This chapter surveys the theory of weak interactions. It relies heavily on Chapter 7, but see Chapters 8 and 9. Section 4.6 of Chapter 4 would be useful background. I begin by reviewing the Feynman Rules for the coupling of leptons to W°, and treat three classic problems in some detail: the beta decay of the muon, the neutrino, and the charged pion. Next, we consider the coupling of quarks to W°, which brings in the Cabibbo angle, the GIM mechanism, and the Kobayashi–Maskawa matrix. In Section 10.6.1, we state the Feynman rules for coupling quarks and leptons to the Z°, and the final section probably the most difficult in this book, shows how all electromagnetic and weak vertex factors can be derived, in the Glashow–Weinberg–Salam electroweak theory.

# 10.1 Charged Leptonic Weak Interactions

The mediators of weak interactions (analogous to photons in QED and gluons in QCD) are the W°'s (W+ and W−) and the Z°. Unlike the photons and gluons, which are massless, these "intermediate vector bosons" are extremely heavy—by far the heaviest elementary particles yet detected. Experimentally,

\[ M_W = 82 \pm 2 \text{ GeV/c}^2, \quad M_Z = 92 \pm 2 \text{ GeV/c}^2 \]  \(10.1\)

Now, a massive particle of spin 1 has three allowed polarization states (m_\ell = 1, 0, -1), whereas a zero massless particle has only two (if z is in the direction of motion, the "longitudinal" polarization m_\ell = 0 does not occur). Thus for photons and gluons, we imposed both the Lorentz condition

\[ \epsilon_{\mu} p_{\mu} = 0 \]  \(10.2\)

(reducing the number of independent components in \( \epsilon \) from 4 to 3) and also
the Coulomb gauge \( E^a = 0 \), so that \( e \cdot B = 0 \), which reduces it further from 3 to 2. However, for the \( W \)'s and the \( Z \) the Lorentz condition alone exhausts the gauge freedom, and we do not invoke the Coulomb gauge. Moreover, the propagator for massive spin-1 particles is no longer simply \( -i \Sigma_{\mu\nu} q^\mu q^\nu \), but rather,

\[
\frac{-i\Sigma_{\mu\nu} \theta_{\mu\nu}(q^2, M_W^2)}{q^2 - M_W^2}(\text{propagator for } W) \quad (10.3)
\]

where \( M_W, M_Z \) are, as the case may be. In practice, \( q^2 \) is ordinarily so much smaller than \( (M_W^2) \) that we may safely use

\[
\frac{g_{\mu\nu}}{(M_W^2)}(\text{propagator for } q^2 \ll (M_W^2)) \quad (10.4)
\]

However, when a process involves energies that are comparable to \( M_W^2 \) we must, of course, revert to the exact expression.

The theory of "charged" weak interactions (mediated by the \( W \)) is simpler than that for "neutral" ones (mediated by the \( Z \)), so for the moment I shall concentrate on the former. In this section we consider the coupling of \( W \)'s to leptons; in the next section we'll discuss their coupling to quarks and hadrons.

The fundamental leptonic vertex is

\[
\begin{array}{c}
\gamma^a \to W^a \to \ell^a \bar{\ell} \bar{\ell} \end{array}
\]

Here an electron, muon, or tau is converted into the associated neutrinos, with emission of a \( W \) (or absorption of a \( W \)). The reverse process (\( \ell \to \gamma \bar{\ell} \)) is also possible, of course, as well as the "neutral" reactions involving antineutrinos. The Feynman rules are the same as for QED (apart from the modifications already mentioned to accommodate the massive mediator), except for the vertex factor

\[
\frac{ig_\mu}{\sqrt{2} \gamma^a(1 - \gamma^5)} \quad \text{(weak vertex factor)} \quad (10.5)
\]

The various \( 2 \)'s are purely conventional, and \( g_\mu \equiv \sqrt{2} g_\mu \) is the "weak coupling constant" (analogous to \( g \) in QED and \( g_\mu \) in QCD). The factor \((1 - \gamma^5)\), however, is of profound importance. For \( \gamma^5 \) alone would yield a vector coupling (like QED or QCD), whereas \( \gamma^a \gamma^5 \) gives an axial vector [see eq. (7.6)]. A theory that adds

\[\text{**It might be better for the plasma propagator as } M_W = 0.\]

\[\text{**Before is \( \alpha \cdot B \) the plasma limit is non-interchangeable, because in one particular\]

\[\text{**where the number of degrees of freedom (i.e., the number of\]

\[\text{**the number of allowed spin orientations) drops sharply from } 2 \left( \text{for } M_W \right) \text{ to } 1 \text{ (for } M_W = 0).\]
10.2 DECAY OF THE MUON

The decay of the muon into three leptons is a weak interaction process. The Feynman diagram for this process is:

\[
\begin{align*}
\Gamma(p_1, p_2, p_3) &= \frac{G_F^2}{8\pi M_{\mu}} \left[ (1 - \gamma^2) \sigma_1(1 - \gamma^2) \sigma_2(1 - \gamma^2) \sigma_3(1 - \gamma^2) \sigma_4(1 - \gamma^2) \right] \\
\end{align*}
\]

where \(G_F\) is the Fermi constant, and \(M_{\mu}\) is the mass of the muon. The total decay rate is given by:

\[
\Gamma = \frac{1}{2\pi} \int d\phi \frac{d\sigma}{d\phi} d\Omega
\]

where \(d\sigma/d\phi\) is the differential cross section and \(d\Omega\) is the solid angle element. The cross section for this process is:

\[
\frac{d\sigma}{d\phi} = \frac{G_F^2}{8\pi M_{\mu}^2} \left( m_{\mu}^2 \right)^2
\]

The decay width is given by:

\[
\Gamma = \frac{1}{2\pi} \int d\phi \frac{d\sigma}{d\phi} d\Omega
\]

where \(d\sigma/d\phi\) is the differential cross section and \(d\Omega\) is the solid angle element. The cross section for this process is:

\[
\frac{d\sigma}{d\phi} = \frac{G_F^2}{8\pi M_{\mu}^2} \left( m_{\mu}^2 \right)^2
\]

The total decay rate is given by:

\[
\Gamma = \frac{1}{2\pi} \int d\phi \frac{d\sigma}{d\phi} d\Omega
\]

where \(d\sigma/d\phi\) is the differential cross section and \(d\Omega\) is the solid angle element. The cross section for this process is:

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\[
\frac{d\sigma}{d\phi} = \frac{G_F^2}{8\pi M_{\mu}^2} \left( m_{\mu}^2 \right)^2
\]
so that
\[
dx = \frac{E_0 E_x \sin \theta \, dt}{c E_0}
\]

Then
\[
\int \sin \theta \, \frac{dt}{E_0} \left( \frac{m c^2 - E_0}{c E_0} \right)
= \frac{c}{E_0} \int \frac{dx}{E_0} = \frac{1}{c} \int \left( \frac{m c^2 - E_0}{c E_0} \right) dx
= \begin{cases} 
\frac{c}{E_0} x, & \text{if } x < \left( \frac{m c^2 - E_0}{c E_0} \right) - x_0 \\
0, & \text{otherwise}
\end{cases}
\]

where
\[
x_0 = \frac{1}{c} \sqrt{E_0^2 + E_0^2 \pm 2 E_0 E_0} = \frac{1}{c} E_0 \mp E_0
\]

The inequality in equation (10.28) can be expressed more neatly:
\[
|E_0 - E_d| < \left( \frac{m c^2}{c E_0} \right) < E_0 + E_0
\]

or, adding \((E_0 + E_0)\) and dividing through by \(2\):
\[
\frac{1}{2} (|E_0 - E_d| + E_0 + E_0) < \left( \frac{m c^2}{c E_0} \right) (10.30)
\]

The term on the left is simply the larger of \(E_0\) and \(x_0\); the other one is necessarily even smaller, so expressions (10.31) is equivalent to these inequalities:
\[
\begin{cases} 
E_0 < m c^2 \\
E_0 > m c^2
\end{cases}
\]

or equivalently,
\[
\frac{1}{E_0} > m c^2
\]

[These constraints make good sense kinematically. Particle 2, for example, gets the maximum possible energy when 3 and 4 encroach diametrically opposite to 1.

In case 2 picks up half the available energy \((m c^2)\), while 3 and 4 share the other half. If there is a nonzero angle between 3 and 4, 2 gets less, and 3 plus 4 get correspondingly more. Thus \(m c^2\) is the maximum energy for any individual outgoing particle, and the minimum total for any pair.]

The inequalities (10.32) specify the limits on the \(E_0\) and \(E_0\) integrals. \(E_0\) runs from \(m c^2 - E_0\) up to \(m c^2\), and \(E_0\) will go from 0 to \(m c^2\). The \(\theta\) and \(\phi\) integrals leave us with
\[
d\Gamma = \frac{\left( \frac{m c^2}{E_0} \right)^2}{(4\pi)^2} \, dE_0 \, dE_0 \, d\theta \, d\phi
\]

Putting in equation (10.20) and carrying out the \(E_0\) integral, we have
\[
d\Gamma = \frac{m c^2 \, \frac{\partial\phi}{\partial E}}{E_0} \, d\phi
\]
10.3 DECAY OF THE NEUTRON

If we put in the observed muon lifetime and mass, we find

\[ G_{\mu}(M_{\mu}) = \left( \frac{E_\gamma}{3M_{\mu}c^2} \right) = 1.166 \times 10^{-3} \text{GeV}^2 \]  

(10.40)

The corresponding value of \( g_\omega \) (less accurately known, at present, because of the experimental uncertainty in \( M_{\omega} \)) is

\[ g_\omega = 0.66 \]  

(10.41)

and hence the "weak fine structure constant" is

\[ a_w = \frac{\alpha}{4\pi} = \frac{1}{29} \]  

(10.42)

This number should come as something of a shock: it is larger than the electromagnetic fine structure constant \( (\alpha) \), by a factor of nearly 6. Weak interactions are feeble not because the intrinsic coupling is small (it isn’t), but because the mediators are so massive—so, more precisely, because we typically work at energies so far below the W mass that the denominator in the propagator \( (q^2 - M_{\omega}c^2) \) is extremely large. New machines are presently under construction that will run at energies close to \( M_{\omega}c^2 \), and in this regime the "weak" interactions will far surpass the electromagnetic ones in strength.

10.3 DECAY OF THE NEUTRON

The success of the muon decay formula (10.35) encourages us to apply the same methods to the decay of the neutron, \( n \to p + e^- + \bar{\nu}_e \). Of course, the neutron and proton are composite particles, but just as the Gott and Rutherford cross sections (which treat the proton as an elementary "nucleon" particle) give a good account of low-energy electron-proton scattering, so we might hope that the diagram

![Diagram](image)

(the same as for muon decay, only with \( n \to p + \pi^0 \) in place of \( n \to x_3 + \pi^0 \)) will afford a reasonable approximation to neutron beta decay. From a calculation of this type, the only new feature is that \( N \) is now a massive particle (a proton, instead of a neutrino). As it happens (Problem 10.4) this does not change the amplitude:

\[ \langle |\mathcal{M}|^2 \rangle = \frac{2}{M_{\omega}c^2} \left( p_1 \cdot p_2 \right) \]  

(10.43)