dicted by Bohr's theory; for the lighter fragments a variation as $v^{16}$ is indicated (L7, B43).

Bell (B7b) has made a more detailed theory of the capture and loss of electrons by fission fragments. He shows that it is a good approximation to treat both processes by classical mechanics. He obtains results for the variation of effective charge with the velocity of the fission fragment, as well as with the nature of the gas through which the fragment passes: the capture cross section is larger for heavier gases because then the gas atom may contain electrons which are moving slowly relative to the fission fragment and thus may easily be captured (similar to Bohr's theory, page 191). Bell's results for the effective charge as a function of velocity are in good agreement with the experiments of Lassen for all gases except $H_2$ and He, for which the theory gives too small a capture cross section; Bell gives reasons which may account for this discrepancy.

If the variation of charge is given by Eq. (30), and if Geiger's relation, range $\sim v^3$, is assumed to hold for constant charge (see page 211), the velocity of the fission particle will decrease linearly with distance as long as the energy loss is due mostly to electronic collisions [first term in Eq. (29)]. With the same assumptions Bohr obtains a relation between the range of a fission fragment and an alpha-particle of the same initial velocity $v$:

$$\frac{R_F}{R_\alpha} \approx 7\frac{A_1}{(Z_1^{\text{eff}})^2} \approx 7\frac{A_1}{Z_1^{2/3}}\left(\frac{e^2}{\hbar v}\right)^2 \qquad (30a)$$

where $A_1$ is the mass number of the fragment.

This relation predicts that the stopping power of various substances for fission particles varies in the same way as for alpha-particles. This is in fair agreement with the experiments of Segrè and Wiegand (S11a) and Lassen (L8). For very light stopping materials deviations appear, and they have been discussed by Bohr (B43, p. 134).

Equation (30a) also predicts the absolute range of fission fragments. In the most recent experiments by Katcoff, Miskel, and Stanley (K3), the range was measured for individual fission products by radiochemical methods. For the heaviest fragment measured, $Eu^{157}$, the mean range in air was 1.79 cm; for the lightest, $Br^{83}$, it was 2.65 cm. These results are in sufficient agreement with Bohr's theory, Eq. (30a). Older experiments are due to the Danish school (B37, B38, B45), to Joliot (J7) and Suzor (S28) in France, and to several others. In these experiments only the light and the heavy groups of fragments were separated.

Figure 12 gives the range-velocity relation according to the experiments of Bøggild *et al.* (B37, B38). The velocity of the fragments was determined by the distribution of nuclear recoils along the track in a cloud chamber, and by a measurement of the lengths and angles of tracks of individual nuclear recoils.



An interesting feature of the stopping of fission fragments is the large straggling which is due to the nuclear collisions in the last part of the range (when the velocity has decreased below $e^2/\hbar = 2 \cdot 10^8$ cm/sec). Bohr (B13, p. 137) has calculated the straggling and finds satisfactory agreement with the experiments of Katcoff *et al.* (K3).

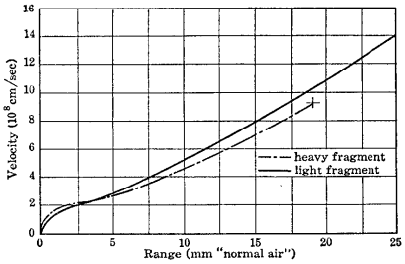


Fig. 12. Range of fission particles, from experiments by Bøggild, Brostrøm, and Lauritsen, Phys. Rev., **59**, 273 (1941).

### B. Relation between Energy Loss and Ionization

**1. Primary and Total Ionization.** An energetic heavy charged particle, in its passage through matter, produces the ionization contributing to its energy loss in two different ways. In the primary collision with the electrons in an atom, the most probable of the ionizing collisions are those in which a relatively slow secondary electron is ejected with kinetic energy smaller than the ionization potential. A small fraction of the ionizing collisions, however, produce secondary electrons of relatively high energy, the maximum energy being $4(m/M)E$ corresponding to a maximum velocity of twice the velocity of the incident heavy particle. For a primary proton of energy $E = 10$ Mev this gives secondaries of maximum energy around 20 kev. These so-called delta-rays are easily visible in cloud chamber photographs of the tracks of ionizing heavy charged particles, and they amount to a few per centimeter near the beginning of the track. The delta-rays themselves then go on to produce further ionization in the atoms of the stopping material, leading to a secondary ionization. The energy distribution of the delta-rays is discussed below (Section 1D).

We have, therefore, to distinguish between the *primary ionization* in which the number of ions produced is equal to the total number of secondary electrons, slow and fast, and the *secondary ionization* produced by the delta-rays. The *total ionization* is the sum of the two. Experimentally, the total ionization is roughly three times the primary ionization. Both types of ionization can be observed in the cloud chamber. To observe the total ionization delayed expansions are used and thereby the tracks are permitted to diffuse in order to get all ions separated; by applying electric clearing fields at the same time, the positive and negative ions can be separated, and this allows a check on the condensation efficiency. To measure primary ionization, sharp tracks are used and clusters of ions are counted rather than single ions. Under these conditions the secondary ions produced by a slow delta-ray will form part of the same cluster together with the primary ion formed when the delta-ray was emitted.

Total ionization is generally measured in ionization chambers or proportional counters (see Section 1B4), whereas primary ionization can be measured, for instance, by using the efficiency of a Geiger counter (Hereford, H15).

The calculation of primary ionization can be done, with any certainty, only for hydrogen [actually hydrogen *atoms* (B12)]. Experiments for hydrogen have been carried out in the cloud chamber by Williams and Terroux (W14, W15) for electrons of velocity $0.5c$. They find 14.7 primary ions per centimeter, whereas the theoretical prediction is 12.6, in satisfactory agreement. At higher energies, up to 13 Mev, Hereford (H15) also found reasonable agreement with theory; in particular he confirmed the theoretical prediction that the primary ionization, like the energy loss, increases with energy in the relativistic region (Sections 1A5 and 2A1). This relativistic increase of ionization had previously been established for nitrogen by Corson and Brode (C16) and then confirmed with greater accuracy by Hazen (H11); in both cases total ionization was measured in the cloud chamber. (See also Sections 1A5 and 2A1.)

**2. Energy Loss per Ion Pair (Total Ionization).** The most remarkable experimental fact about the total ionization is that the energy loss per ion formed, $w$, is very nearly independent of the energy $E$ of the primary. It is also nearly the same for alpha-particles, protons, electrons, etc. Moreover, the energy $w$ is not very different for different stopping gases (see Table 6) and, contrary to naive expectation, is smallest for the rare gases whose ionization potentials are the highest of all atoms.

These facts have been explained by a very reasonable semi-quantitative theory by Fano (F1). Instead of the energy of the primary particle, Fano considers the total energy *available* for ionization, whether it



resides in the primary particle or in a delta-ray. In a collision in which the atom is excited (cross section $\sigma_e$) by either the primary or a delta-ray, the available energy is reduced by the excitation energy $W_e$. Like-

TABLE 6

ENERGY LOSS $w$ PER ION PAIR

| Gas | $Z$ | $w$ for Po Alpha-Particles † according to reference: | | | | $w$ for 340-Mev protons (B1) | Ionization Potential $I$ | $w/I$ † |
|---|---|---|---|---|---|---|---|---|
| | | (A3) | (G16) | (S6) | (S25) | | | |
| Hydrogen | 1 | | 35.0 | .... | 35.1 | 35.3 | 15.6 | 2.24 |
| Helium | 2 | | 30.2 | .... | .... | 29.9 | 24.5 | 1.23 |
| Nitrogen | 7 | $36.3 \pm 0.4$ | .... | 36.2 | 36.2 | 33.6 | 15.5 | 2.34 |
| Oxygen | 8 | | .... | 32.3 | .... | 31.5 | 12.5 | 2.58 |
| Neon | 10 | | .... | 27.1 | 29.0 | 28.6 | 21.5 | 1.26 |
| Argon | 18 | | .... | 24.3 | 27.7 | 25.5 | 15.7 | 1.55 |
| Krypton | 36 | | .... | 22.3 | 25.6 | .... | 13.9 | 1.60 |
| Xenon | 54 | | .... | 21.3 | 22.9 | .... | 12.1 | 1.76 |
| Air I † | .. | $34.7 \pm 0.5$ | 35.6 | 34.8 | 35.7 | 33.3 | .... | .... |
| Air II † | .. | 34.7 | 34.7 | 34.7 | 34.7 | .... | .... | .... |
| $CH_4$ | .. | .... | .... | 29.3 | | | | |
| $C_2H_4$ | .. | .... | .... | 26.9 | | | | |
| $CCl_4$ | .. | $26.8 \pm 1.0$ | .... | .... | | | | |
| CO | .. | .... | .... | 33.9 | | | | |
| $CO_2$ | .. | .... | .... | 33.8 | | | | |

† The absolute values for substances other than air have all been adjusted to the value of Alder, Huber, and Metzger (A3) for air, 34.7 ev. This standard value has been listed for all authors in the line "Air II." The directly measured values for air of the various authors are listed in the line "Air I." The adjusted values of Schmieder (S6) have been used, where available, in calculating $w/I$.

wise, if the atom is ionized and the kinetic energy of the ejected electron is less than the ionization potential $I$ (cross section $\sigma_{i1}$), the total energy given to the atomic electron $W_{i1}$ is lost from the available energy. However, if a delta-ray of kinetic energy greater than $I$ is produced ($\sigma_{i2}$), its kinetic energy is still available for further ionization, and the expenditure of available energy is considered to be only the ionization potential $I$. The average amount of (available) energy spent per ion is then

$$w = \frac{\sigma_e W_e + \sigma_{i1} W_{i1} + \sigma_{i2} I}{\sigma_{i1} + \sigma_{i2}} \tag{31}$$

Now the *ratio* of the various cross sections, especially of $\sigma_e$ and $\sigma_{i1}$, does not change much with the energy of the ionizing particle; neither do the average energies $W_e$ and $W_{i1}$. This explains in a qualitative way the constancy of $w$ with energy. An important contributing reason for this constancy is that $\sigma_{i2}$ is small. In first approximation, then, Eq. (31) can be calculated by finding the total energy loss in all low-energy collisions ($e$ and $i1$) and the cross section $\sigma_{i1}$. In second approximation Fano treats the contribution of the secondary electrons more accurately.

The theory of Fano explains the experimental result that the energy spent per ion is nearly the same for all substances, irrespective of their ionization potential. To see this, consider on one side the rare gases, which have very high ionization potentials. Their excited states all lie in a very narrow energy region close to the ionization potential. Therefore the total transition probability to all discrete states $\sigma_e$ [cf. also Eq. (11)] is small, and nearly every inelastic collision leads to ionization. Thus, while $W_{i1}$ in Eq. (31) is large (since it must be greater than $I$), the energy "wasted" in excitation, $\sigma_e W_e$, is small, and $w$ is only slightly greater than $I$. Conversely, for an alkali atom, for which the ionization potential is small, an overwhelming fraction of the collisions leads to the excitation of the first excited state (e.g., B14, p. 467) and a large amount of energy is "wasted" in excitation, so that $w$ is very much greater than $I$.

Fano made explicit calculations for hydrogen, which has an intermediate ionization potential, and helium, which has a large $I$. He found almost exactly the same values for $w$, namely 36 ev for hydrogen and 38 ev for helium, whereas the ionization potential of helium is 1.8 times greater than that of hydrogen. The number for hydrogen is very close to the experimental value for $H_2$ (Table 6); on the other hand, the calculated value for helium is still too high, since the experimental result is only 30 ev, only 25 percent more than the ionization potential.

It has already been stated that the energy loss per ion pair is nearly independent of the energy of the primary particle. There is clearly no theoretical reason for this rule to hold *exactly*; it can at best be made plausible by the arguments of Fano. From his arguments it may be expected that $w$ is most nearly constant for rare gases. In these the excitation energies are all very close to the ionization potential, and most of the ionizing collisions also involve energy transfers close to $I$, as shown by the experimental fact that the energy per ion, $w$, is only slightly greater than $I$. Therefore, even a considerable change in the distribution of collisions between various kinds will not change $w$ very much.

This conclusion is confirmed by the experimental fact that the *ratio* of the number of ions formed in the three gases helium, neon, and argon



is independent of the energy for alpha particles up to 5 Mev. This result was deduced in a survey article by Gray (G16) from old experiments (1925) of Gurney (G18), carried out with alpha-particles from polonium stopped by various thickness of air. From the experiments it can be concluded that $w$ either varies in exactly the same way with energy for the three gases (which is obviously unreasonable) or is constant for all three.

A direct experimental proof that the energy $w$ per ion pair is independent of energy in *argon* has been given by Jesse, Forstat, and Sadauskis at the Argonne Laboratory (J3, J4, J5). They have shown that, for a number of natural alpha-particles of energies between 5 and 9 Mev, the number of ions formed in argon is proportional to the energy of the particle within 0.5 percent, which probably represents the accuracy of the ionization measurements (there is no trend of the small discrepancy with energy). They have obtained the same result (within 0.3 percent) for the combined ionization and the combined energy of the two particles emitted in the reaction $B^{10} + n = Li^7 + He^4$. Franzen, Halpern, and Stephens (F13) have also obtained the same result for the combined ionization and energy of the two particles emitted in the reaction $N^{14} + n = C^{14} + H$, and in the reaction $He^3 + n = H^3 + H$, all of these reactions being produced by slow neutrons. Unless there is some strange and accidental compensation, we must conclude that the ionization in argon is exactly proportional to the energy, irrespective of the energy and of the kind of particle involved.[1]

On the other hand, for *air* Jesse *et al.* have shown that $w$ varies appreciably with particle energy. They have analyzed the experiments on the ionization in air of Stetter (S25), of Holloway and Livingston (H21), and their own (J3) and have obtained the ratio of the ionizations in argon and air as a function of particle energy. If the ionization in air is multiplied by this ratio, a result proportional to the particle energy is obtained. In this way the measurements of Holloway and Livingston (H21), who measured the ionization of alpha-particles in air as a function of the range, can be corrected and thus converted into a range-energy relation. This procedure, suggested and carried out by Jesse and

[1] However, Leachman (L14a) has shown that for fission fragments the energy per ion pair in argon is greater than for protons, by about 12 percent for the lighter and 5 percent for the heavier fragment. Knipp and Ling (K9a) have attributed this effect to a deficiency in ionization by the heavy recoil atoms projected by the fission fragments. Another reason may be energy loss associated with the capture and loss of electrons by the fragment (Section 1A6) which is not accompanied by the full amount of ionization in the medium. Finally, Jesse has pointed out that, in the case of the very heavily ionizing fission fragments, there may be lack of saturation in the observed ionization current due to columnar ionization, even in argon.

Sadauskis *et al.* (J4, J6; see also B20), was used in establishing the range-energy relation for slow alpha-particles in Section 1A3.

Recently, Kimura *et al.* (K4a) have repeated the experiments of Holloway and Livingston with different electric fields applied to the ionization chamber. At moderate field (500 volts/cm) they reproduce the result of

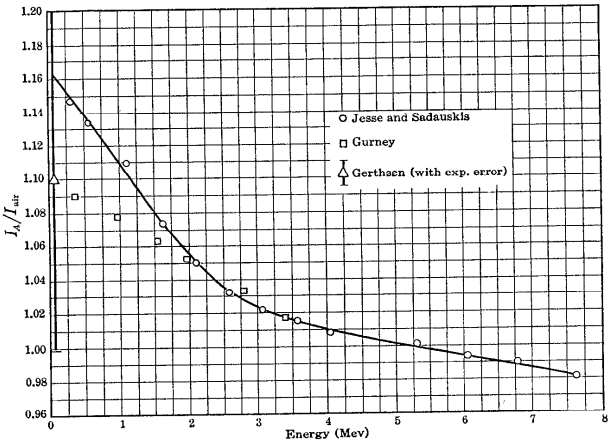


Fig. 13. Ratio of number of ions produced in argon and in air by particles of various energies. The ratio has arbitrarily been set equal to one for polonium alpha-particles. Since the energy $w$ per ion pair is believed to be constant for argon, the curve gives the variation of $w$ for air with energy.

Holloway and Livingston, but at high field (11,400 v/cm) they find a curve for ionization *vs.* distance which is more similar to the corrected curve of true energy loss *vs.* distance. They conclude that the true value of $w$ in air may be more nearly constant with energy than the investigations of Jesse *et al.* indicated, but that the heavy columnar ionization produced by slow alpha-particles makes it very difficult to collect all the ions in air unless extremely high electric fields are applied. This conclusion does not, of course, affect the conversion factor from the data of Holloway and Livingston to ionization in argon.

In Fig. 13 we give the conversion factor from total ionization in air to ionization in argon. The conversion factor was arbitrarily set equal to



one for polonium alpha-particles. We have used the data of Jesse *et al.*, supplemented by the old results of Gurney (G19). The value given for zero energy in Fig. 13 has been estimated on the basis of the experiments of Gerthsen (G8) with hydrogen canal rays. He finds that the energy per ion for slow protons ($<50$ kev) is about $10 \pm 10$ percent higher than for natural alpha-particles. There is no reason why there should be any difference between protons and alpha-particles (both being heavy) in the *distribution* between exciting and ionizing collisions and therefore in $w$ (the *number* of collisions per centimeter is, of course, another matter). We therefore assume that there is no rapid increase of $w$ at very low energies, in contrast to (G16) and (C19).

The $w$ so far considered, for instance in Fig. 13, was obtained by dividing the total energy of the particle by the total number of ions formed. This is the quantity of the greatest practical importance which must be used to convert ionization in air into energy. A theoretically more significant quantity is the energy $w'$ per ion pair for a *small* change of particle energy. It is related to $w$ as follows:

$$\frac{1}{w'} = \frac{dI}{dE} = \frac{d}{dE}\left(\frac{E}{w}\right) = \frac{1}{w}\left(1 - \frac{d \log w}{d \log E}\right) \tag{32}$$

where $I$ is the total number of ions formed by a particle of initial energy $E$. For the highest energy interval measured by Stetter (S25), i.e., the difference between the alpha-particles of RaC′ and ThA (see Table 2 of reference J5), the value of $w'$ in air is about 7 percent less than $w$ for polonium alpha-particles, or 32.3 ev.

More information on the change with energy at high energies is provided by the experiments by Bakker and Segrè (B1), who used 340-Mev protons which are also listed in Table 6. Their value for air, 33.3 ev, is lower than that for polonium alpha-particles and in sufficient agreement with the $w'$ value of Stetter just mentioned (32), with the commonly quoted value for electrons, 32 ev, and with Eq. (32a). For nitrogen a greater decrease with increasing energy is found. For the rare gases, helium, neon, and argon, the high-energy value of Bakker and Segrè agrees well with the mean values of the determinations for polonium alpha-particles. For these gases the value of $w$ is especially sensitive to small impurities; this may explain the relatively wide variation between individual experimental results.

Returning to energies below that of polonium alpha-particles, the energy per ion, $w$, is independent of the particle energy $E$ also for helium and neon, since it is independent of $E$ for argon and since the ratio of ionization in argon and the other rare gases is independent of

energy (G16, G19). This is plausible from Fano's theory (see above). However, it follows from Gurney's experiments, as well as from those of Bakker and Segrè, that $w$ is also independent of energy for hydrogen, and there is no obvious theoretical reason for this.[1]

The value of $w$ for *electrons* was measured in argon by Curran, Angus, and Cockcroft (C24) and by Nicodemus (N3), and in air by Pigge (P6), and Gerbes (G6). In argon, Curran *et al.* find a constant value for $w$, in agreement with the results for protons. Nicodemus finds 26.9 ev, very close to the value for alpha-particles adopted in Table 6. In air at high energies, Gerbes (and other authors) find practically the same value of $w$ as for alpha-particles,[2] as might be expected. At low energies Pigge finds substantially higher values, going up to 42 ev at 300 ev. Gerbes has summarized these data in the empirical formula

$$w = 31.6 + 5.3E'^{-\frac{1}{2}} \tag{32a}$$

where $E'$ is in kev. Gray and Cranshaw and Harvey have used similar formulas for alpha-particles. The increase of $w$ for slow electrons is not unreasonable, but the increase for heavy particles seems to be less than indicated by a formula of the type of (32a).

The most recent, and probably best, *absolute* values for $w$ in nitrogen and air for polonium alpha-particles have been obtained by Alder, Huber, and Metzger (A3); these values are given in Table 6 and used as standards. Gray, in his survey article (G16), and Stetter (S25) favor for air a value higher by about 1 ev.

For the other substances listed in Table 6 we have taken the values relative to air from various authors as listed below, together with the absolute value for air from Alder *et al.* For many substances, values are available both from Schmieder (S6) and from Stetter (S25); they have both been given in order to show the experimental variation of $w$. This is especially large for rare gases in which the ionization is highly sensitive to small impurities (private communication from Jesse). For nitrogen and hydrogen the agreement of various authors is good. The values from Gray (G16) are based on experiments by Naidu (N1) for helium and by Gurney (G19) and Taylor for hydrogen. Rossi and Staub (R9, p. 227) give additional values.

**3. Fluctuations of Ionization.** For the evaluation of measurements of the energy of a particle by its total ionization, the fluctuation of the number of ions for a given initial energy is of interest. This problem has also been studied by Fano (F2). Let $p_k$ be the fraction of inelastic collisions associated with a loss $w_k$ of available energy, where the available energy is defined (as in the theory of the preceding subsection) as the energy of the particle plus that of all delta-rays still capable of ionizing; $w_k$ can therefore not exceed $2I$. Also let $p' = \Sigma p_k^{(i)}$ be the total probability of ionization; then, if $N_0$ is the expected number of ions, the mean square fluctuation of the actual number is, according to Fano,

$$<(N - N_0)^2>_{\text{av}} = \frac{N_0}{p'}\left[\sum_k p_k^{(i)}\left(1 \quad \frac{w_k}{w}\right)^2 + \Sigma p_k^{(e)}\left(\frac{w_k}{w}\right)^2\right] \tag{33}$$

where the first sum goes only over the ionizing, the second only over the exciting collisions. It is convenient to introduce the average energy loss $W_i$, for all ionizing collisions:

$$\Sigma p_k^{(i)} w_k^{(i)} = p' W_i \tag{33a}$$

and similarly for the exciting collisions:

$$\Sigma p_k^{(e)} w_k^{(e)} = (1 - p') W_e \qquad \Sigma p_k^{(e)} = 1 - p' \tag{33b}$$

so that

$$w = W_i + W_e \frac{1 - p'}{p'} \tag{33c}$$

Then Eq. (33) may be written

$$\frac{<(N - N_0)^2>_{\text{av}}}{N_0} \equiv F = \frac{1 - p'}{p'^2}\frac{W_e}{w^2} + \frac{1}{w^2 p'} \cdot \left[\sum_{\text{ion}} p_k^{(i)}(w_k^{(i)} - W_i)^2 + \sum_{\text{exc.}} p_k^{(e)}(w_k^{(e)} - W_e)^2\right] \tag{33d}$$

Here the first term arises from the fact that some collisions are ionizing while others are not, and it goes to zero if there are only ionizing collisions; the other two terms arise from the fluctuation of energy loss in the ionizing and the exciting collisions, respectively. In general, the first term predominates (by a factor 10 or so). Since $wp' > W_e$ (Eq. 33c), the fluctuation factor $F$ is in general *less* than $1 - p'$, which is considerably less than the result $F = 1$ which would be obtained for random events. The fluctuations should be particularly small for helium because $w$ is very close to $I$; the numerical values for $w$ and $I$ set an upper limit of about $\frac{1}{4}$ for $1 - p'$ and thus for $F$.

Hanna, Pontecorvo, and Kirkwood (H8, K5) have studied the fluctuation in the size of the pulses produced in a proportional counter filled with argon by low-energy electrons (17.4, 2.8, and 0.25 kev). The average number of ions directly produced by an electron of 2.8 kev is $N_0 = 100$

[1] Gray's belief that there are theoretical reasons for constancy in hydrogen is not justified because there is no direct theory for $w$ but only plausibility arguments.

[2] To be exact, $w$ for electrons agrees with the differential value $w'$ for alpha-particles, see above.

(see Table 6), the measured standard deviation is ±9 percent, corresponding to ±9 ions. Theoretically, this fluctuation consists of two parts, the fluctuation in the number of ions directly produced by the electron and the fluctuation in the amplification in the counter. The former is $\sqrt{FN_0}$ with $F$ given by Eq. (33d); Fano has found that $F$ should be about ⅓ for argon; therefore this fluctuation should be about $\sqrt{33}$. The fluctuation in the gas amplification has been calculated by Snyder (S18) and found to be $\sqrt{N_0}$. The total fluctuation should therefore be $\sqrt{(1+F)N_0} \approx 12$. The observed fluctuation is *less* than this. Qualitatively, this supports Fano's result that $F$ is considerably smaller than 1; quantitatively, there is at present no theoretical explanation of the very small fluctuation found by Hanna *et al.*[1] Their result for 17.4 kev is similarly small.

**4. Experimental Applications.** The near constancy of the energy loss per ion provides a very useful experimental tool. In first approximation it may be assumed that the energy of a particle is proportional to the total number of ions produced. This method has been widely used for the determination of particle energies: (1) by Holloway and Livingston (H21) for the purpose of establishing a range-energy relation; (2) by many authors for the measurement of the energy of particles arising from nuclear reactions, e.g., of the protons from the photoelectric disintegration of the deuteron (S26); and (3) recently by Cranshaw and Harvey (C19) and by Jesse *et al.* (J3, J4) for the determination of energies of alpha-particles. It is most accurate to use the ionization in argon (or another rare gas) for this purpose rather than that in air (see Section 1B2). The experiments of Holloway and Livingston were done in air; they were corrected by means of Fig. 13 to establish the range-energy relation (see Section 1A3).

The distribution of total ionization along the path of a particle is, in first approximation, the same as that of the rate of energy loss, $-dE/dx$. It can be measured for individual particles in a shallow ionization chamber with linear amplifier (H21). The result is a Bragg curve of ionization *versus* residual range. Figure 14 shows the Bragg curve as obtained by Holloway and Livingston for polonium alpha-particles.

[1] In contrast to Hanna *et al.*, Stetter (S25) found fluctuations in the number of ions which were considerably *greater* than $\sqrt{N}$. For example, for polonium alphas in argon Stetter's root mean square fluctuation is 6500; with $N$ = 186,000, this is $15\sqrt{N}$. Moreover, the fluctuation increases rapidly with increasing atomic number, from $4.5\sqrt{N}$ for hydrogen to $40\sqrt{N}$ for xenon (!); the number of ions does not show a Gaussian distribution, and its variation is correlated with the range straggling. These results are almost certainly wrong, but a repetition of Stetter's experiments would be desirable.



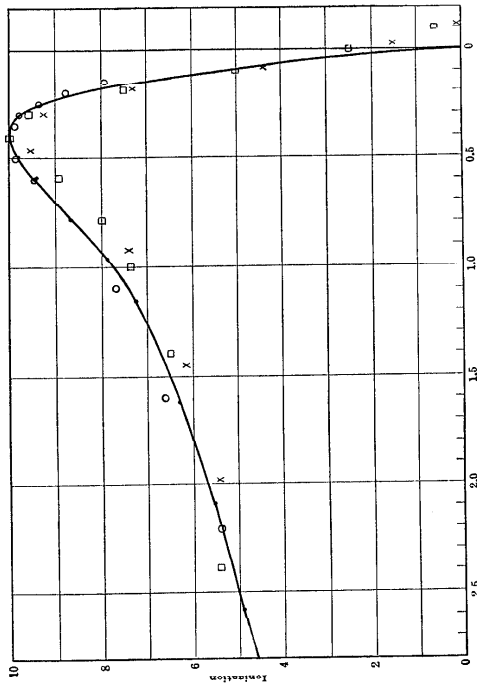


Fig. 14. Ionization of an alpha-particle as a function of its residual range, according to experiments by Holloway and Livingston, Phys. Rev., **54**, 29 (1938), Fig. 9.

Starting at high energy, the ionization rises, reaches a maximum about 0.4 cm from the end of the range, and then drops sharply to zero at the end of the range.

To be distinguished from the ionization curve for an individual particle is the Bragg curve for *all* the particles emitted by a radioactive source as a function of the distance from the source. This curve is modified by the range straggling (see Section 1C) of the particles which has the effect of giving a tail to the curve and of smearing out the maximum. This curve is easier to measure because it involves the measurement of the total ionization current produced by many particles rather than the small ionization pulse produced by a single particle. All the earlier measurements were concerned with this over-all Bragg curve.

Since the ionization per centimeter of path is a function of the velocity of the particle, it can be used to determine the velocity. This measurement may be combined with one of the momentum by magnetic deflection; combination of the two data gives the mass of the particle. This method has been widely used for mass determinations, especially of mesons. Because it is difficult to measure the ionization accurately, the method is inferior to a determination of magnetic deflection and range (F14).

The most accurate determination of particle ranges is done by means of ionization chambers with linear amplifiers. By applying a bias, only those particles which form more than a given number of ions in the chamber (depending on the bias applied) are measured, i.e., only those particles whose energy exceeds a certain minimum energy $E_m$ when entering the chamber. The distance between the source and the front face of the ion chamber is then the difference of the ranges for the initial particle energy $E_i$ and the energy $E_m$. It is most convenient to make relative measurements, using the same chamber for the particles of unknown range and for other particles of the same kind whose range is accurately known (for the use of this technique, see, e.g., C3).

### C. Straggling. Determination of the Mean Range from the Extrapolated Range.

**1. Fluctuations in the Energy Loss and in the Range.** In Section 1A1 we have given a formula for the *average* energy loss per centimeter suffered by a charged particle in traversing some stopping material. Actually, any given particle loses its energy in small but finite amounts, taking a large but finite number of collisions to reduce its energy by some prescribed amount. If we were therefore to examine particles of the same initial energy which have all traveled the same length of path in the stopping material, we should find a statistical fluctuation in the amount of energy lost. Similarly, there would be a statistical fluctuation in the range of these particles. These fluctuation effects are known as "straggling."



Let us denote by $(\Delta E^2)_x = [(E^2)_{\text{av}} - (E_{\text{av}})^2]_x$ the mean square fluctuation in the energy lost by particles of the same initial energy traversing a distance $X$, and by $(\Delta X^2)_E = [(X^2)_{\text{av}} - (X_{\text{av}})^2]_E$ the mean square fluctuation in the distance traversed by particles which have lost the energy $E$. Then, if $dX$ and $dE$ denote a very small distance and energy loss, respectively, and if $dE$ is the mean energy loss corresponding to the distance $dX$, we have, obviously,

$$(\Delta E^2)_{dX} = \left(\frac{dE}{dX}\right)^2 (\Delta X^2)_{dE} \tag{34}$$

where $dE/dX$ is the average rate of energy loss. The fluctuation $\Delta X^2$ for a *finite* energy loss $E$ is then obtained by integration of Eq. (34):

$$(\Delta X^2)_E = \int_0^E (\Delta X^2)_{dE} = \int (\Delta E^2)_{dX} \left(\frac{dE}{dX}\right)^{-2}$$

$$= \int \frac{d}{dX}(\Delta E^2) \left(\frac{dE}{dX}\right)^{-2} dX = \int \frac{d}{dX}(\Delta E^2) \left(\frac{dE}{dX}\right)^{-3} dE \tag{35}$$

The fluctuation $\Delta E^2$ for a small distance can be calculated directly from the theory. By applying the Born approximation as in the theory of stopping, Livingston and Bethe (L18) obtained

$$\frac{d}{dX}[(E^2)_{\text{av}} - (E_{\text{av}})^2]_X = 4\pi e^4 z^2 N \left(Z' + \sum_n k_n \frac{I_n Z_n}{mv^2} \log \frac{2mv^2}{I_n}\right) \tag{36}$$

Here $Z'$ is the total number of "effective" electrons defined as the number of electrons in the atom, $Z$, excluding those in the inner shells for which $I_n > 2mv^2$. The sum goes over the shells which are *not* excluded, $Z_n$ is the number of electrons in the $n$th shell, $I_n$ their average excitation energy, and the $k_n$ certain constants between $\frac{2}{3}$ and $\frac{4}{3}$. For high energies the sum over $n$ may be neglected, and $Z'$ replaced by $Z$, and the classical formula of Bohr (B41) is obtained:

$$\frac{d}{dX}\Delta E^2 = 4\pi e^4 z^2 NZ \tag{36a}$$

The corrections in (36) compared with (36a) amount to an increase by about 20 percent for 8-Mev alpha-particles in mica (B11), and 50 percent around 4 Mev. At higher energies, the corrections become negligible.

Using Eq. (34) and integrating Eq. (37), we obtain, from the initial energy $E_0$ of the particle to zero, the mean square fluctuation in the range:

$$(R - R_0)_{\text{av}}^2 = [(R^2)_{\text{av}} - R_0^2]$$

$$= \int_0^{E_0} \frac{4\pi e^4 z^2 NZ'}{(dE/dx)^3} \left\{1 + \sum_n \frac{k_n I_n Z_n}{mv^2 Z'} \log \frac{2mv^2}{I_n}\right\} dE \quad (37)$$

In the calculations for air which are summarized in Fig. 15a, the constants are set equal to $\frac{4}{3}$ for both the $K$ and $L$ shells. $R_0$ is the *mean range* as given in all the previous range-energy relations.

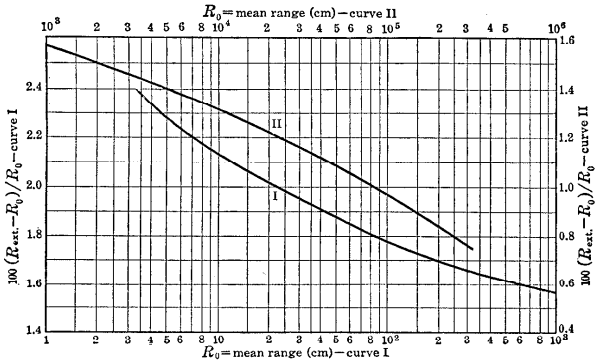


Fig. 15a. Straggling of protons, $S$, in air, in percent of their range. Abscissa, range of proton.

The foregoing formulas take into account only collisions with electrons. Elastic collisions with the atoms as a whole (so called nuclear collisions) contribute very little to the stopping of alpha-particles, protons, etc. They contribute somewhat more to the fluctuations in energy loss because the energy loss in a "nuclear" collision is relatively very large; but, for the same reason, these collisions do not give rise to a normal (Gaussian) distribution of the energy but rather to a "tail" in the energy distribution consisting of a very few particles which have lost considerably more energy than the average.

In fission products near the end of their range, the nuclear collisions are most important even for the energy loss, and they give rise to a large straggling (about 10 percent of the total range). All these problems are discussed in detail by Bohr (B43).



**2. Relation between the Mean Range and the Extrapolated Number-Distance Range.** Since the range of any one of a group of initially monoenergetic particles can be regarded as the sum of a very large number of statistically independent displacements corresponding to a succession of small energy losses, we should expect that the probability distribution of the ranges about their mean value $R_0$ is given by a Gaussian function.[1] The width of the Gaussian will be proportional to the mean square fluctuation $(R - R_0)_{\text{av}}^2$. Thus the probability of finding a particle with range between $R$ and $R + dR$ is

$$p(R)\,dR = \frac{1}{\alpha\sqrt{\pi}} e^{-(R-R_0)^2/\alpha^2}\,dR \quad (38)$$

where

$$(R - R_0)_{\text{av}}^2 = \int p(R)(R - R_0)^2\,dR = \tfrac{1}{2}\alpha^2 \quad (38a)$$

Experimentally, it is not very convenient to make direct measurement of the number of particles whose ranges end in the interval from $R$ to $R + dR$.[2] Instead the number of particles which reach a certain distance $r$ from the source, i.e., particles whose range is greater than $r$, are usually measured. An ionization chamber whose front face is at the distance $r$ from the source, connected to an amplifier and counting circuits which in principle detect ionization pulses of all sizes, no matter how small, would be the ideal detecting device. The natural noise background of the amplifier, however, provides a fundamental limitation to the instrument. Therefore it is more practical to measure the number of particles which have a certain minimum residual energy $E_1$ when entering the chamber; according to Section 1B, this corresponds to a certain minimum ionization and therefore to a certain bias. Such experiments will give the range of the particles up to the point where their energy has been reduced to $E_1$ rather than their complete range. This makes no difference for the following consideration.

With instruments of this type, an experimental number-distance curve is obtained. The recording of the detector is proportional to the

[1] A more accurate theory of the distribution function is given by Landau (L2); for further discussion see Section 2A2.

[2] See, however, Rutherford, Ward, and Wynn-Williams, *Proc. Roy. Soc.* (*London*), **129**, 211 (1930), in which two adjacent shallow ionization chambers are used with opposite applied potentials and a common collecting grid.

function

$$P(r) = \int_r^\infty p(R)\,dR = \frac{1}{2}\left(1 - \text{Erf}\,\frac{r - R_0}{\alpha}\right) \tag{39}$$

where $\text{Erf}\,(x) = (2/\sqrt{\pi})\int_0^x \exp(-t^2)\,dt$. From Eq. (39) and from the curve of $P(r)$ shown in Fig. 15b, we see that $P(r) = \frac{1}{2}$ for $r = R_0$, so that the mean range $R_0$ may be defined as that distance which is reached by just one-half of the particles. In order to apply this definition experimentally, however, it is necessary to know that the beam of particles is initially homogeneous in energy. Although this condition is very nearly met by the best accelerators which produce collimated beams of monoenergetic particles (Van de Graaf generator or linear accelerator), it is often not satisfied. For example, the beam of a cyclotron shows considerable energy variations, and the same is true of particles emitted in a nuclear reaction, the energy variation in this case being due to a possible stopping in the material of the target and to the dependence of the energy on the angle of emission in the reaction.

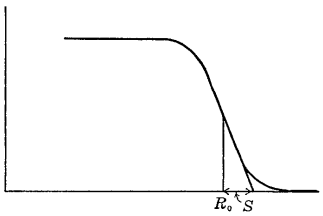


Fig. 15b. Number-distance curve for monoenergetic particles.

For an inhomogeneous beam the half-intensity point has no special importance. It is therefore preferable to measure what is called the *extrapolated number-distance range*. This is obtained by drawing the tangent at the steepest point on the essentially straight descending portion of the number-distance curve. The intersection of this tangent with the range axis gives the extrapolated range (cf. Fig. 15b). The range distribution function $P(r)$ for homogeneous particles has its steepest slope of $-1/\alpha\sqrt{\pi}$ at $r = R_0$. Since $P(R_0) = \frac{1}{2}$, the extrapolated range is

$$R_{\text{extr}} = R_0 - \frac{P(R_0)}{P'(R_0)} = R_0 + \frac{1}{2}\alpha\sqrt{\pi} \tag{40}$$



We define the "straggling" $S$ as the difference between extrapolated number-distance range and mean range. Since $S = \frac{1}{2}\alpha\sqrt{\pi}$, we obtain

$$S^2 = \tfrac{1}{2}\pi(R - R_0)_{\text{av}}^2 \tag{41a}$$

$$p(R)\,dR = \frac{1}{2S}\,e^{-(\pi/4S^2)(R-R_0)^2}\,dR \tag{41b}$$

In comparing the values of $S$ calculated from Eq. (37) with the results of experiment, we must expect to find disagreement for low energies where the theory of stopping is inaccurate. Thus, for the very low energies where the charged particle begins to capture and lose electrons intermittently, the straggling will in general be increased. However, if we calculate the difference of $S^2$ for two particle energies, making use of Eq. (37), we should improve the agreement with experiment since the low-energy straggling cancels out. For the low-energy alpha-particles of Po and the high-energy ones of ThC′, experiment gives $S = 0.043$ cm and 0.111 cm, respectively. The experimental value of $S_{\text{ThC}'}^2 - S_{\text{Po}}^2$ is thus $105 \times 10^{-4}$ cm$^2$, which is to be compared with the theoretical result $121 \times 10^{-4}$ cm$^2$. We therefore calculate differences of $S^2$ theoretically and then add the empirical low-energy results. The result is shown in Fig. 15a in which straggling is given as a function of the range of the particles.

At high energies, the straggling formula (37) reduces to the simple formula of Bohr (36a). This formula has been well verified by Bloembergen and Van Heerden (B34a), who used monochromatic protons of 50 to 115 Mev. Mather and Segrè found that about 75 percent of the straggling of the 340-Mev protons from the Berkeley cyclotron was contributed by the theory of Eqs. (35) and (36a), the remainder presumably being due to the energy inhomogeneity of the cyclotron beam of about $\pm 2$ Mev.

For high energies, the number-distance curve is not given by Fig. 15b because nuclear collisions reduce the number of particles appreciably before the end of their range. Mather and Segrè (M9) also observed some unexplained deviations from the expected ionization *vs.* distance curve. They further pointed out that multiple scattering contributes to the experimental straggling.

From Eq. (37) we can easily obtain the dependence of the straggling on mass and charge of the particle being stopped. Since $dE/dx$ is proportional to $z^2$ and $dE \sim M$, all other factors being functions of the velocity, we find

$$S = \sqrt{M}\,z^{-2}g(v) \tag{42}$$

By using Eq. (16), the straggling of particles of atomic weight $M$, charge

$z$, range $R$ can be expressed in terms of the straggling of protons of range $z^2R/M$ (i.e., of the same velocity) by

$$\left(\frac{S}{R}\right)_{M,z,R} = \frac{1}{\sqrt{M}}\left(\frac{S}{R}\right)_{1,1,z^2R/M} \tag{42a}$$

The main practical interest of the straggling is the calculation of mean ranges from measurements of the extrapolated range. For a homogeneous group of particles we simply have to subtract $S$ from the extrapolated range. For an inhomogeneous group of known inhomogeneity we may still define the extrapolated range by the tangent to the steepest portion of the number-range curve, and we may calculate the relation between this extrapolated curve and the mean range. This calculation was carried out by Livingston and Bethe for several practical cases. The first of these is a nuclear reaction in a thick target; then the incident particle loses energy in the target and the energy of the emitted particle depends on that of the incident one, and thus on the exact depth in the target where the reaction occurred. The result is an inhomogeneous beam of particles whose energy distribution can be calculated if the excitation function of the reaction is known. (Usually it is sufficient to use the Gamow penetration function.) The second case which has been treated is the dependence of the energy of the particles emitted in a reaction on the angle of emission (see Part VI of Volume II). Since the detector always admits a finite angular interval, this again gives a beam of calculable inhomogeneity. In either case, the extrapolated range will differ from the mean range by a calculable fraction of $S$.

In some of the older experiments it was not possible to observe the pulses due to individual particles, but instead the total ionization was measured (cf. Section 1B4). This leads to a curve of ionization *versus* distance (Bragg curve) which can also be extrapolated to obtain the "extrapolated ionization range." Some of the best range determinations for natural alpha-particles by the Cavendish Laboratory were made in this way. Holloway and Livingston (H21) have discussed these experiments and the relation of extrapolated ionization range to mean range in detail. Rado (R1) attempted a similar discussion for protons but found too much variation in the experimental data between observers. All the recent experiments were made by counting single pulses; they do not require any corrections.

### D. Elastic and Inelastic Scattering of Heavy Particles by Atoms

**1. Elastic Scattering.** In the previous discussion of the stopping of heavy charged particles, except for the stopping of the fission fragments, we have neglected any energy loss through collisions with the nuclei of the atoms of the stopping material. This has been justified in Section 1A6. Nevertheless, the cloud chamber photographs of the tracks of heavy particles occasionally show marked changes in direction, usually accompanied by another track forming a fork. This is to be interpreted as a close nuclear collision in which a recoil nucleus is produced. At high energies the nuclear scattering is due mostly to the nuclear forces. At lower energies the nuclear forces play no essential role, since the electrostatic repulsion between the incident heavy particle and the nucleus keeps the two particles beyond the effective range of the nuclear forces.



We shall consider only the low-energy scattering processes due to the Coulomb forces. The frequency of such collisions is governed by the well-known Rutherford scattering formula. If the mass $M_1$ of the incident charged particle is small compared with the mass of the nucleus $M_2$, and if $z$ and $Z$ are the charges of these two particles, respectively, the differential cross section for scattering into the solid angle $2\pi \sin\theta\, d\theta$ with no change of velocity $v$ is

$$d\Phi_0(\theta) = \frac{2\pi e^4 z^2 Z^2}{16E^2 \sin^4(\theta/2)} \sin\theta\, d\theta \tag{43}$$

$$d\Phi_0(\theta) = \frac{0.8139 z^2 Z^2}{E_{\text{Mev}}{}^2} \cdot \frac{\sin\theta\, d\theta}{\sin^4(\theta/2)} \cdot 10^{-26}\ \text{cm}^2 \tag{43a}$$

where $E = \frac{1}{2}M_1v^2$ is the kinetic energy of the incident particle and $\theta$ is the angle of scattering from the incident direction. In Eq. (43a) $E$ is measured in Mev.

Formula (43) is actually not correct for extremely small angles, since it does not allow for the scattering by the electrons in the atom. This effect can be accounted for by replacing $Z$ in Eq. (43) by $Z - F(\theta)$, where $F(\theta)$ is the atom form factor well known in the scattering of x-rays. For heavy charged particles $F(\theta)$ is completely negligible for any observable angle. Its inclusion in formula (43), however, prevents the cross section from becoming infinite at $\theta = 0$ because of the shielding effect of the atomic electrons.

Integration of Eq. (43) over angles from $\theta$ to $\pi$ gives the cross section for scattering into angles exceeding $\theta$. Thus

$$\int_\theta^\pi d\Phi_0(\theta) = \frac{\pi e^4 z^2 Z^2}{4E^2} \cot^2\frac{1}{2}\theta \tag{44}$$

Multiplying this cross section by the number of nuclei per cubic centimeter, $N$, of the stopping material gives the total number of scatterings

per centimeter greater than $\theta$. For air at 15°C and 1 atmos this gives

$$N\int_{\theta}^{\pi} d\Phi_0(\theta) = z^2 \frac{0.406 \times 10^{-4}}{E_{\text{Mev}}{}^2} \cot^2 \frac{1}{2}\theta \text{ per cm} \tag{45}$$

For a 1-Mev proton the probability of scattering through an angle greater than 10° is 0.53 percent per centimeter.

The multiple scattering of heavy particles can also be derived from Eq. (43). The result is exactly the same as for electrons; see Section 2E.

For the elastic scattering of heavy charged particles by light atoms, formula (43) cannot be used directly but rather should be interpreted as applying to the coordinate system in which the center of gravity of the incident particle and scattered atom is at rest. If the transformation from the center-of-gravity system to the laboratory system is performed, the cross section is

$$d\Phi_0(\theta) = \frac{2\pi e^4 z^2 Z^2}{4E^2} \frac{\sin\theta\, d\theta}{\sin^4\theta} \frac{(M_2\cos\theta \pm \sqrt{M_2{}^2 - M_1{}^2\sin^2\theta})^2}{M_2\sqrt{M_2{}^2 - M_1{}^2\sin^2\theta}} \tag{46}$$

For $M_2 > M_1$ only the positive sign should be used before the square root; for $M_1 > M_2$ (which occurs practically only for a few cases, e.g., collisions of alpha-particles or deuterons in hydrogen) Eq. (46) should be calculated for a given angle $\theta$; both positive and negative signs should be used and the results added. Note that for $M_2 > M_1$ all angles of scattering from 0° to 180° are possible. For $M_1 > M_2$ the maximum angle of scattering is $\theta_{\max} = \text{arc sin } M_2/M_1$.

The Rutherford scattering formula (43) is rigorously correct both in classical theory and in quantum theory with one exception—the scattering of identical particles. Only the quantum theory formula takes proper account of the indistinguishability of the scattered particle and the particle initially at rest. Different results are obtained for the cross section according as the particles have spin 0 (alpha-particles) or ½ (protons). For spin 0, according to Mott (M18), we have for the differential cross section in the laboratory system

$$d\Phi_0(\theta) = \frac{2\pi z^4 e^4 \cos\theta \sin\theta\, d\theta}{E^2} \times \left[\frac{1}{\sin^4\theta} + \frac{1}{\cos^4\theta} + \frac{2}{\cos^2\theta\sin^2\theta}\cos\left(\frac{z^2e^2}{\hbar v}\log\tan^2\theta\right)\right] \tag{47a}$$



and for particles of spin ½

$$d\Phi_0(\theta) = \frac{2\pi z^4 e^4 \cos\theta \sin\theta\, d\theta}{E^2} \times \left[\frac{1}{\sin^4\theta} + \frac{1}{\cos^4\theta} - \frac{1}{\cos^2\theta\sin^2\theta}\cos\left(\frac{z^2e^2}{\hbar v}\log\tan^2\theta\right)\right] \tag{47b}$$

The classical Rutherford formula does not contain the last terms in Eqs. (47a) and (47b) which depend on $\hbar$. The experiments verify the quantum theory results very well.

**2. Inelastic Scattering; Secondary Electrons.** In every primary ionizing collision between a charged particle and an atom, one or more electrons are ejected. The more energetic of these electrons are responsible for the secondary ionization which always accompanies a primary ionization process (cf. Section 1B). It is therefore of interest to examine the numbers and energy distribution of these secondary electrons.

We restrict the consideration to the so-called delta-rays, electrons ejected with energy large compared with the ionization potential. The binding of these electrons can therefore be neglected and their collisions with the incident heavy particles evaluated by means of the Rutherford scattering formula. The cross section, Eq. (43), is valid provided that $E$ and $\theta$ are understood to refer to the center-of-gravity coordinate system for the incident particle and the atomic electron. This means that $E = \frac{1}{2}mv^2$, and a simple consideration shows that the angle of ejection of the electron in the laboratory system is given by $\Theta = (\pi - \theta)/2$ and that the energy $W$ of the ejected electrons is

$$W = 2mv^2 \sin^2\frac{\theta}{2} = 2mv^2\cos^2\Theta \tag{48}$$

where $v$ is the velocity of the incident heavy charged particle; the electron mass has been neglected in comparison to that of the incident particle. From Eqs. (48) and (43) we find that the cross section for ejection of a delta-ray with energy between $W$ and $W + dW$ is

$$d\Phi = \frac{2\pi e^4 z^2}{mv^2}\frac{dW}{W^2} \tag{49}$$

and that the cross section for finding an electron between angle $\Theta$ and $\Theta + d\Theta$ with the incident direction is

$$d\Phi = \frac{2\pi e^4 z^2}{m^2v^4}\frac{\sin\Theta\, d\Theta}{\cos^3\Theta} \tag{49a}$$

The electrons in the forward direction are the most energetic and have an energy nearly $4(m/M)E$, where $E$ is the energy of the primary particle. Most of the electrons are emitted at larger angles with correspondingly smaller energy, the maximum angle being 90° for zero energy.

To find the number of delta-rays emitted per centimeter of path we must multiply the cross section, Eq. (49) or (49a), by the number of electrons in the atoms per cubic centimeter of the stopping material. Taking protons in air at 15°C and 760 mm pressure, the number of delta-rays per centimeter with energy between $W$ and $W + dW$ (measured in kev) is

$$dN = \frac{0.091}{(v/c)^2} \frac{dW_{\text{kev}}}{W_{\text{kev}}{}^2} \tag{50}$$

where $W$ is understood to be less than or equal to its maximum allowed value. For a 10-Mev incident proton $W_{\max} = 21.7$ kev, and the number of delta-rays per centimeter from 15 to 21.7 kev is 0.09; from 10 to 15 kev, 0.14; and from 5 to 10 kev, 0.43. The number of delta-rays per centimeter of path is inversely proportional to the proton energy.

The observation of the number of delta-rays per centimeter of path has recently proved very useful in the establishment of the charge of heavy nuclei in the cosmic radiation at the top of the atmosphere (B53, F11); it is in this case much more reliable than a determination of the ionization.

## SECTION 2. PENETRATION OF BETA-RAYS THROUGH MATTER

### A. Energy Loss

For electrons of relatively low energy (less than the so-called critical energy for the stopping material—cf. Section 2A4), the energy loss in matter is due to excitation and ionization of the bound electrons in the stopping substance, just as for the heavy particles. Indeed, the energy loss per centimeter for a proton and that for an electron of the same velocity, for low velocities, are not very different. For high energies, however, the energy loss of the electrons is due to an entirely different mechanism, namely loss by the emission of electromagnetic radiation in the electric field of the nuclei of the stopping material. According to the classical electromagnetic theory a charge which undergoes an acceleration $a$ emits radiant energy at a rate $(2/3)(e^2/c^3)a^2$. An electron in the Coulomb field of a nucleus can experience a large acceleration in virtue of its small mass, the acceleration being proportional to the nuclear charge $Z$ divided by the mass $m$. The resulting radiation, or bremsstrahlung as it is called, is the dominant influence in the energy loss of fast electrons. It plays no role in the stopping of heavy particles, since the dependence of the acceleration $a$ on $1/M$ gives a factor $(m/M)^2$ compared with the effect for electrons.



**1. Energy Loss by Inelastic Collisions.** The theory of the energy loss of an electron by inelastic encounters with the electrons in the stopping material parallels the treatment for heavy particles and has been worked out by Bethe as part of the general theory (B12, B14). The expression for the ionization energy loss per centimeter of electrons differs from that for protons in two important respects. The first modification of formula (3a), which represents the energy loss for heavy particles, consists of the replacement of the term $2mv^2$ in the log by $mv^2$. This change arises from the fact that the reduced mass of a two-electron system is $\frac{1}{2}m$, whereas the reduced mass of the system consisting of a heavy particle and an electron is essentially the mass of the electron. Thus the formula given in (B12) for the energy loss of electrons per centimeter by collisions is

$$-\frac{dE}{dx} = \frac{4\pi e^4 N}{mv^2} Z \log \frac{mv^2}{I}$$

where all the symbols are defined as in Eq. (3a) and $v$ is a non-relativistic velocity.

This is not quite correct, however, since insufficient consideration is given to the ultimate indistinguishability of the two electrons emerging from the ionizing collision. If the electron emerging with the higher energy is defined as the primary one, the maximum energy loss in any collision is $\frac{1}{4}mv^2$ and not $\frac{1}{2}mv^2$. With this definition and with the Mott scattering cross section for identical particles of spin $\frac{1}{2}$, Eq. (47b), the energy loss is corrected to (cf. B14)

$$-\frac{dE}{dx} = \frac{4\pi e^4 NZ}{mv^2} \log_e \frac{mv^2}{2I} \sqrt{\frac{e}{2}} \tag{51}$$

where the $e$ under the log is the natural base of logarithms. The correction does not usually amount to more than 10 percent. Comparison with experiments on the energy loss in various gases gives very good results, as shown by Williams (W10, W17). Since the differences between Eqs. (51) and (3) lie in relatively small factors in the logarithmic term, protons and electrons of the same non-relativistic velocity will lose energy at about the same rate on the average.

For relativistic velocities the energy loss per centimeter, corresponding to Eq. (4) for heavy particles, has been worked out by Bethe (B14) on