b-435), we see that a wave linearly polarized in the z direction (the direction perpendicular to both $\mathbf{k}$ and $\mathbf{B}_0$) can exist if

$$v_{ph} = \frac{\omega}{k} = V_a \cos \gamma \tag{5b-436}$$

This wave is called the pure Alfvén wave. Solving of Eqs. (5b-433) and (4b-434) gives the phase velocity of mhd waves containing components u_x and u_y:

$$v_{ph} = \frac{\omega}{k} = \frac{1}{\sqrt{2}} \{(V_a{}^2 + U^2) \pm [(V_a{}^2 + U^2) - 4V_a{}^2U^2 \cos^2 \gamma]^{\frac{1}{2}}\}^{\frac{1}{2}} \tag{5b-437}$$

The plus and minus signs refer to fast and slow mhd waves. The above results are applicable only if $\omega \ll \omega_i$, $\omega \ll \omega_{ci}$ and $V_a \ll c$, where ω_i is the ion plasma frequency, ω_{ci} is the ion cyclotron frequency, and c is the velocity of light in vacuum. Hence, the dispersion characteristics of high-frequency waves must be found from the full set of equations for the hydrodynamic-continuum model.[1] A sketch of phase velocity vs. frequency for waves in a fully ionized plasma is given in Fig. 5b-20.

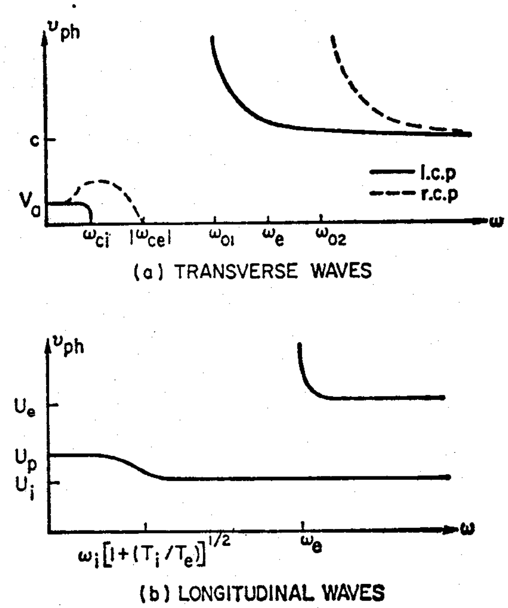

FIG. 5b-20. Phase velocity vs. frequency for waves in a fully ionized plasma according to the hydrodynamic-continuum model. Waves are assumed to be propagating in the direction of $\mathbf{B}_0$, the applied magnetic field. $|\omega_{ce}| = eB_0/m_e$,

$$\omega_{ci} = eB_0/m_i, \quad \omega_e = (e^2n_0/m_e\epsilon_0)^{\frac{1}{2}}$$
$$\omega_i = (m_e/m_i)^{\frac{1}{2}}\omega_e$$

and $U_p = \gamma K(T_e + T_i)/m_i$ (the plasma sound speed). V_a is the Alfvén velocity. $U_e = \gamma KT_e/m_e$, $U_i = \gamma KT_i/m_i$, γ is the ratio of specific heats at constant pressure and constant volume, and K is the Boltzmann's constant. T_e, T_i, m_e, m_i are respectively the electron temperature, the ion temperature, the electron mass, and the ion mass. The above curves are valid only if $T_e \gg T_i$ and the phase velocity of the wave is not close to the thermovelocity of ions or electrons.

Longitudinal Electron Landau Waves. Let us now consider the problem of the propagation of small-amplitude longitudinal waves in an electron plasma with no uniform applied static magnetic field by the use of the Boltzmann-Vlasov equation. Assuming that

$$f = f_0(\mathbf{v}) + f_1(\mathbf{v})e^{ikx-i\omega t} \qquad |f_1| \ll f_0 \tag{5b-438}$$

$$\mathbf{E} = E\mathbf{e}_x e^{ikx-i\omega t} \tag{5b-439}$$

where f_0 is the equilibrium distribution function for electrons, and substituting Eqs. (5b-438) and (5b-439) into Eqs. (5b-415) and (5b-416), one has

$$\left[\frac{\omega_p{}^2}{n_0k^2}\int \frac{(\partial f_0/\partial v_x)\, d^3v}{v_x - \omega/k} - 1\right] E = 0 \tag{5b-440}$$

where n_0 is the equilibrium electron density. Setting the quantity in the square brackets to zero gives the dispersion equation for the longitudinal electron waves. The solution of this dispersion equation has been obtained for the case when f_0 is the maxwellian velocity distribution for a stationary plasma; i.e.,

$$f_0 = n_0 e^{-v^2/a^2}/\pi^{\frac{3}{2}}a^3$$

[1] The description of the propagation characteristics of waves according to the hydrodynamic-continuum model is not valid when the phase velocity of a particular mode of interest is close to the thermovelocity of ions or electrons. In that case, Boltzmann's equations must be used. See B. D. Fried and R. W. Gould, ***Phys. Fluids* 4**, 139 (1961).

with $a^2 = 2KT/m_e$, K is the Boltzmann's constant, and T is the temperature:

$$k^2 = -k_D{}^2 \left(1 - 2C \int_0^C e^{z^2-C^2}\, dz + i\pi^{\frac{1}{2}} C e^{-C^2}\right) \tag{5b-441}$$

where $C = \omega/ka$, and $k_D{}^2 = 2\omega_p{}^2/a^2$ is the Debye wave number. The integral in the above equation is called the dispersion function and has been tabulated.[1] The last term, which is imaginary, is known as the Landau damping term. When $\omega/k \to \infty$, Eq. (5b-441) may be written as

$$\omega^2 = \omega_p{}^2 + \frac{3KTk^2}{m_e} + \cdots - \frac{2i\pi^{\frac{1}{2}}\omega_p{}^5}{k^3a^3} e^{-\omega_p{}^2/k^2a^2} \tag{5b-442}$$

Hence the longitudinal waves will decay in a *collisionless* electron plasma. The Landau damping characteristics are also present for transverse waves.[2]

Motion of a Charged Particle in Electromagnetic Fields. The motion of a charged particle in electromagnetic fields is governed by the following equation:

$$m \frac{d\mathbf{v}}{dt} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{5b-443}$$

where m, q, $\mathbf{v}$, $\mathbf{E}$, and $\mathbf{B}$ are respectively the mass of the particle, the charge, the velocity, the applied electric field, and the applied magnetic field. Some important behaviors of a charged particle in such an applied field are listed below:

IN CONSTANT AND UNIFORM **E** AND **B** FIELDS

1. The particle rotates about the **B** direction at a gyrofrequency $(\omega_c) = |qB/m|$ and with a radius $|V_0/\omega_c|$, where V_0 is the initial velocity of the particle in a plane normal to the **B** direction.
2. The particle possesses a drift velocity, $\mathbf{v}_D = \mathbf{E} \times \mathbf{B}/B^2 + m(\mathbf{g} \times \mathbf{B})/qB^2$, where $\mathbf{g}$ is the uniform gravitational field.
3. There is a constant acceleration in the **B** direction unless q and **E** are perpendicular to **B**. (In the last case the particle drifts in the **B** direction with its initial velocity.)

IN A NONUNIFORM **B** FIELD. The particle possesses a drift velocity,

$$\mathbf{v}_D = \frac{\nabla_\perp B}{\omega_c B}\left(\frac{1}{2} V_\perp{}^2 + V_\parallel{}^2\right) \mathbf{e}_D$$

where $\nabla_\perp B$ is the gradient of the scalar B in the plane perpendicular to **B**, $V_\perp$ and $V_\parallel$ are respectively the initial velocities perpendicular and parallel to the magnetic field **B**, and $\mathbf{e}_D$ is a unit vector in the direction $\mathbf{B} \times \nabla B$.

ADIABATIC INVARIANCE OF $\boldsymbol{\mu}$. When the applied magnetic field changes slowly with space or time,

$$\frac{d\boldsymbol{\mu}}{dt} = 0$$

where $\boldsymbol{\mu}$ is the magnetic moment for the changed particle and $\boldsymbol{\mu} = -w_\perp \mathbf{B}/B^2$ with $w_\perp = \frac{1}{2} m V_\perp{}^2$, which is the kinetic energy of the motion perpendicular to **B**.

ENERGY CONSERVATION IN A STATIONARY FIELD

$$\frac{d}{dt}\left(\tfrac{1}{2} m v^2 + q\Phi\right) = 0$$

[1] B. D. Fried and S. D. Conte, "The Plasma Dispersion Function," Academic Press, Inc., New York, 1961.

[2] See the treatment by Bernstein and Harris on waves in a hot plasma with an applied static magnetic field. [I. B. Bernstein, *Phys. Rev.* **109**, 10 (1958); E. G. Harris, *J. Nuclear Energy, Pt. C.* **2**, 138 (1961).] Also, for the treatment of waves in hot plasma-filled waveguides, see H. H. Kuehl, G. E. Stewart, and C. Yeh, *Phys. Fluids* **8**, 723 (1965).

where Φ is the potential energy per unit charge. The above equation indicates that the sum of kinetic and potential energies stays constant in a stationary field with $\mathbf{E} = -\nabla\Phi$.

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5b-14. Skin Effect. At high frequencies currents in a conductor tend to concentrate on the surface and decay approximately exponentially into the conductor. The concentration increases as frequency, conductivity, or permeability increases. The result is an increased resistance and decreased internal inductance at frequencies for which the effect is significant.

The basic equations governing the skin-effect phenomena are the Maxwell's equations applied to good conductors. A good conductor is defined by the following characteristics: the free-charge term is zero, i.e., $\rho = 0$; conduction current is given by Ohm's law, $\mathbf{J} = \sigma\mathbf{E}$, where σ is the conductivity; displacement current is negligible in comparison with conduction current, $\omega\epsilon \ll \sigma$. Under this assumption, Maxwell's equations are:

$$\begin{aligned} \nabla \times \mathbf{E} &= i\omega\mu\mathbf{H} \qquad & \nabla \cdot \mathbf{D} &= 0 \\ \nabla \times \mathbf{H} &= \sigma\mathbf{E} \qquad & \nabla \cdot \mathbf{B} &= 0 \end{aligned} \tag{5b-444}$$

with $\mathbf{B} = \mu\mathbf{H}$, $\mathbf{D} = \epsilon\mathbf{E}$, and $\mathbf{J} = \sigma\mathbf{E}$. A time dependence of $e^{-i\omega t}$ has been assumed for all field components and suppressed. Combining these equations and assuming that ϵ, μ, σ are independent of the position vector (i.e., a homogeneous medium), one has

$$\nabla^2\mathbf{P} - \tau^2\mathbf{P} = 0 \tag{5b-445}$$

where $\mathbf{P}$ may be $\mathbf{E}$, or $\mathbf{H}$, or $\mathbf{J}$; and $\tau^2 = -i\omega\mu\sigma = -2i/\delta^2$. $\delta = (2/\omega\mu\sigma)^{\frac{1}{2}}$ is called the skin depth; it is a measure of the decaying characteristics of fields within a conductor. The surface resistivity R_s is defined as $R_s = 1/\sigma\delta = (\omega\mu/2\sigma)^{\frac{1}{2}}$. Data for δ and R_s as functions of frequency are given for several common materials in Table 5b-5. The boundary conditions at the surface between a good dielectric and a good conductor are $\mathbf{n} \cdot \mathbf{J} = 0$ and $\mathbf{J} = \sigma\mathbf{E}_0$, where $\mathbf{n}$ is normal to the surface, and $\mathbf{E}_0$ is the applied field at the surface. The boundary conditions at the surface between two good conductors are the continuity of tangential electric and magnetic fields.

The internal impedance Z_i of a good conductor is defined as the ratio of the electric field at the surface to total current. The time-averaged power dissipated as Joules heat within the volume V is $\frac{1}{2}\int\sigma|E|^2\,dv$.

Formulas for Several Simple Conductors. PLANE SEMI-INFINITE CONDUCTOR. The plane conductor extends from $x = 0$ to $x = \infty$, and E_0 is an applied field in the z direction at $x = 0$.

$$J_z = \sigma E_0 e^{-x/\delta} e^{ix/\delta} \tag{5b-446}$$

$$Z_i = R_i - i\omega L_i = (1 - i)R_s \tag{5b-447}$$

TABLE 5b-5. SKIN-EFFECT QUANTITIES FOR CONDUCTORS

Metal	Resistivity* ρ(ohm-m)10^8	Relative* permeability at 0.002 weber/m²	$\delta\sqrt{\nu}$ δ = depth of penetration, m, ν = frequency, Hz	$10^7 R_s/\sqrt{\nu}$ R_s = surface resistivity, ohms/m²
Aluminum	2.828	1	0.085	3.33
Brass (65.8 Cu, 34.2 Zn)	6.29†	1	0.126	4.99
Brass (90.9 Cu, 9.1 Zn)	3.65†	1	0.096	3.79
Graphite	1,000	1	1.592	62.81
Chromium	2.6†	1	0.081	3.21
Copper	1.724	1	0.066	2.61
Gold	2.22†	1	0.075	2.96
Lead	22	1	0.236	9.32
Magnesium	4.6	1	0.108	4.26
Mercury	95.8†	1	0.493	19.43
Nickel	7.8	100	0.014	55.71
Phosphor bronze	7.75†	1	0.140	5.54
Platinum	9.83†	1	0.158	6.22
Silver	1.629	1	0.064	2.55
Tin	11.5	1	0.171	6.73
Tungsten	5.51	1	0.118	4.67
Zinc	5.38†	1	0.117	4.60
Magnetic iron	10	200	0.011	90.9
Permalloy (78.5 Ni, 21.5 Fe)	16	8,000	0.0022	727
Supermalloy (5 Mo, 79 Ni, 16 Fe)	60	10^5	0.0012	4,880
Mumetal (75 Ni, 2 Cr, 5 Cu, 18 Fe)	62	20,000	0.0029	2,140

* Values from Pender and McIlwain, "Electrical Engineers' Handbook," 4th ed., John Wiley & Sons, Inc., New York, 1950.
† Values at 0°C; others at 20°C.

SOLID ROUND WIRE. For a solid round conductor of radius a with applied axial electric field E_0 at the surface, we have

$$J_z = \sigma E_0 \frac{J_0(i^{\frac{1}{2}}r/\delta)}{J_0(i^{\frac{1}{2}}a/\delta)} \tag{5b-448}$$

$$Z_i = R_i - i\omega L_i = \frac{-R_s}{\sqrt{2}a\pi} \frac{J_0(i^{\frac{1}{2}}a/\delta)}{J_0'(i^{\frac{1}{2}}a/\delta)} \tag{5b-449}$$

where $J_0(i^{\frac{1}{2}}a/\delta)$ is a Bessel function of order zero with complex argument.[1] For $a/\delta \ll 1$,

$$Z_i \simeq \frac{1}{\pi a^2 \sigma}\left[1 + \frac{1}{48}\left(\frac{a}{\delta}\right)^4\right] - i\frac{\omega\mu}{8\pi} \tag{5b-450}$$

for $a/\delta \gg 1$,

$$Z_i \simeq \frac{(1-i)R_s}{2\pi a} \tag{5b-451}$$

[1] S. Ramo, J. R. Whinnery, and T. Van Duzer, "Fields and Waves in Communication Electronics," chap. 5, John Wiley & Sons, Inc., New York, 1965: S. J. Haefner, *Proc. IRE* **25**, 434 (1937); H. A. Wheeler, *ibid.* **43**, 805 (1955).

Formulas are also available for tabular conductors and rectangular conductors as well as coated conductors.[1] An example of skin depth and high-frequency resistance of copper is given in Fig. 5b-21.

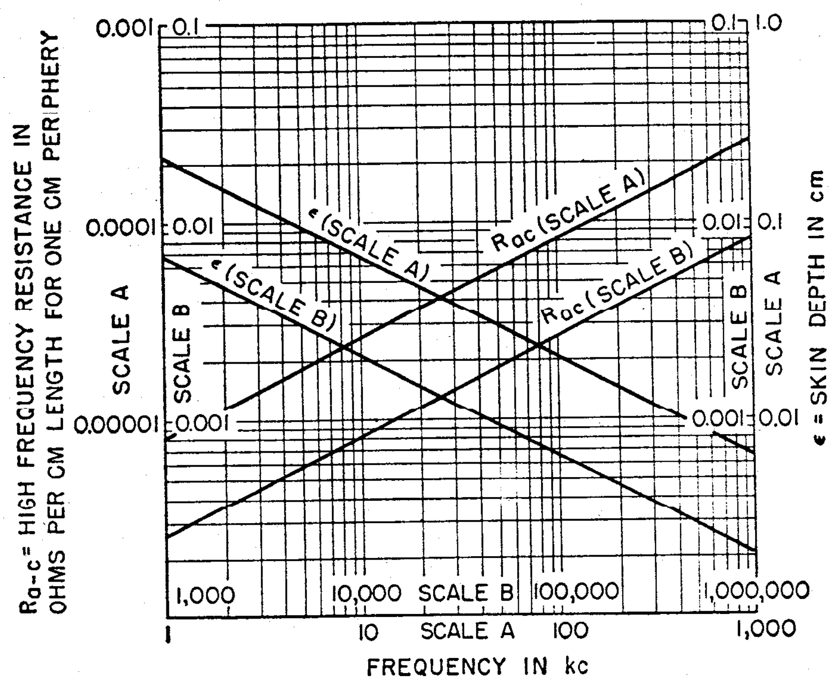

FIG. 5b-21. Skin depth and high-frequency resistance of copper. *(From F. E. Terman, "Radio Engineers' Handbook," p. 35, McGraw-Hill Book Company, New York, 1943.)*

Transient Penetration in the Plane Conductor. If a constant magnetic field H_0 is suddenly applied at time $t = 0$ to the surface of a semi-infinite plane conductor, field at depth x, time $t > 0$ is

$$H(x,t) = H_0 \left[1 - \operatorname{erf} \left(\frac{x}{2} \sqrt{\frac{\mu\sigma}{t}} \right) \right] \tag{5b-452}$$

If the applied field increases linearly with time, $H(0,t) = Ct$ for $t > 0$:

$$H(x,t) = Ct \left\{ \left(1 + \frac{\mu\sigma x^2}{2t} \right) \left[1 - \operatorname{erf} \left(\frac{x}{2} \sqrt{\frac{\mu\sigma}{t}} \right) \right] - x \sqrt{\frac{\mu\sigma}{\pi t}} \exp \left(\frac{-\mu\sigma x^2}{4t} \right) \right\} \tag{5b-453}$$

Anomalous Skin Effect. At sufficiently low temperatures and high frequencies, the mean free path of the electrons in a good conductor becomes greater than the classically predicted skin depth, and the classical skin-effect equations break down. Thus, the radio-frequency skin conductivity is practically independent of bulk conductivity (measured at direct current) when the mean free path of the electrons is sufficiently long. Data for Na, Cu, Ag, Au, Pt, W, Al, Pb, and Sn have been given by Pippard, Chambers, and Dingle.[2]

[1] S. Ramo, J. R. Whinnery, and T. Van Duzer, "Fields and Waves in Communication Electronics," chap. 5, John Wiley & Sons, Inc., New York, 1965: S. J. Haefner, *Proc. IRE* **25**, 434 (1937); H. A. Wheeler, *ibid.* **43**, 805 (1955).

[2] R. B. Dingle, *Physica* **19**, 348 (1953); R. G. Chambers, *Nature* **165**, 239 (1950); A. B. Pippard, *Proc. Roy. Soc. (London)*, ser. A, **191**, 385 (1947).