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ON THE SOLID ANGLE SUBTENDED BY A CIRCULAR DISC

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The solid angle subtended by a circular disc from a point is approximated by the analytical expression for the solid angle for n-sided, regular polygon of area equal to that of the disc. When the point is farther from the plane of the disc than 0.1 the disc radius, polygons of 20 and 36 sides are required for approximating the solid angle subtended by the disc to within 1 and 0.1% respectively. In addition, the average solid angles subtended by circular discs from circular discs are calculated and given in tabular form for a range of disc sizes.

1. Introduction

Knowledge of the solid angle subtended by a circular disc of known diameter at a point located at a known distance from the center of the disc is required in a variety of problems involving the measurement of nuclear radiation; for example, in the mathematical modeling of nuclear gauges[1,2]). More commonly, this solid angle Ω appears in the calculation of the "geometry factor", f_G for a cylindrical detector and a radiation source, defined to be the fraction of the source radiation which is incident on the circular face of the detector:

$$f_G = \Omega/4\pi, \tag{1}$$

if the source is radiating isotropically.

This general problem has been treated by several authors[3–9]). In the notation of fig. 1:

$$\Omega = \int_{r=0}^{R} \int_{\phi=0}^{2\pi} (\cos\theta/d^2)\, r\, \mathrm{d}r\, \mathrm{d}\phi\,. \tag{2}$$

Except for the simple case in which the point is located on the axis of the disc, no closed-form expression for this double integral has been obtained. Even in cases where infinite-series representations have been developed, none of them converge rapidly over the practical range of values of h/D and R/D, the two parameters which uniquely determine Ω[6]). Evaluation of the integral either by term-by-term integration of a series expansion[3–6]) or by numerical quadrature methods[7]) result in lengthy computations and are not suitable for incorporation in mathematical models which often require many calculations of Ω. An alternate approach is to prepare a table of values of Ω over a range of values of h/D and R/D and to obtain the in-between cases by interpolation. Masket et al.[9]) have prepared such a table; however the solid angle varies too rapidly to permit accurate interpolation.

Jaffey[5]) used the approach of applying a relatively slowly varying correction factor to a simple approximate expression for the solid angle. For small values of h/R, the correction factors varied too rapidly for accurate interpolation. Gardner and Carnesale[10]) developed a method wherein they replaced the circular disc with a square of equal area and obtained an approximate expression good to within one percent for all $h/D \geqslant 0.1$, except for a region of R/D within the interval $0.1 < R/D < 2.0$. For this region, they have presented a table of multiplicative correction factors. These factors, however, range from 1 to 2.24 and are not easily amenable to an empirical fit in terms of the variables R/D and h/D. Therefore, it is still necessary to perform interpolation in order to obtain the correction factor.

This paper presents a variation of the approach used by Gardner and Carnesale[10]). The circle is replaced

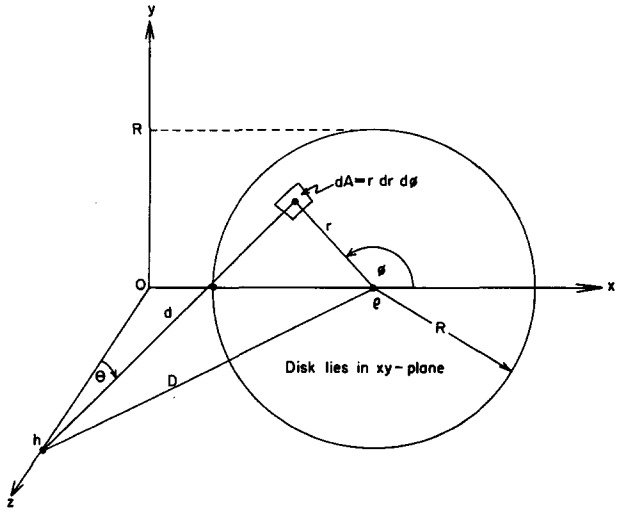

Fig. 1. Coordinate system and notation for circle.

by an equal-area polygon[12]) with an even number of sides n. The double integral can be integrated analytically for polygons. The final formula for the solid angle is a completely analytic expression involving only $n/2-1$ terms.

A typical case where this analytical approximation for the solid angle is useful is when a circular source is facing a circular disc so that the planes of the two are parallel and the center point of each is intersected by the same line drawn normal to the surfaces. This problem is encountered often in radiation detection and in sputtering experiments involving circular target and collector. Burtt[4]) has given a series approximation for this case, but it does not converge for large values of the ratio of source radius to the source-disc distance. A table of values has been generated for this case using the polygon approximation in the integration over the face of the source.

2. Derivation of polygon solid angle equations

The equal-area polygon that replaces the circle is shown in fig. 2. The left-most side is taken to be vertical because, at least in the case of a small number of sides, we have found that such a configuration gives the best results.

For the notation of fig. 2 the solid angle subtended by the differential area $\mathrm{d}A$ is given by

$$\mathrm{d}\Omega = \frac{h\,\mathrm{d}x\,\mathrm{d}y}{(x^2+y^2+h^2)^{\frac{3}{2}}}. \tag{3}$$

The solid angle subtended by the area with first quadrant under the side AB is given by

$$\Omega_{\mathrm{AB}} = \int_{x=a}^{b}\int_{y=0}^{mx+c} \frac{h\,\mathrm{d}x\,\mathrm{d}y}{(h^2+x^2+y^2)^{\frac{3}{2}}}, \tag{4}$$

integrating over y,

$$\Omega_{\mathrm{AB}} = \int_a^b \frac{(mx+c)\,\mathrm{d}x}{(x^2+h^2)\,[(1+m^2)x^2+2mcx+c^2+h^2]^{\frac{1}{2}}}. \tag{5}$$

The above integral may be evaluated using the bilinear transformation

$$x = (\mu t+\nu)/(t+1), \tag{6}$$

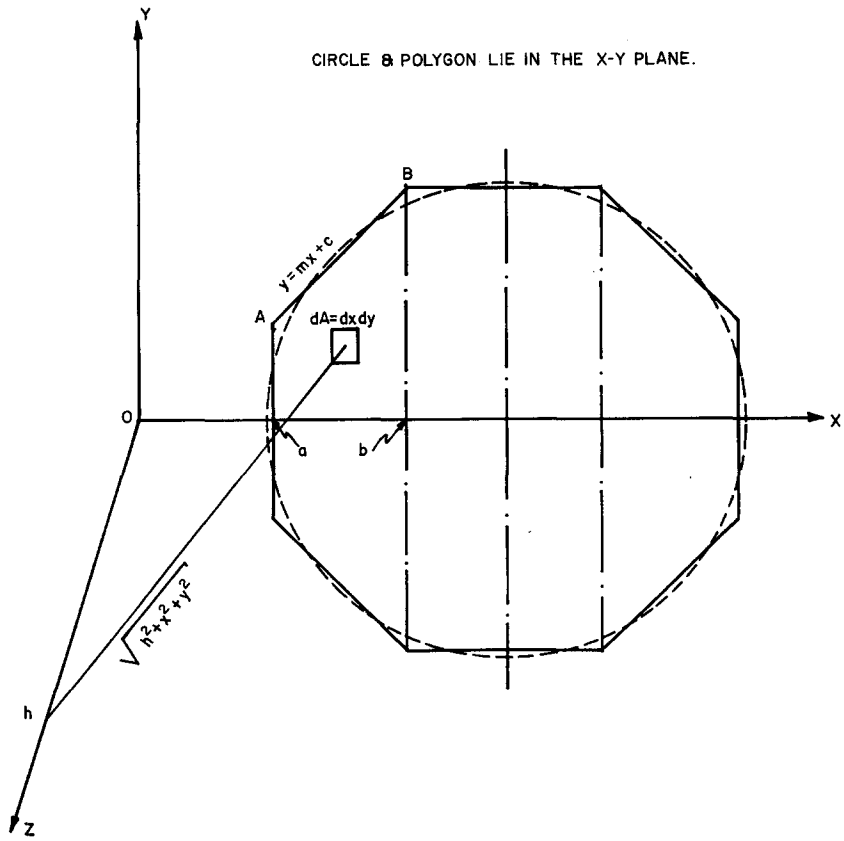

Fig. 2. Coordinate system for an equal-area polygon (arbitrarily chosen to have 8 sides) replacing the circle.

as shown by Hardy[11]) and choosing the constants μ and ν such that

$$\mu\nu = -h^2 \tag{7}$$

and,

$$(1+m^2)\mu\nu + mc(\mu+\nu) + (c^2+h^2) = 0. \tag{8}$$

This reduces the integral to a tabulated form.

Without going into the details, the final results may be stated as follows:

If the sides are numbered 1, 2, ... starting from the left of the polygon as shown in fig. 3, the solid angle subtended by the trapezium under the rth side is given by

$$\Omega_r = 2(E_r - F_r), \tag{9}$$

where

$$E_r = \arctan \frac{h(x_{r+1}^2 + y_r^2 - 2m_r x_r y_r + m_r^2 x_r^2 + h^2 + m_r^2 x_{r+1}^2 + 2m_r x_{r+1} y_r - 2m_r^2 x_{r+1} x_r)^{\frac{1}{2}}}{m_r(h^2 + x_{r+1} x_r) - x_{r+1} y_r} \tag{10}$$

and

$$F_r = \arctan \frac{h(x_r^2 + y_r^2 + h^2)^{\frac{1}{2}}}{m_r(x_r^2 + h^2) - x_r y_r}. \tag{11}$$

Note that E_r and F_r should be taken from 0 to π.

Adding up the contributions from all the $(n-2)/2$ non-vertical sides, we get the total solid angle for the n-sided polygon for any specified h, R, and ρ:

$$\Omega(h, R, \rho) = 2 \sum_{r=1}^{(n-2)/2} (E_r - F_r). \tag{12}$$

The subscripted variables in eqs. (10) and (11) are given by the following relations:

$$m_r = \tan[(n-4r)\pi/2n], \quad 1 \leqslant r \leqslant (n-2)/2, \tag{13}$$

$$y_r = \frac{\frac{1}{2} p \sin[(2r-1)\pi/n]}{\sin(\pi/n)}, \quad 1 \leqslant r \leqslant n/2, \tag{14}$$

$$p = 2R[(\pi/n)\,\mathrm{tg}(\pi/n)]^{\frac{1}{2}}, \tag{15}$$

$$x_r = \rho - d_r, \quad 1 \leqslant r \leqslant n/2, \tag{16}$$

$$\rho = (D^2 - h^2)^{\frac{1}{2}}. \tag{17}$$

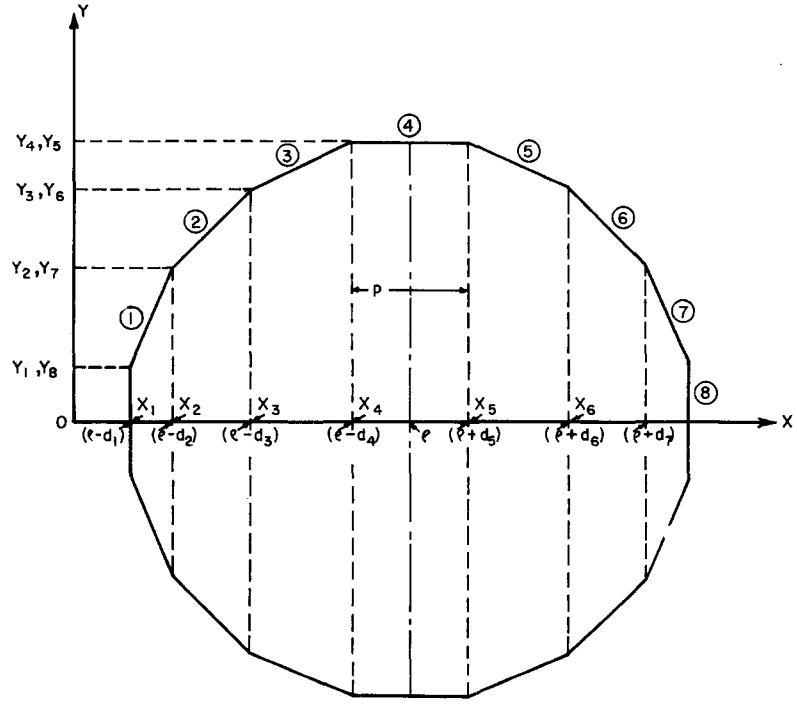

Fig. 3. Notation for equal-area polygon (arbitrarily chosen to have 16 sides) replacing circle.

For $n/2$ even:

$$d_r = \tfrac{1}{2}p + \frac{p\sin(\pi/4-\pi r/n)}{\sin(\pi/n)} \times \sin(\pi r/n+\pi/4-\pi/n), \quad 1 \leqslant r \leqslant (n-4)/4;$$

$$d_r = \tfrac{1}{2}p, \qquad r = n/4; \tag{18}$$

$$d_r = -d_l, \qquad (n+4)/4 \leqslant r \leqslant n/2;$$

where $l = (n+2)/2 - r$.

For $n/2$ odd:

$$d_r = \frac{p\sin(\pi/4-\pi/2n+\pi r/n)}{\sin(\pi/n)}, \times \sin(\pi/4+\pi/2n-\pi r/n), \quad 1 \leqslant r \leqslant (n-2)/4;$$

$$d_r = 0, \qquad r = (n+2)/4; \tag{19}$$

$$d_r = -d_l, \qquad (n+6)/4 \leqslant r \leqslant n/2;$$

where $l = (n+2)/2 - r$.

3. Approximation of circular disc solid angle by regular polygons

A general computer program for calculating the solid angle subtended by a regular polygon with n sides (n even) with area equal to that of a circle with radius R has been written. This program has been written in FORTRAN as a subroutine. The subroutine input consists of the circle radius R, the vertical distance from the point to the plane of the circle h, the lateral distance from the point to the center line of the circle ρ, and the number of sides of the polygon n. The output is the solid angle subtended by the polygon. This subroutine is available from the authors upon request.

This computer program has been used to calculate representative values of the solid angle subtended by an n-sided regular polygon in the region in which the equal-area square approximation of the circle[10]) was most inaccurate; viz. $h/D \geqslant 0.1$ and $0.1 \leqslant R/D \leqslant 2$. The results indicate that the accuracy for approximating a circle improves monotonically with an increasing number of polygon sides. The calulated values were compared to the table of values given by Masket, Macklin and Schmitt[9]). It was found that a 36-sided polygon was necessary to give 0.1% accuracy and a 20-sided polygon was sufficient for 1.0% accuracy. This large number of sides is only required at an h/D of 0.1 and an R/D of about 1. For other cases the number of sides could be greatly reduced. The computer time required for calculating a 36-sided polygon

TABLE 1

Circular disc to circular disc solid angle values for various values of S/h and R/h.

R/h	S/h 0.1	0.3	0.6	0.9	1.2	1.5	1.8	2.1	2.4	2.7	3.0
0.1	0.03095	0.02925	0.02476	0.01986	0.01567	0.01242	0.00997	0.00818	0.00671	0.00562	0.00477
0.2	0.12116	0.11481	0.09776	0.07880	0.06239	0.04954	0.03980	0.03243	0.02682	0.02248	0.01908
0.3	0.26328	0.25045	0.21526	0.17492	0.13924	0.11091	0.08925	0.07282	0.06025	0.05053	0.04290
0.4	0.44680	0.42708	0.37143	0.30515	0.24470	0.19577	0.15795	0.12906	0.10689	0.08970	0.07618
0.5	0.65997	0.63413	0.55894	0.46532	0.37664	0.30304	0.24532	0.20085	0.16656	0.13988	0.11887
0.6	0.89147	0.86103	0.76970	0.65039	0.53234	0.43134	0.35064	0.29780	0.23903	0.20094	0.17087
0.7	1.13154	1.09827	0.99552	0.85472	0.70853	0.57890	0.47299	0.38940	0.32402	0.27271	0.23208
0.8	1.37246	1.33801	1.22890	1.07237	0.90146	0.74364	0.61123	0.50503	0.42117	0.35498	0.30236
0.9	1.60860	1.57431	1.46338	1.39749	1.10699	0.92312	0.76401	0.63394	0.53006	0.44749	0.38158
1.0	1.83613	1.80299	1.69389	1.52472	1.32082	1.11458	0.92977	0.77527	0.65020	0.54996	0.46954
1.2	2.25712	2.22791	2.12937	1.96798	1.75641	1.52114	1.29270	1.09086	0.92173	0.78331	0.67081
1.4	2.62808	2.60346	2.51930	2.37672	2.17806	1.93761	1.68279	1.44153	1.22982	1.01554	0.90407
1.6	2.95056	2.93026	2.86038	2.73981	2.56524	2.33980	2.08017	1.81417	1.56660	1.34999	1.16650
1.8	3.22972	3.21311	3.15579	3.05603	2.90856	2.70986	2.46499	2.19324	1.92198	1.67254	1.45443
2.0	3.47156	3.45798	3.41108	3.32925	3.20716	3.03871	2.82114	2.56226	2.28368	2.01134	1.76304
2.2	3.68179	3.67064	3.63219	3.56511	3.46478	3.32491	3.13924	2.90661	2.63803	2.35664	2.08615
2.4	3.86541	3.85622	3.82453	3.76934	3.68692	3.57171	3.41685	3.21650	2.97185	2.69712	2.41599
2.6	4.02668	4.01905	3.99275	3.94708	3.87909	3.78425	3.65634	3.48823	3.27513	3.02110	2.74323
2.8	4.16911	4.16271	4.14074	4.10268	4.04625	3.96786	3.86237	3.72305	3.54286	3.31863	3.05769
3.0	4.29558	4.29018	4.27168	4.23974	4.19257	4.12738	4.04007	3.92495	3.77498	3.58357	3.34978

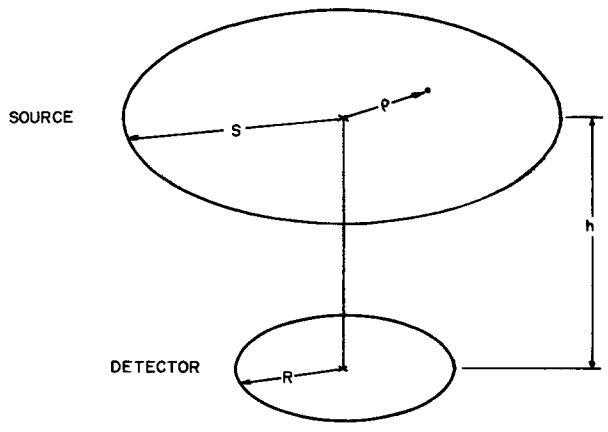

Fig. 4. Notation for circular disc facing a circular disc.

with this program was about 60 msec on the IBM 360 Model 75 computer.

4. The solid angle subtended by a circular disc from a circular disc

The polygon approximation of the solid angle subtended by a circular disc from a point has been used to generate a table of solid angle values for the case of a circular disc facing a circular disc. The two discs are parallel to each other and their centers are intersected by a single line normal to their surfaces. The arrangement of the two discs and the notation used is given in fig. 4. The pertinent solid angle is given by:

$$\Omega_D(S/h, R/h) = \int_0^S \Omega(h, R, \rho)\,\rho\,d\rho \Big/ \int_0^S \rho\,d\rho \qquad (20)$$

or

$$\Omega_D(S/h, R/h) = (2/S^2)\int_0^S \Omega(h, R, \rho)\,\rho\,d\rho\,. \qquad (21)$$

The integration indicated by eq. (21) has been performed using a standard Simpson's rule numerical method on the computer. The resulting values for a range of S/h from 0 to 3.0 and a range of R/h from 0.1 to 3.0 are presented in table 1. The interval size for the integrations for ρ was always 0.01 so that the error due to the numerical integration is of the order of $-(10^{-10}\,h^5/144)\,(d^4y/d\rho^4)$ where y is $2\,\Omega(h, R, \rho)\,\rho/S^2$ and is in the range from y at $\rho = 0$ to y at $\rho = S/h$. Integrations with different interval sizes indicated that the controlling error was that due to evaluating $\Omega(h, R, \rho)$, so the estimated accuracy of the values in table 1 is $\leqslant 0.1\%$.

5. Future work

An approximation for the solid angle subtended by a right circular cylinder is presently being developed. This should be useful in processing radiation survey results in cases where the detector is a right circular cylinder and is 100 percent efficient for the radiation being detected.

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