Radiation Detection Measurement p92-95
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This result shows that the standard deviation expected for the sum of all the counts is the same as if the measurement had been carried out by performing a single count, extending over the entire period represented by all the independent counts.

Now if we proceed to calculate a mean value from these N independent measurements,

$$\bar{x} = \frac{\Sigma}{N} \tag{3-43}$$

Equation (3-43) is an example of dividing an error-associated quantity (Σ) by a constant (N). Therefore, Eq. (3-40) applies and the expected standard deviation of this mean value is given by

$$\sigma_{\bar{x}} = \frac{\sigma_\Sigma}{N} = \frac{\sqrt{\Sigma}}{N} = \frac{\sqrt{N\bar{x}}}{N}$$

$$\boxed{\sigma_{\bar{x}} = \sqrt{\frac{\bar{x}}{N}}} \tag{3-44}$$

Note that the expected standard deviation of any single measurement x_i is

$$\sigma_{x_i} = \sqrt{x_i}$$

Because any typical count will not differ greatly from the mean, $x_i \cong \bar{x}$, and we therefore conclude that the mean value based on N independent counts will have an expected error that is smaller by a factor $\sqrt{N}$ compared with any single measurement on which the mean is based. A general conclusion is that, if we wish to improve the statistical precision of a given measurement by a factor of 2, we must invest four times the initial counting time.

Case 5. Combination of Independent Measurements with Unequal Errors

If N independent measurements of the same quantity have been carried out and they do not all have nearly the same associated precision, then a simple average (as discussed in Case 4) no longer is the optimal way to calculate a single "best value." We instead want to give more weight to those measurements with small values for σ_{x_i} (the standard deviation associated with x_i) and less weight to measurements for which this estimated error is large.

Let each individual measurement x_i be given a weighting factor a_i and the best value $\langle x \rangle$ computed from the linear combination

$$\langle x \rangle = \frac{\sum_{i=1}^{N} a_i x_i}{\sum_{i=1}^{N} a_i} \tag{3-45}$$

We now seek a criterion by which the weighting factors a_i should be chosen in order to minimize the expected error in $\langle x \rangle$.

For brevity, we write

$$\alpha \equiv \sum_{i=1}^{N} a_i$$

so that

$$\langle x \rangle = \frac{1}{\alpha} \sum_{i=1}^{N} a_i x_i$$

Now apply the error propagation formula [Eq. (3-37)] to this case:

$$\begin{aligned}
\sigma_{\langle x \rangle}^2 &= \sum_{i=1}^{N} \left(\frac{\partial \langle x \rangle}{\partial x_i} \right)^2 \sigma_{x_i}^2 \\
&= \sum_{i=1}^{N} \left(\frac{a_i}{\alpha} \right)^2 \sigma_{x_i}^2 \\
&= \frac{1}{\alpha^2} \sum_{i=1}^{N} a_i^2 \sigma_{x_i}^2
\end{aligned}$$

$$\sigma_{\langle x \rangle}^2 = \frac{\beta}{\alpha^2} \tag{3-46}$$

where

$$\beta \equiv \sum_{i=1}^{N} a_i^2 \sigma_{x_i}^2$$

In order to minimize $\sigma_{\langle x \rangle}$, we must minimize $\sigma_{\langle x \rangle}^2$ from Eq. (3-46) with respect to a typical weighting factor a_j:

$$0 = \frac{\partial \sigma_{\langle x \rangle}^2}{\partial a_j} = \frac{\alpha^2 \dfrac{\partial \beta}{\partial a_j} - 2\alpha\beta \dfrac{\partial \alpha}{\partial a_j}}{\alpha^4} \tag{3-47}$$

Note that

$$\frac{\partial \alpha}{\partial a_j} = 1 \qquad \frac{\partial \beta}{\partial a_j} = 2 a_j \sigma_{x_j}^2$$

Putting these results into Eq. (3-47), we obtain

$$\frac{1}{\alpha^4}\left(2\alpha^2 a_j \sigma_{x_j}^2 - 2\alpha\beta\right) = 0$$

and solving for a_j, we find

$$a_j = \frac{\beta}{\alpha} \cdot \frac{1}{\sigma_{x_j}^2} \tag{3-48}$$

If we choose to normalize the weighting coefficients,

$$\sum_{i=1}^{N} a_i \equiv \alpha = 1$$

$$a_j = \frac{\beta}{\sigma_{x_j}^2}$$

Putting this into the definition of β, we obtain

$$\beta = \sum_{i=1}^{N} a_i^2 \sigma_{x_i}^2 = \sum_{i=1}^{N} \left(\frac{\beta}{\sigma_{x_i}^2} \right)^2 \sigma_{x_i}^2$$

or

$$\beta = \left(\sum_{i=1}^{N} \frac{1}{\sigma_{x_i}^2} \right)^{-1} \tag{3-49}$$

Therefore, the proper choice for the normalized weighting coefficient for x_j is

$$\boxed{a_j = \frac{1}{\sigma_{x_j}^2} \left(\sum_{i=1}^{N} \frac{1}{\sigma_{x_i}^2} \right)^{-1}} \tag{3-50}$$

We therefore see that *each data point should be weighted inversely as the square of its own error.*

Assuming that this optimal weighting is followed, what will be the resultant (minimum) error in $\langle x \rangle$? Because we have chosen $\alpha = 1$ for normalization, Eq. (3-46) becomes

$$\sigma_{\langle x \rangle}^2 = \beta$$

In the case of optimal weighting, β is given by Eq. (3-49). Therefore,

$$\boxed{\frac{1}{\sigma_{\langle x \rangle}^2} = \sum_{i=1}^{N} \frac{1}{\sigma_{x_i}^2}} \tag{3-51}$$

From Eq. (3-51), the expected standard deviation $\sigma_{\langle x \rangle}$ can be calculated from the standard deviations σ_{x_i} associated with each individual measurement.

V. OPTIMIZATION OF COUNTING EXPERIMENTS

The principle of error propagation can be applied in the design of counting experiments to minimize the associated statistical uncertainty. To illustrate, consider the simple case of measurement of the net counting rate from a long-lived radioactive source in the presence of a steady-state background. Define the following:

$S \equiv$ counting rate due to the source alone without background

$B \equiv$ counting rate due to background

The measurement of S is normally carried out by counting the source plus background (at an average rate of $S + B$) for a time T_{S+B} and then counting background alone for a time T_B. The net rate due to the source alone is then

$$S = \frac{N_1}{T_{S+B}} - \frac{N_2}{T_B} \tag{3-52}$$

where N_1 and N_2 are the total counts in each measurement.

Applying the results of error propagation analysis to Eq. (3-52), we obtain

$$\sigma_S = \left[\left(\frac{\sigma_{N_1}}{T_{S+B}} \right)^2 + \left(\frac{\sigma_{N_2}}{T_B} \right)^2 \right]^{1/2}$$

$$\sigma_S = \left(\frac{N_1}{T_{S+B}^2} + \frac{N_2}{T_B^2} \right)^{1/2}$$

$$\sigma_S = \left(\frac{S+B}{T_{S+B}} + \frac{B}{T_B} \right)^{1/2} \tag{3-53}$$

If we now assume that a fixed total time $T = T_{S+B} + T_B$ is available to carry out both measurements, the above uncertainty can be minimized by optimally choosing the fraction of T allocated to T_{S+B} (or T_B). We square Eq. (3-53) and differentiate

$$2\sigma_S \, d\sigma_S = -\frac{S+B}{T_{S+B}^2} dT_{S+B} - \frac{B}{T_B^2} dT_B$$

and set $d\sigma_S = 0$ to find the optimum condition. Also, because T is a constant, $dT_{S+B} + dT_B = 0$. The optimum division of time is then obtained by meeting the condition

$$\left. \frac{T_{S+B}}{T_B} \right|_{\text{opt}} = \sqrt{\frac{S+B}{B}} \tag{3-54}$$

A figure of merit that can be used to characterize this type of counting experiment is the inverse of the total time, or $1/T$, required to determine S to within a given statistical accuracy. If certain parameters of the experiment (such as detector size and pulse acceptance criteria) can be varied, the optimal choice should correspond to maximizing this figure of merit.

In the following analysis, we assume that the optimal division of counting times given by Eq. (3-54) is chosen. Then we can combine Eqs. (3-53) and (3-54) to obtain an expression for the figure of merit in terms of the fractional standard deviation of the source rate, defined as $\epsilon \equiv \sigma_S/S$

$$\frac{1}{T} = \epsilon^2 \frac{S^2}{(\sqrt{S+B} + \sqrt{B})^2} \tag{3-55}$$

Equation (3-55) is a useful result that can be applied to analyze the large category of radiation measurements in which a signal rate S is to be measured in the presence of a steady-state background rate B. For example, it predicts the attainable statistical accuracy (in terms of the fractional standard deviation ϵ) when a total time T is available to measure the signal plus background and the background alone. The assumption has been made that this time is subdivided optimally between the two counts. Note that, in common with simple counting measurements, the time required varies as the inverse square of the fractional standard deviation desired for the net signal rate precision. Cutting the predicted statistical error of a measurement in half requires increasing the available time by a factor of 4.