Chapter 10

Weak Interactions

This chapter surveys the theory of weak interactions. It relies heavily on Chapter 7, but not on Chapters 8 and 9; Section 4.6 of Chapter 4 would be useful background. I begin by stating the Feynman Rules for the coupling of leptons to $W^{\pm}$, and treat three classic problems in some detail: the beta decays of the muon, the neutron, and the charged pion. Next, we consider the coupling of quarks to $W^{\pm}$, which brings in the Cabibbo angle, the GIM mechanism, and the Kobayashi–Maskawa matrix. In Section 10.6 I state the Feynman rules for coupling quarks and leptons to the Z^0, and the final section (probably the most difficult in this book) shows how all electromagnetic and weak vertex factors can be derived, in the Glashow–Weinberg–Salam electroweak theory.

10.1 CHARGED LEPTONIC WEAK INTERACTIONS

The mediators of weak interactions (analogous to photons in QED and gluons in QCD) are the W's (W^+ and W^-) and the Z^0. Unlike the photon and gluons, which are massless, these "intermediate vector bosons" are extremely heavy—by far the heaviest elementary particles yet detected. Experimentally,

$$M_W = 82 \pm 2 \text{ GeV}/c^2, \qquad M_Z = 92 \pm 2 \text{ GeV}/c^2 \qquad (10.1)$$

Now, a *massive* particle of spin 1 has *three* allowed polarization states ($m_s = 1, 0, -1$), whereas a free *massless* particle has only *two* (if z is the direction of motion, the "longitudinal" polarization $m_s = 0$ does not occur). Thus for photons and gluons, we imposed both the Lorentz condition

$$\epsilon^{\mu} p_{\mu} = 0 \qquad (10.2)$$

(reducing the number of independent components in ϵ^{μ} from 4 to 3) and also the Coulomb gauge ($\epsilon^0 = 0$, so that $\epsilon \cdot \mathbf{p} = 0$, which reduces it further from 3 to 2). However, for the W's and the Z the Lorentz condition alone exhausts the gauge freedom, and we do *not* invoke the Coulomb gauge. Moreover, the propagator for massive spin-1 particles is no longer simply $-ig_{\mu\nu}/q^2$, but rather,*

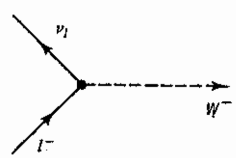
$$\frac{-i(g_{\mu\nu} - q_\mu q_\nu/M^2c^2)}{q^2 - M^2c^2} \qquad \text{(propagator for } W \text{ and } Z) \qquad (10.3)$$

where M is M_W or M_Z, as the case may be. In practice, q^2 is ordinarily so much smaller than $(Mc)^2$ that we may safely use

$$\frac{ig_{\mu\nu}}{(Mc)^2} \qquad \text{(propagator for } q^2 \ll (Mc)^2) \qquad (10.4)$$

However, when a process involves energies that are comparable to Mc^2 we must, of course, revert to the exact expression.

The theory of "charged" weak interactions (mediated by the W's) is simpler than that for "neutral" ones (mediated by the Z), so for the moment I shall concentrate on the former. In this section we consider the coupling of W's to *leptons;* in the next section we'll discuss their coupling to quarks and hadrons. The fundamental leptonic vertex is



Here an electron, muon, or tau is converted into the associated neutrino, with emission of a W^- (or absorption of W^+). The reverse process ($\nu_l \to l^- + W^+$) is also possible, of course, as well as the "crossed" reactions involving antileptons. The Feynman rules are the same as for QED (apart from the modifications already mentioned to accommodate the massive mediator), except for the vertex factor

$$\frac{-ig_w}{2\sqrt{2}}\gamma^\mu(1-\gamma^5) \qquad \text{(weak vertex factor)} \qquad (10.5)$$

The various 2's are purely conventional, and $g_w = \sqrt{4\pi\alpha_w}$ is the "weak coupling constant" (analogous to g_e in QED and g_s in QCD). The factor $(1-\gamma^5)$, however, is of profound importance, for γ^μ alone would yield a *vector* coupling (like QED or QCD), whereas $\gamma^\mu\gamma^5$ gives an *axial* vector [see eq. (7.68)]. A theory that adds

* It might bother you that this does not reduce to the photon propagator as $M \to 0$. For particles of spin 1 (or higher) the massless limit is notoriously treacherous, because in one critical respect it is not a continuous procedure. The number of degrees of freedom (that is, the number of allowed spin orientations) drops abruptly from $2s+1$ (for $M \neq 0$) to 2 (for $M = 0$). There are ways of formulating the theory that allow a smooth transition to $M = 0$, but only at the cost of introducing spurious nonphysical states.

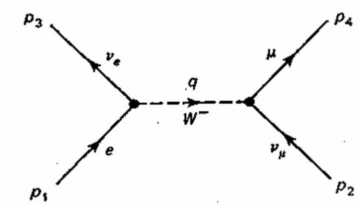
a *vector* to an *axial* vector is bound to violate the conservation of parity, and this is precisely what happens in the weak interactions (Chap. 4, Sect. 4.6).*

EXAMPLE 10.1 Inverse Muon Decay

Consider the process

$$\nu_\mu + e^- \to \mu^- + \nu_e$$

represented (in lowest order) by the diagram



Here $q = p_1 - p_3$, and for any experiment likely in the near future $q^2 \ll M_W^2c^2$, so we can safely use the simplified propagator (10.4), and the amplitude is

$$\mathcal{M} = \frac{g_w^2}{8(M_Wc)^2}[\bar{u}(3)\gamma^\mu(1-\gamma^5)u(1)][\bar{u}(4)\gamma_\mu(1-\gamma^5)u(2)] \qquad (10.6)$$

Applying Casimir's trick (7.123), we find

$$\sum_{\text{spins}} |\mathcal{M}|^2 = \left(\frac{g_w^2}{8(M_Wc)^2}\right)^2 Tr[\gamma^\mu(1-\gamma^5)(\not{p}_1 + m_ec)\gamma^\nu(1-\gamma^5)\not{p}_3]$$
$$\times\, Tr[\gamma_\mu(1-\gamma^5)\not{p}_2\gamma_\nu(1-\gamma^5)(\not{p}_4 + m_\mu c)] \qquad (10.7)$$

The trace theorems of Chapter 7, Section 7.7, yield

$$8[p_1^\mu p_3^\nu + p_1^\nu p_3^\mu - g^{\mu\nu}(p_1 \cdot p_3) - i\epsilon^{\mu\nu\lambda\sigma}p_{1\lambda}p_{3\sigma}] \qquad (10.8)$$

for the first trace, and

$$8[p_{2\mu}p_{4\nu} + p_{2\nu}p_{4\mu} - g_{\mu\nu}(p_2 \cdot p_4) - i\epsilon_{\mu\nu\kappa\tau}p_2^\kappa p_4^\tau] \qquad (10.9)$$

for the second. It follows that†

$$\sum_{\text{spins}} |\mathcal{M}|^2 = 4\left(\frac{g_w}{M_Wc}\right)^4 (p_1 \cdot p_2)(p_3 \cdot p_4) \qquad (10.10)$$

Actually, we want the sum over final spins but the *average* over initial

* In fact, the violation is "maximal," in the sense that the two terms are equally large. When parity violation was first considered, a factor of the form $(1 + \epsilon\gamma^5)$ was used, but experiments soon dictated that $\epsilon = -1$. (See Problem 10.1.) We call it a "V–A" ("vector minus axial vector") coupling. Fermi's original theory of beta decay was a pure *vector* theory (like QED), and although others proposed scalar, pseudoscalar, tensor, or pure axial couplings, it was not until 1956 that anyone seriously contemplated *mixing* terms of different parity.

† Note that $\epsilon^{\mu\nu\lambda\sigma}\epsilon_{\mu\nu\kappa\tau} = -2(\delta^\lambda_\kappa\delta^\sigma_\tau - \delta^\lambda_\tau\delta^\sigma_\kappa)$. (See Problem 7.33.)

spins. The electron has *two* spin states, but the neutrino (as we learned in Chapter 4, Section 4.6) has only *one* (if you like, the incident neutrinos are *always* polarized, since they only come "left-handed"). So

$$\langle|\mathcal{M}|^2\rangle = 2\left(\frac{g_w}{M_W c}\right)^4 (p_1 \cdot p_2)(p_3 \cdot p_4) \qquad (10.11)$$

If we now go to the *CM* frame, and neglect the mass of the electron

$$\langle|\mathcal{M}|^2\rangle = 8\left(\frac{g_w E}{M_W c^2}\right)^4 \left\{1 - \left(\frac{m_\mu c^2}{2E}\right)^2\right\} \qquad (10.12)$$

where E is the incident electron (or neutrino) energy. The differential scattering cross section [eq. (6.42)] is isotropic (all scattering angles equally likely)

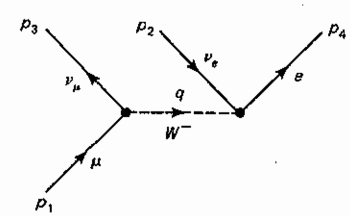
$$\frac{d\sigma}{d\Omega} = \frac{1}{2}\left(\frac{\hbar c g_w^2 E}{4\pi (M_W c^2)^2}\right)^2 \left\{1 - \left(\frac{m_\mu c^2}{2E}\right)^2\right\}^2 \qquad (10.13)$$

and the total cross section is

$$\sigma = \frac{1}{8\pi}\left[\left(\frac{g_w}{M_W c^2}\right)^2 \hbar c E\right]^2 \left\{1 - \left(\frac{m_\mu c^2}{2E}\right)^2\right\}^2 \qquad (10.14)$$

10.2 DECAY OF THE MUON

Electron-neutrino scattering is not the easiest thing in the world to study experimentally, but the closely related process, muon decay ($\mu \to e + \nu_\mu + \bar{\nu}_e$), is the cleanest of all weak interaction phenomena, theoretically and experimentally. The Feynman diagram



leads to the amplitude

$$\mathcal{M} = \frac{g_w^2}{8(M_W c)^2}[\bar{u}(3)\gamma^\mu(1-\gamma^5)u(1)][\bar{u}(4)\gamma_\mu(1-\gamma^5)v(2)] \qquad (10.15)$$

from which we obtain, as before,

$$\langle|\mathcal{M}|^2\rangle = 2\left(\frac{g_w}{M_W c}\right)^4 (p_1 \cdot p_2)(p_3 \cdot p_4) \qquad (10.16)$$

In the muon rest frame, $p_1 = (m_\mu c, \mathbf{0})$, we have

$$p_1 \cdot p_2 = m_\mu E_2 \qquad (10.17)$$

and since $p_1 = p_2 + p_3 + p_4$

$$(p_3 + p_4)^2 = p_3^2 + p_4^2 + 2p_3 \cdot p_4 = m_e^2 c^2 + 2p_3 \cdot p_4$$
$$= (p_1 - p_2)^2 = p_1^2 + p_2^2 - 2p_1 \cdot p_2 = m_\mu^2 c^2 - 2p_1 \cdot p_2 \qquad (10.18)$$

from which it follows that

$$p_3 \cdot p_4 = \frac{(m_\mu^2 - m_e^2)c^2}{2} - m_\mu E_2 \qquad (10.19)$$

The algebra will be simpler later on, at no significant cost in accuracy, if we set $m_e = 0$, so that

$$\langle|\mathcal{M}|^2\rangle = \left(\frac{g_w}{M_w c}\right)^4 m_\mu^2 E_2 (m_\mu c^2 - 2E_2) \qquad (10.20)$$

Now, the decay rate is given by the Golden Rule (6.15):*

$$d\Gamma = \frac{\langle|\mathcal{M}|^2\rangle}{2\hbar m_\mu}\left(\frac{c\, d^3\mathbf{p}_2}{(2\pi)^3 2E_2}\right)\left(\frac{c\, d^3\mathbf{p}_3}{(2\pi)^3 2E_3}\right)\left(\frac{c\, d^3\mathbf{p}_4}{(2\pi)^3 2E_4}\right)(2\pi)^4 \delta^4(p_1 - p_2 - p_3 - p_4) \qquad (10.21)$$

where $E_2 = |\mathbf{p}_2|c$, $E_3 = |\mathbf{p}_3|c$, and $E_4 = |\mathbf{p}_4|c$. To begin with, we peel apart the delta function:

$$\delta^4(p_1 - p_2 - p_3 - p_4) = \delta\left(m_\mu c - \frac{E_2}{c} - \frac{E_3}{c} - \frac{E_4}{c}\right)\delta^3(\mathbf{p}_2 + \mathbf{p}_3 + \mathbf{p}_4) \qquad (10.22)$$

and perform the $\mathbf{p}_3$ integral:

$$d\Gamma = \frac{\langle|\mathcal{M}|^2\rangle c^3}{16(2\pi)^5 \hbar m_\mu}\frac{(d^3\mathbf{p}_2)(d^3\mathbf{p}_4)}{E_2 E_3 E_4}\delta\left(m_\mu c - \frac{E_2}{c} - \frac{E_3}{c} - \frac{E_4}{c}\right) \qquad (10.23)$$

where E_3 now stands for $|\mathbf{p}_2 + \mathbf{p}_4|c$. Next we'll do the $\mathbf{p}_2$ integral. Setting the polar axis along $\mathbf{p}_4$ (which is *fixed*, for the purposes of the $\mathbf{p}_2$ integration), we have

$$\left(\frac{E_3}{c}\right)^2 = |\mathbf{p}_2 + \mathbf{p}_4|^2 = \mathbf{p}_2^2 + \mathbf{p}_4^2 + 2\mathbf{p}_2 \cdot \mathbf{p}_4$$
$$= \frac{1}{c^2}(E_2^2 + E_4^2 + 2E_2 E_4 \cos\theta) \qquad (10.24)$$

and

$$d^3\mathbf{p}_2 = \left(\frac{E_2}{c}\right)^2 \frac{dE_2}{c}\sin\theta\, d\theta\, d\phi \qquad (10.25)$$

The ϕ integral is trivial ($\int d\phi = 2\pi$); to carry out the θ integration, let

$$x \equiv \frac{1}{c}\sqrt{E_2^2 + E_4^2 + 2E_2 E_4 \cos\theta} = \frac{E_3}{c} \qquad (10.26)$$

* Note that this is a *three* body decay, so we have to go all the way back to the Golden Rule.

so that

$$dx = -\frac{E_2 E_4 \sin\theta\, d\theta}{cE_3} \tag{10.27}$$

Then

$$\int_0^\pi \frac{\sin\theta\, d\theta}{E_3}\,\delta\!\left(m_\mu c - \frac{E_2}{c} - \frac{E_3}{c} - \frac{E_4}{c}\right)$$
$$= \frac{c}{E_2E_4}\int_{x_-}^{x_+}\delta\!\left(m_\mu c - x - \frac{E_2}{c} - \frac{E_4}{c}\right)dx$$
$$= \begin{cases} \dfrac{c}{E_2E_4}, & \text{if } x_- < \left(m_\mu c - \dfrac{E_2}{c} - \dfrac{E_4}{c}\right) < x_+ \\ 0, & \text{otherwise} \end{cases} \tag{10.28}$$

where

$$x_\pm \equiv \frac{1}{c}\sqrt{E_2^2 + E_4^2 \pm 2E_2E_4} = \frac{1}{c}\,|E_2 \pm E_4| \tag{10.29}$$

The inequality in equation (10.28) can be expressed more neatly:

$$|E_2 - E_4| < (m_\mu c^2 - E_2 - E_4) < E_2 + E_4 \tag{10.30}$$

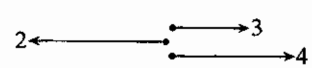
or, adding $(E_2 + E_4)$ and dividing through by 2:

$$\tfrac{1}{2}\{|E_2 - E_4| + E_2 + E_4\} < \tfrac{1}{2}m_\mu c^2 < (E_2 + E_4) \tag{10.31}$$

The term on the left is simply the larger of E_2 and E_4; the other one is necessarily even smaller, so expression (10.31) is equivalent to *three* inequalities:

$$\left\{\begin{array}{r} E_2 < \tfrac{1}{2}m_\mu c^2 \\ E_4 < \tfrac{1}{2}m_\mu c^2 \\ (E_2 + E_4) > \tfrac{1}{2}m_\mu c^2 \end{array}\right\} \tag{10.32}$$

[These constraints make good sense kinematically: Particle 2, for example, gets the maximum possible energy when 3 and 4 emerge diametrically opposite to it:



In this case 2 picks up half the available energy ($\tfrac{1}{2}m_\mu c^2$), while 3 and 4 share the other half. If there is a nonzero angle between 3 and 4, 2 gets less, and 3 plus 4 get correspondingly more. Thus $\tfrac{1}{2}m_\mu c^2$ is the maximum energy for any *individual* outgoing particle, and the *minimum* total for any *pair*.]

The inequalities (10.32) specify the limits on the E_2 and E_4 integrals: E_2 runs from $\tfrac{1}{2}m_\mu c^2 - E_4$ up to $\tfrac{1}{2}m_\mu c^2$, and E_4 will go from 0 to $\tfrac{1}{2}m_\mu c^2$. The θ and ϕ integrals leave us with

$$d\Gamma = \frac{\langle|\mathcal{M}|^2\rangle c}{(4\pi)^4\hbar m_\mu}\,dE_2\,\frac{d^3\mathbf{p}_4}{E_4^2} \tag{10.33}$$

Putting in equation (10.20) and carrying out the E_2 integral, we have

$$d\Gamma = \left(\frac{g_w}{4\pi M_W c}\right)^4 \frac{m_\mu c}{\hbar}\frac{d^3\mathbf{p}_4}{E_4^2}\int_{1/2m_\mu c^2 - E_4}^{1/2m_\mu c^2} E_2(m_\mu c^2 - 2E_2)\,dE_2$$
$$= \left(\frac{g_w}{4\pi M_W c}\right)^4 \frac{m_\mu c}{\hbar}\left(\frac{m_\mu c^2}{2} - \frac{2}{3}E_4\right)d^3\mathbf{p}_4 \tag{10.34}$$

Finally, writing

$$d^3\mathbf{p}_4 = 4\pi\left(\frac{E_4}{c}\right)^2\frac{dE_4}{c}$$

and dropping the subscript ($E \equiv E_4$ is the electron energy), we obtain

$$\frac{d\Gamma}{dE} = \left(\frac{g_w}{M_W c}\right)^4 \frac{m_\mu^2 E^2}{2\hbar(4\pi)^3}\left(1 - \frac{4E}{3m_\mu c^2}\right) \tag{10.35}$$

This tells us the energy distribution of the electrons emitted in muon decay. It fits the experimental spectrum perfectly (Fig. 10.1). The total decay rate is

$$\Gamma = \left(\frac{g_w}{M_W c}\right)^4 \frac{m_\mu^2}{2\hbar(4\pi)^3}\int_0^{1/2m_\mu c^2} E^2\left(1 - \frac{4E}{3m_\mu c^2}\right)dE$$
$$= \left(\frac{m_\mu g_w}{M_W}\right)^4 \frac{m_\mu c^2}{12\hbar(8\pi)^3} \tag{10.36}$$

and hence the lifetime of the muon is

$$\tau = \frac{1}{\Gamma} = \left(\frac{M_W}{m_\mu g_w}\right)^4 \frac{12\hbar(8\pi)^3}{m_\mu c^2} \tag{10.37}$$

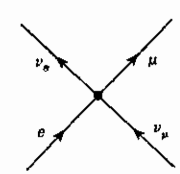
Notice that g_w and M_W do not appear *separately*, either in the muon lifetime formula or in the electron-neutrino scattering cross section; only their *ratio* occurs. It is traditional, in fact, to express weak interaction formulas in terms of the "Fermi coupling constant"

$$G_F \equiv \frac{\sqrt{2}}{8}\left(\frac{g_w}{M_W c^2}\right)^2(\hbar c)^3 \tag{10.38}$$

Thus the muon lifetime is written

$$\tau = \frac{192\pi^3\hbar^7}{G_F^2 m_\mu^5 c^4} \tag{10.39}$$

In Fermi's original theory of beta decay (1933) there *was* no *W*; the interaction was supposed to be a direct four-particle coupling, represented in the Feynman language by a diagram of the form



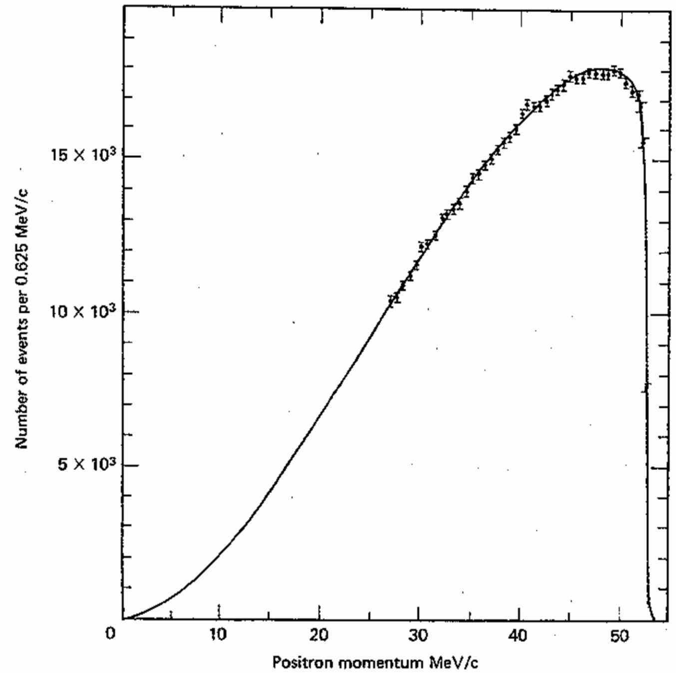
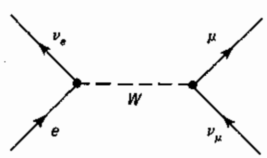


Figure 10.1 Experimental spectrum of positrons in $\mu^+ \to e^+ + \nu_e + \nu_\mu$. The solid line is the theoretically predicted spectrum based on equation (10.35), corrected for electromagnetic effects. (*Source:* M. Bardon et al., *Phys. Rev. Lett.* **14**, 449 (1965).)

From the modern perspective, Fermi's theory combined the W propagator with the two vertex factors, in the diagram



to make an effective four-particle coupling constant G_F. It worked, but only because the W is so heavy that expression (10.4) is a good approximation to the true propagator (10.3),* and in fact it was recognized already in the fifties that Fermi's theory could not be valid at high energies. The idea of a weak mediator (analogous to the photon) was suggested by O. Klein as far back as 1938.

* Fermi also thought the coupling was pure vector, as I mentioned in the footnote (*) on p. 303. Despite these defects (for which Fermi could scarcely be blamed; after all, he invented the theory at a time when the neutrino was a wild speculation and the Dirac equation itself was quite new) Fermi's theory was astonishingly prescient, and all subsequent developments have been relatively small adjustments to it.

If we put in the observed muon lifetime and mass, we find that

$$G_F/(\hbar c)^3 = \frac{\sqrt{2}}{8}\left(\frac{g_w}{M_W c^2}\right)^2 = 1.166 \times 10^{-5}/\text{GeV}^2 \qquad (10.40)$$

The corresponding value of g_w (less accurately known, at present, because of the experimental uncertainty in M_W) is

$$g_w = 0.66 \qquad (10.41)$$

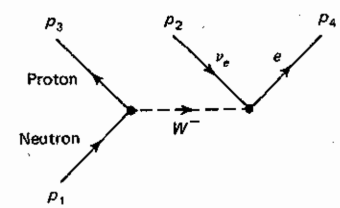
and hence the "weak fine structure constant" is

$$\alpha_w = \frac{g_w^2}{4\pi} = \frac{1}{29} \qquad (10.42)$$

This number should come as something of a shock: It is *larger* than the electromagnetic fine structure constant ($\alpha = \frac{1}{137}$), by a factor of nearly five! Weak interactions are feeble *not* because the *intrinsic* coupling is small (it *isn't*), but because the *mediators* are so *massive*—or, more precisely, because we typically work at energies so far below the W mass that the denominator in the propagator $(q^2 - M_W^2 c^2)$ is extremely large. New machines are presently under construction that will run at energies close to $M_W c^2$, and in this regime the "weak" interactions will far surpass the electromagnetic ones in strength.

10.3 DECAY OF THE NEUTRON

The success of the muon decay formula (10.35) encourages us to apply the same methods to the decay of the neutron, $n \to p + e + \bar{\nu}_e$. Of course, the neutron and proton are composite particles, but just as the Mott and Rutherford cross sections (which treat the proton as an elementary "Dirac" particle) give a good account of low-energy electron-proton scattering, so we might hope that the diagram



(the same as for muon decay, only with $n \to p + W^-$ in place of $\mu \to \nu_\mu + W^-$) will afford a reasonable approximation to neutron beta decay. From a calculational point of view the only new feature is that 3 is now a massive particle (a proton, instead of a neutrino). As it happens (Problem 10.4) this does not change the amplitude:

$$\langle |\mathcal{M}|^2 \rangle = 2\left(\frac{g_w}{M_W c}\right)^4 (p_1 \cdot p_2)(p_3 \cdot p_4) \qquad (10.43)$$