negative. A negative phase shift means that the radial wave function is "pushed out" in comparison with the force-free wave function.

In similar fashion, we see that a negative potential makes ϵ_l negative and δ_l positive. This means that the radial wave function is "pulled in" by the attractive potential.

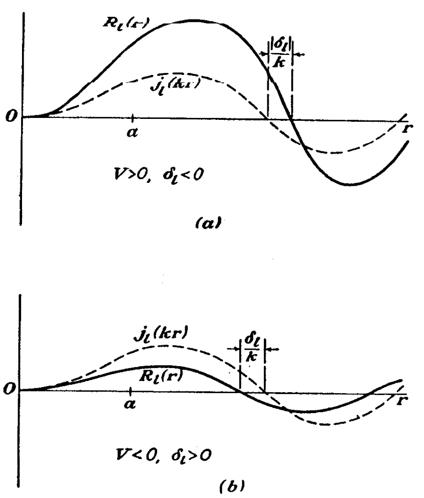

FIG. 18. Schematic plots of the effects of (a) positive (repulsive) potential, and (b) negative (attractive) potential, on the force-free radial wave function $j_l(kr)$; the range of the potential is a in each case. $R_l(r)$ is drawn arbitrarily to start out like $j_l(kr)$ at $r = 0$, and is bent up more rapidly in (a) so that it has a greater amplitude and a retarded phase (pushed out) with respect to $j_l(kr)$. In (b), $R_l(r)$ bends over sooner, and thus has a smaller amplitude than $j_l(kr)$ and an advanced phase (pulled in). The amplitudes have no direct physical significance, whereas the phases determine the scattering. The difference between neighboring nodes of j_l and R_l is not precisely equal to the phase shift divided by k (as indicated) until j_l has gone through several oscillations and attained its asymptotic form.

These conclusions are valid even when l is not large compared to ka and δ_l is not small. This may be seen graphically by comparing $j_l(kr)$ and $R_l(r)$ when they are arbitrarily made to start out in the same way at $r = 0$. Figure 18(a) shows a schematic comparison for positive V, and Fig. 18(b) for negative V.

Ramsauer-Townsend Effect. The construction in Fig. 18(b) suggests that an attractive potential might be strong enough so that one of the radial partial waves is pulled in by just half a cycle and its phase shift is 180°. If this were the case, the corresponding term in the expression (19.11) for $f(\theta)$ would vanish, and there would be no contribution to the scattering. It is clear from the foregoing discussion that the phase shift is largest for $l = 0$. The possibility then arises that ka can be small enough and the attractive potential large enough in magnitude so that $\delta_0 = 180°$ and all other phase shifts are negligibly small. In such a case, the scattered amplitude $f(\theta)$ vanishes for all θ, and there is no scattering.

This is the explanation[1] of the *Ramsauer-Townsend effect*, the extremely low minimum observed in the scattering cross section of electrons by rare-gas atoms at about 0.7 electron-volt bombarding energy.[2] A rare-gas atom, which consists entirely of closed shells, is relatively small,

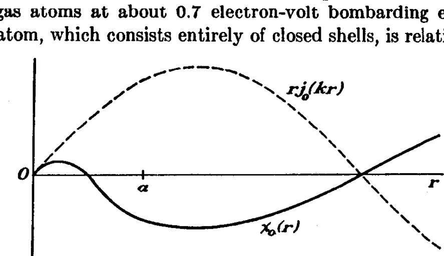

FIG. 19. Schematic plot of the effect of the potential of a rare-gas atom of "radius" a on the $l = 0$ partial wave of an incident electron that has the minimum cross section observed in the Ramsauer-Townsend effect. As in Fig. 18, the actual and force-free wave functions start out in the same way at $r = 0$; the former is "pulled in" by 180° of phase. In an actual case, the quantity ka would be somewhat smaller than is indicated here.

and the combined force of nucleus and atomic electrons exerted on an incident electron is strong and sharply defined as to range. Thus it is reasonable to expect that a situation such as that illustrated in Fig. 19 could occur. Here the partial wave with $l = 0$ has exactly a half cycle more of oscillation inside the atomic potential than the corresponding force-free wave, and the wave length of the electron is large enough in comparison with a so that higher l phase shifts are negligible. It is clear that this minimum cross section will occur at a definite energy, since the shape of the wave function inside the potential is insensitive to the relatively small bombarding energy whereas the phase of the force-free wave function depends rapidly on it.

Physically, the Ramsauer-Townsend effect may be thought of as a diffraction of the electron around the rare-gas atom, in which the wave function inside the atom is distorted in just such a way that it fits on

[1] This explanation, suggested by N. Bohr, was shown to be quantitatively reasonable by H. Faxén and J. Holtsmark, *Zeits. f. Physik*, **45**, 307 (1927).

[2] The experimental results are summarized by R. Kollath, *Phys. Zeits.*, **31**, 985 (1931).

smoothly to an undistorted wave function outside. This effect is analogous to the perfect transmission found at particular energies in the one-dimensional problem considered earlier [see discussion of Eq. (17.5)]. Unlike the situation in one dimension, however, the Ramsauer-Townsend effect cannot occur with a repulsive potential, since ka would have to be at least of order unity to make $\delta_0 = -180°$, and a potential of this large range would produce higher l phase shifts.

Scattering by a Perfectly Rigid Sphere. As a first example of the method of partial waves, we compute the scattering by a perfectly rigid sphere, which is represented by the potential $V(r) = +\infty$ for $r < a$, and $V(r) = 0$ for $r > a$. The solution for $r > a$ is just Eq. (19.7). The boundary condition, obtained in Sec. 8, that $u(a,\theta) = 0$, is equivalent to the requirement that all the radial functions vanish at $r = a$. The phase shifts may then be obtained by setting either $R_l(a)$ given by (19.7) equal to zero, or γ_l in (19.14) equal to infinity:

$$\tan \delta_l = \frac{j_l(ka)}{n_l(ka)} \tag{19.20}$$

The calculation of the scattering is particularly simple in the low-energy limit: $ka = 2\pi a/\lambda \ll 1$. Then substitution of (15.7) into (19.20) gives as an approximation for the phase shifts

$$\tan \delta_l \cong -\frac{(ka)^{2l+1}}{(2l+1)[1 \cdot 1 \cdot 3 \cdot 5 \cdot \cdot \cdot (2l-1)]^2} \tag{19.21}$$

Thus δ_l falls off very rapidly as l increases, in agreement with (19.19). All the phase shifts vanish as $k \to 0$; however, the $l = 0$ partial wave gives a finite contribution to the scattering because of the factor $1/k^2$ that appears in (19.12) and (19.13). We thus obtain

$$\sigma(\theta) \cong a^2, \qquad \sigma \cong 4\pi a^2 \tag{19.22}$$

The scattering is spherically symmetrical, and the total cross section is four times the classical value.

In the high-energy limit ($ka \gg 1$), we might expect to get the classical result, since it is then possible to make wave packets that are small in comparison with the size of the scattering region, and these can follow the classical trajectories without spreading appreciably. This corresponds to the ray limit in the wave theory of light or sound. The differential scattering cross section is rather difficult to find, and we only indicate the computation of the leading term in the total cross section. Substitution of (19.20) into (19.13) gives

$$\sigma = \frac{4\pi}{k^2} \sum_{l=0}^{\infty} \frac{(2l+1) j_l^2(ka)}{j_l^2(ka) + n_l^2(ka)} \tag{19.23}$$

We can make use of asymptotic expansions of Bessel functions that are valid when the argument is large and the order is smaller than, of the order of, and larger than the argument.[1] The calculation shows that most of the contribution to the sum in (19.23) comes from

$$l < (ka) - C(ka)^{\frac{1}{3}},$$

where C is a number of order unity; the leading term here is $\frac{1}{2}(ka)^2$. The other two parts of the sum, for $(ka) - C(ka)^{\frac{1}{3}} < l < (ka) + C(ka)^{\frac{1}{3}}$, and for $l > (ka) + C(ka)^{\frac{1}{3}}$, each contribute terms of order $(ka)^{\frac{4}{3}}$, and hence may be neglected in the high-energy limit. Thus

$$\sigma \cong 2\pi a^2 \tag{19.24}$$

which is twice the classical value.

The reason for the apparently anomalous result (19.24) is that the asymptotic form of the wave function is so set up in Eq. (18.10) that in the classical limit the scattering is counted twice: once in the true scattering (which turns out to be spherically symmetric as it is in the classical problem), and again in the shadow of the scattering sphere that appears in the forward direction, since this shadow is produced by interference between the incident plane wave e^{ikz} and the scattered wave $f(\theta)e^{ikr}/r$ [see also the discussion of Eq. (19.14)]. However, so long as ka is finite, diffraction around the sphere in the forward direction actually takes place, and the total measured cross section (if the measurement can be made so that it includes the strong forward maximum) is approximately $2\pi a^2$.

Scattering by a Square Well Potential. As a second example of the method of partial waves, we consider the somewhat more complicated problem of the scattering from the spherically symmetric square well potential illustrated in Fig. 13 of Sec. 15. The interior ($r < a$) wave function that is finite at $r = 0$ is seen by analogy with Eq. (15.11) to be

$$R_l(r) = B_l j_l(\alpha r), \qquad \alpha = \left[\frac{2\mu(E + V_0)}{\hbar^2}\right]^{\frac{1}{2}} \tag{19.25}$$

Thus the phase shifts are given by Eq. (19.14), where the ratio of slope to value of the lth partial wave at $r = a$ is

$$\gamma_l = \frac{\alpha j_l'(\alpha a)}{j_l(\alpha a)} \tag{19.26}$$

[1] Watson, *op. cit.*, Chap. VIII.