David Bohm Quantum Theory Prentice-Hall 1951

Thus, for large negative times, we obtain an ingoing packet. Since only positive values of r exist, the ingoing packet disappears at $t = 0$, after which it is replaced by an outgoing packet. The center of the outgoing packet corresponding to the lth wave occurs where

$$\frac{\partial \phi_l'}{\partial k} = 0 \qquad \text{where} \qquad \phi_l' = kr - \frac{\hbar k^2}{2m}t + 2\delta_l$$

or at

$$r = vt - 2\frac{\partial \delta_l}{\partial k}$$

The outgoing packet thus appears with a time delay (or advance) of $2\frac{\partial \delta_l}{\partial k}$, which is the result of the action of the potential (see, for example, Sec. 29, and Chap. 11, Sec. 19).

We conclude from the above that the incident packet is identical with the incident part of a packet of plane waves, but that the outgoing packet will be modified by the actions of the potential.

47. Formula for Scattering Cross Section. To obtain the strength of the scattered wave, we note that even if there were no potential, there would still be an outgoing wave, which is just the outgoing part of a plane wave. The test for a scattered wave is to see whether the outgoing packet has been modified. We therefore obtain the asymptotic form of the scattered wave by subtracting from the actual outgoing wave the outgoing wave that would be present if there were no potential. That is, according to (77) and (78),

$$F_{\text{scatt}} = \sum \frac{e^{ikr}}{r}\frac{(e^{2i\delta} - 1)P_l(\cos\theta)(2l+1)}{2ik} = \frac{e^{ikr}}{r}f(\theta) \qquad (79)$$

where F_{scatt} is the asymptotic form of the scattered wave. The complete asymptotic wave function is now

$$e^{ikz} + \frac{f(\theta)e^{ikr}}{r}$$

Comparing with eq. (45), we see that the cross section is

$$\sigma = |f(\theta)|^2 = \frac{1}{k^2}\left|\sum_l \frac{(2l+1)}{2}P_l(\cos\theta)(e^{2i\delta_l} - 1)\right|^2 \qquad (80)$$

The above formula yields the angular-dependent cross section, once we know δ_l. (The latter must be obtained by solving Schrödinger's equation.) This angular dependence arises, in part, from the interference of waves of different l. For example, suppose we have scattered waves with $l = 0$ alone. Then there is no angular dependence, i.e., the cross section is spherically symmetric. With $l = 1$ alone, the cross section is proportional to $\cos^2\theta$. If both are present, as in $f(\theta) = a + b\cos\theta$,

then

$$\sigma = |a|^2 + |b|^2\cos^2\theta + (ab^* + ba^*)\cos\theta$$

A few typical curves are shown in Fig. 19. Thus, the angular dependence of the cross section involves interference between different l terms. If higher angular momenta are included, the pattern may grow still more complex. In the classical limit ($l \to \infty$) one can form a packet of waves of different l in such a way that they build up to a maximum at a definite value of θ. This corresponds to a classical orbit in which particles come in with a definite collision parameter and scatter through a definite angle.

48. Total Cross Section. To find the total cross section, we integrate σ over all solid angle, using the orthogonality of the $P_l(\cos\theta)$ and the

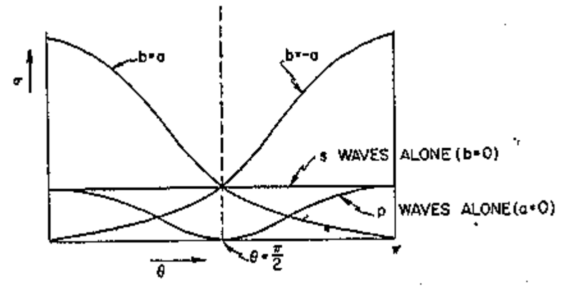

FIG. 19

normalization conditions [Chap. 14, eq. (52a)]. We obtain for the total cross section,

$$S = \sum_l \frac{4\pi}{k^2}(2l+1)\sin^2\delta_l \qquad (81a)$$

The above result means that in the total cross section, the various partial waves do not interfere. It is only in determining the angular distribution that they interfere.

The maximum cross section corresponding to a given value of l is

$$(S_l)_{\max} = \frac{4\pi(2l+1)}{k^2} \qquad (81b)$$

This will occur if $\delta_l = \pi/2$. Writing $k = 2\pi/\lambda$, we obtain

$$(S_l)_{\max} = \frac{(2l+1)\lambda^2}{\pi} \qquad (81c)$$

For s waves, for example, the maximum cross section corresponds to a circle of radius λ/π, and for higher l, it is still higher. This cross section can actually be produced by a scatterer that is much smaller than λ, provided that conditions are such as to make $\delta_l = \pi/2$.

49. Calculation of Phase for Impenetrable Sphere. Because the sphere is impenetrable, we must have $\psi = 0$ at the edge of the sphere, which we assume has a radius a. For s waves, the differential equation outside the sphere is just $-d^2g/dr^2 = k^2g$. The solution is

$$g_0 = A \sin (kr + \delta)$$

To have $g = 0$ at $r = a$, we must have $\delta_0 = -ka$. The partial scattering cross section for s waves is therefore

$$S = \frac{4\pi}{k^2} \sin^2 ka \tag{82a}$$

For higher angular momenta, the solutions are

$$g = \sqrt{kr}\,[AJ_{l+\frac{1}{2}}(kr) + BJ_{l-\frac{1}{2}}(kr)]$$

[Note that since the origin is now excluded, $J_{-l-\frac{1}{2}}(kr)$ must now be retained.] The boundary condition at $r = a$ yields

$$g(a) = 0 \qquad \text{or} \qquad \frac{B_l}{A_l} = -\frac{J_{l+\frac{1}{2}}(ka)}{J_{-l-\frac{1}{2}}(ka)}$$

The phase can be calculated from the asymptotic form of the wave function [see eq. (70)].[a] For large r,

$$\begin{aligned} g &\sim A \cos \left[kr - \left(l + \frac{1}{2}\right)\frac{\pi}{2} - \frac{\pi}{4}\right] + B \cos \left[kr + \left(l + \frac{1}{2}\right)\frac{\pi}{2} - \frac{\pi}{4}\right] \\ &= A \sin \left(kr - \frac{l\pi}{2}\right) + (-1)^l B \cos \left(kr - \frac{l\pi}{2}\right) \\ &= \sqrt{A^2 + B^2} \sin \left(kr - \frac{l\pi}{2} + \delta_l\right) \end{aligned}$$

where
$$\tan \delta_l = (-1)^l \frac{B_l}{A_l}$$

Special Case: $ka \ll 1$. If the wave length is so large that $ka \ll 1$, it is readily seen that the values of δ_l for successive l rapidly become very small. This is because particles of a given angular momentum, l, will scatter heavily only if it is likely that they strike the potential, or only if $pa \gtrsim \hbar l$ or $ka \gtrsim l$ (see Sec. 44). This can also be shown by evaluating δ_l from the formula given above.

Problem 2: Using the series expansion of Bessel's functions,* evaluate $\tan \delta_l$ for small ka, and show that

$$\frac{\delta_1}{\delta_0} \ll 1, \qquad \frac{\delta_l + 1}{\delta_l} \ll 1$$

[a] *Ibid.*

For small ka, the cross section is therefore given almost entirely by the s waves. Thus, we can use eq. (82a). With the expansion of $\sin^2 ka$, this equation yields

$$S \cong 4\pi a^2 \tag{82b}$$

Note that this result is four times the classical result for a hard sphere* (eq. 3). The increase is the result of quantum-mechanical diffraction effects.

It is of some interest to follow the transition from quantum to classical scattering, since the cross section must drop from $4\pi a^2$ to πa^2 as this transition takes place. Quantum scattering occurs when $ka \ll 1$, i.e., when $\lambda \gg 2\pi a$. As the wavelength goes below the size of the sphere, the first effect will be to introduce waves of higher angular momentum, so that the cross section becomes angular dependent. As the wavelength is made still shorter, however, and the classical region is approached, the cross section once again becomes spherically symmetrical, with a value reduced to πa^2. except for a region near $\theta = 0$ with an angular width of the order of $\Delta\theta \cong \lambda/2\pi a$. The polar intensity pattern is shown for large λ in Fig. 20. The large projection in the forward direction is essentially a diffraction effect, containing a total cross section of πa^2. Thus, for very short wavelengths, the total cross section is $2\pi a^2$, in contrast to the value of $4\pi a^2$, obtained with very long wavelengths. In the classical limit, however, the wavelength becomes so short that the large projection near the forward direction corresponds to deflections too small to produce significant results. Thus, for all practical purposes, the effective classical cross section is only πa^2.

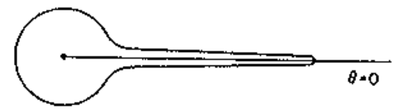

FIG. 20

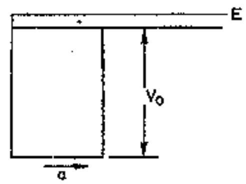

FIG. 21

Problem 3: With a sphere of radius 1 cm, and electrons of energy 1 ev, compute the angular width of the diffraction pattern in the forward direction and show that it is too small to be important in practice. (Use Huyghens' principle, as in optics.)

50. Application of Exact Method to Scattering from Square Well for s Waves. Consider a square well of radius a, depth V_0, as shown in Fig. 21. Suppose that particles are incident with energy E. We wish to compute the cross section, restricting ourselves to s waves only. This restriction will be valid only if $ka \ll 1$. Inside the well the radial equation is

* In eq. (3), a is by definition equal to only half the distance between centers of the spheres, whereas in (82b), it is equal to the full distance between centers. (We are considering only identical particles here.)

$$-\frac{d^2g}{dr^2} = k_1^2 g \tag{83}$$

where $$k_1^2 = (E + V_0)\frac{2m}{\hbar^2}$$

Since $g(r)$ must vanish at the origin,* the most general admissible solution is

$$g = A \sin k_1 r \tag{83a}$$

where A is an arbitrary constant. Outside the well, the most general solution is

$$g = B \sin (kr + \delta_0) \tag{83b}$$

where $$k^2 = \frac{2m}{\hbar^2} E$$

B and δ must be obtained by requiring that $g(r)$ and $g'(r)$ be continuous at $r = a$. To solve for δ alone, however, it is sufficient to make g'/g continuous. Setting

$$\frac{g'}{g} = \alpha \tag{84}$$

we obtain

$$k \cot (ka + \delta_0) = \alpha \tag{84a}$$

where $$\alpha = k_1 \cot k_1 a$$

or $$\tan (ka + \delta_0) = \frac{\tan ka + \tan \delta_0}{1 - \tan ka \tan \delta_0} = \frac{k}{\alpha} \tag{84b}$$

Solving for $\tan \delta_0$, we obtain

$$\tan \delta_0 = \frac{\dfrac{k}{\alpha} - \tan ka}{1 + \dfrac{k}{\alpha}\tan ka} = \frac{k\left(\dfrac{1}{\alpha} - \dfrac{\tan ka}{k}\right)}{1 + \dfrac{k}{\alpha}\tan ka} \tag{84c}$$

The total cross section is [see eq. (81a)]

$$S = \frac{4\pi \sin^2 \delta_0}{k^2} = \frac{4\pi}{k^2(1 + \cot^2 \delta_0)} = \frac{4\pi}{k^2 + \dfrac{(\alpha + k \tan ka)^2}{\left(1 - \alpha\dfrac{\tan ka}{k}\right)^2}} \tag{85}$$

By evaluating α from eq. (84a), one obtains the cross section.

51. Ramsauer Effect. We observe from eq. (85) that if the scattering phase is equal to some integral multiple of π for nonzero k, the cross section vanishes. If δ is an integral multiple of π, then $\tan \delta_0 = 0$. For a square well, we obtain the condition for the vanishing of $\tan \delta_0$ from eq. (84c):

* See Chap. 15, Sec. 3.

$$\frac{1}{\alpha} = \frac{\tan ka}{k} \tag{86}$$

Obtaining α from eq. (84a) one finds

$$\frac{\tan k_1 a}{k_1} = \frac{\tan ka}{k}$$

For small k, $ka \ll 1$. Replacement of $\tan ka$ by ka then yields

$$\tan (k_1 a) \cong k_1 a$$

For small k, k_1 is given approximately by $\sqrt{2mV_0}/\hbar$. If V_0 and a are such that the eq. (86) is satisfied, the scattering cross section will be zero, and if it is nearly satisfied, the cross section will be very small. This vanishing of the scattering cross section for a non-zero potential is peculiar to the wave properties of matter. It would occur, for example, with light waves which were being scattered from small transparent spheres with a high index of refraction, so chosen that the $\sin \delta_0$ corresponding to the scattered wave vanished. This means, essentially, that the contributions of the various parts of the potential to the scattered wave [see Sec. 26] interfere destructively, leaving only an unscattered wave. Although this result was derived for a square well, it can easily be extended to any well that has the property that it is fairly localized in space. This is because the vanishing of the phase is determined by the cumulative phase shifts suffered by the wave throughout the entire well, so that it is always possible to obtain a phase shift of $n\pi$ by properly choosing the magnitude and range of the potential.

For slow electrons scattered from noble gas atoms, it turns out that the $\sin \delta_0$ is very small and the cross section for electron-atom scattering is therefore much smaller than the gas-kinetic cross section. This effect is known as the Ramsauer effect. As the electron energy is increased, the phase of the scattered wave changes, and, eventually, at higher energies above 25 ev the usual gas-kinetic cross section is approached.

The Ramsauer effect is somewhat analogous to the transmission resonances obtained in the one dimensional potential (see Chap. 11, Sec. 9). The analogy, however, is not complete, because the condition for the Ramsauer effect [eq. (86)] is not exactly the same as that for a transmission resonance in a one-dimensional well [eq. (50), Chap. 11]. The reason for the difference is that in the one-dimensional case we define the transmitted wave as the total wave that comes through the well. In the scattering problems, we have an incident wave that converges on the well. Some of it enters the well and some of it is reflected at the edge of the well. The net effect is to produce an outgoing wave, whose phase depends on what happens to the wave at the well. The question of how much of this outgoing wave corresponds to a scattered wave depends on how large a phase shift it has suffered relative to the outgoing wave which would have been present in the absence of a potential. Thus we see that the intensity of the scattered wave depends on properties of the potential that are somewhat different from those determining the intensity of that part of the wave that is transmitted through the potential and out again on the other side. The vanishing of the cross section in the Ramsauer effect is, as we have already seen, a result of the fact that the contributions of different parts of the potential all add up in such a way as to produce a wave that cannot be distinguished from one which has not been inside a potential at all.

52. Approximation for Small k. For small k, we can expand the expression for $\tan \delta_0$, retaining only terms up to order k^3. We obtain

$$\tan \delta_0 \cong k \left[\frac{\left(\frac{1}{\alpha} - a\right) - \frac{k^2 a^3}{3}}{1 + k^2 \frac{a}{\alpha}} \right] \tag{87}$$

We see that as $k \to 0$, the phase also approaches zero. The sign of the phase at small k depends on the sign of $\frac{1}{\alpha} - a$.

If k is so small that $k^2a^2 \ll 1$ and $k^2a/\alpha \ll 1$, then the above expression simplifies to

$$\tan \delta_0 \cong k \left(\frac{1}{\alpha} - a\right) \tag{87a}$$

The cross section is (in this approximation)

$$S \cong \frac{4\pi}{k^2 + \frac{\alpha^2}{(1 - \alpha a)^2}} = \left(\frac{2\pi\hbar^2}{m}\right) \frac{1}{E + \frac{\hbar^2\alpha^2}{2m}\frac{1}{(1 - \alpha a)^2}} \tag{88}$$

To obtain a good idea of the low-energy cross section, we need only obtain α, which is defined in eq. (84a).

53. Application to Nuclear Scattering. We shall now make some applications in the field of nuclear scattering. Before doing this, however, we wish to point out that very little certain knowledge exists concerning nuclear forces. The main reason for studying the problem in this book is to illustrate how one uses the quantum theory to try to make advances in new fields, where the fundamentals are still uncertain. In this way, we hope to show that the application of the theory is not necessarily always restricted to the mere calculation of various kinds of numerical results, on the basis of a known and defined theory.

As has been stated in Chap. 11, Sec. 3, evidence exists indicating that the potential energy of a neutron in the field of a proton can be represented by a well that is about 20 mev deep and has a radius of the order of 2.8×10^{-13} cm. This well is almost certainly not exactly square, but many of its main features can be represented roughly with the aid of a square well.

We can obtain, however, many important results without making any specific assumptions about the shape of the well, other than that beyond some radius, which is of the order of 3×10^{-3} cm, the potential is small enough to be neglected. For this reason, it is convenient to separate the problem of solving Schrödinger's equation into two parts, namely, that of solving the problem inside the well and that of solving it outside. Since there is no appreciable potential outside, the solution is just that for a free particle [see eq. (83b)]. Inside the well, the general problem of solving the wave equation is complicated, but the result of this procedure, starting with $g = 0$ at $r = 0$, will always be to determine the ratio $g'/g = \alpha$ at the point $r = a$. All that remains to be done is to make g'/g continuous at the point a by proper choice of the phase δ.

For s waves, the procedure is exactly the same as that leading to eq. (84), so that the same equations hold, provided that we interpret α as the ratio, $(g'/g)_{r=a}$ obtained by solving Schrödinger's equation with the actual potential, whatever it may be.

54. Approximate Expression of Low-energy Cross Section in Terms of Binding Energy of Deuteron. Although we cannot solve for α directly unless we know the details of the shape of the potential, we shall nevertheless be able to obtain a good deal of information about α by comparing the observed cross sections with those which are predicted as a function of α. In this work, we shall use an approximate value of α, obtained with the aid of the observed result that there is a bound state of the deuteron at $E = -2.23$ mev. The value of α for a bound state is easily calculated from the fact that outside the potential the wave function is just a decaying exponential (for s waves), $g = A \exp(-\sqrt{2mB}\, r/\hbar)$ where B is the binding energy. Thus, we obtain for the value of α_0 in the bound state

$$\alpha_0 = -\frac{\sqrt{2mB}}{\hbar}$$

We observe that α_0 must be *negative* for a bound state; this is because the wave function inside the well has gone past a maximum and is decreasing with radius to meet a decaying exponential at $r = a$.

Now, the potential is of the order of 20 mev deep; hence α_0 undergoes only a small change as E is increased from -2.23 mev to a value of zero or slightly above, simply because the wavelength at any particular point is not changed much by this small fractional increase in kinetic energy.

For example, with a square well $\alpha_0 = k_1 \cot k_1 a$ is changed by about 20 per cent as E is increased from -2.16 mev to zero.*

Approximating α_0 by its value in the bound state, we obtain for the cross section (from eq. 88)

$$S \cong \left(\frac{2\pi\hbar^2}{m}\right)\frac{1}{E + \dfrac{B}{(1-\alpha_0 a)^2}} \tag{89}$$

In terms of the actual proton mass and the energy in the laboratory system of co-ordinates, which is twice the relative energy, we obtain

$$S = \frac{4\pi\hbar^2}{m_L}\frac{1}{\dfrac{E_L}{2} + \dfrac{B}{(1-\alpha_0 a)^2}} \tag{90}$$

where E_L is the energy in the laboratory system, and m_L is the actual proton mass.

Problem 14: Evaluate the above cross section at $E = 0$. (The result is of the order of 3×10^{-24} cm².)

The above cross section has a maximum at $E = 0$, and it decreases more or less uniformly thereafter. The approximation is fairly good (to about 25 per cent) up to $E \cong 5$ mev. At higher energies, more accurate formulae must be used. Furthermore, the p waves begin to come in, and these will affect both the total cross section and the angular dependence.

55. Spin-dependent Forces. We have obtained, in the previous section, a general approximate expression for the low-energy neutron-proton scattering cross section expressed as a function of the binding energy of the deuteron only; it is independent of the details of the shape of the potential function. Comparison with experiment should therefore provide a good check on the validity of our basic ideas of nuclear forces. Experiment shows that the low energy cross section is of the order of 20×10^{-24} cm², whereas our predictions are of the order of only 3×10^{-24} cm².

This discrepancy was explained by Wigner, who observed that to obtain a larger zero-energy cross section, one must, according to eq. (88), have a well for which the value of α is far below that which is obtained from the deuteron binding energy. To obtain such a low value of α, one needs a well which is shallower than that needed to yield the correct deuteron binding energy. This will cause the wave function to curve less within the region of the potential, and therefore to reach $r = a$ with a smaller slope. A comparison between the properties of a well deep

* Our procedure is therefore to evaluate α_0 empirically for a slightly negative value of the energy, and to use this value as an approximation to α_0 for slightly positive values of the energy.

enough to explain the deuteron binding energy and a well that explains scattering by yielding a small value of α is shown in Fig. 22.

In order to reconcile the different potential depths demanded by scattering data and by deuteron binding energy, he suggested that the nuclear forces were spin dependent, in such a way that when the spins of the particles are parallel the well is deeper than when they are antiparallel. Now it is known on independent grounds* that in the deuteron, the neutron and proton spins are parallel. On the other hand, in a beam of incident neutrons, for example, the relative orientations of spin to that of any proton in the target are random, so that both possibilities occur. This means that the large scattering cross section is produced in those cases of antiparallel orientation, while the parallel orientation has a deep enough well to explain the binding energy of the deuteron.

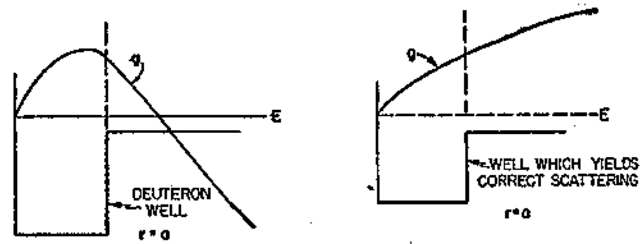

Fig. 22

In a beam in which neutron spins are oriented at random, the neutron spin will be parallel on the average, to that of an arbitrary proton for $\frac{3}{4}$ of the incident particles, and antiparallel for $\frac{1}{4}$ of them. This result follows from a study of the properties of the spin variables, which shows that there are three times as many ways to make the spins parallel as there are to make the spins antiparallel.† This means that the total cross section is

$$S = \tfrac{3}{4}S_p + \tfrac{1}{4}S_a$$

where S_p and S_a are respectively the cross sections for parallel and antiparallel spins.

Setting S equal to its observed value of 21×10^{-24} cm², and

$$S_p = 3 \times 10^{-24} \text{ cm}^2$$

we obtain

$$S_a \cong 75 \times 10^{-24} \text{ cm}^2$$

This is indeed a rather large cross-section. The cross-sectional area of the potential well is only about 0.3×10^{-24} cm². The possibility of so large a cross section comes entirely from the wave properties of matter and is, as we shall see, connected with the existence of a resonance near

* H. A. Bethe, *Elementary Nuclear Physics*.

† See, for example, Chap. 17, Sec. 10.