CHAPTER

# 2

# Theory of Electromagnetic Interactions

**2.1.** ***General remarks.*** Theoretical physicists have not yet succeeded in their attempts to formulate the principles of quantum electrodynamics in a completely general manner, free from internal contradictions. They have, however, established a formalism that answers unambiguously most problems arising in the study of electromagnetic interactions between radiation and matter. Whenever the theoretical predictions have been submitted to experimental test, they have been found to be accurate, within the limits of the experimental errors and the mathematical approximations made in the development of the theory. Confidence in the theory of electromagnetic interactions has grown to the point where one may grant its validity beyond the limits of experimental accuracy and perhaps even apply it to fields where experimental tests are still lacking. In the past, study of high-energy phenomena, cosmic rays in particular, was mainly a means for testing the theory of electromagnetic interactions. Today, however, one may justifiably use the results of this theory as a basis for the interpretation of the observed phenomena.

A rigorous derivation of the theoretical formulae lies beyond the scope of the present volume. In many cases, however, we shall try to justify these formulae by means of semi-quantitative derivations based largely on classical models. This procedure provides a physical interpretation for the laws expressed by the theoretical formulae and thus develops an intuitive "feeling" for the phenomena associated with the passage of high-energy particles through matter. We believe that this purpose is important, because one must often rely on such intuition to grasp the significance of a set of experimental data or to devise new methods for the solution of a given problem.

In the study of electromagnetic interactions we encounter two different kinds of entities: *electromagnetic fields* and *particles*. The classical Maxwell theory, leading to the concept of *electromagnetic waves*, fully describes the macroscopic *electromagnetic field*. In the microscopic realm, however, the





field obeys quantum laws whose significance, in certain cases, we may regard as intuitive by thinking of the electromagnetic field as a flux of *photons*. The *particles*, e.g., *electrons*, *mesons*, *protons*, are both the sources of the electromagnetic field and the recipients of its effects. Most of these particles appear in a dual capacity, namely, as radiation quanta and as constituents of matter. Their electromagnetic properties depend on their electric charges and magnetic moments. Their mechanical properties depend on their masses and their spins.

In the rigorous sense, we should always treat the interactions between two particles in terms of the electromagnetic fields set up by the particles and the effects of these fields on the particles themselves. This remark applies to both classical and quantum electrodynamics. If we look at the corpuscular aspect of the electromagnetic field, we may say that electromagnetic interactions should always be described as processes of photon emission and absorption. However, many cases arise in classical electrodynamics where one can calculate the interaction of two charged bodies in terms of the relatively simple Coulomb forces acting between their charges, rather than in terms of the more general electromagnetic field. Correspondingly there occur problems of quantum electrodynamics wherein one can neglect emission or absorption of photons and describe the electromagnetic interactions between particles by means of suitable fields of force. In fact, even when photons are specifically included in the formulation of the problem, one generally proceeds by first computing the mechanical behavior of the particles concerned without reference to emission or absorption of photons, and later introducing radiation phenomena as a perturbation.

With the above considerations in mind, we now proceed to a classification of the elementary electromagnetic phenomena that are of importance in the interactions with matter of high-energy radiation quanta.

Consider first the various phenomena that take place when a charged particle passes in the neighborhood of an atom.

If the distance of closest approach is large compared with the dimensions of the atom, the atom reacts as a whole to the variable field set up by the passing particle. The result is an *excitation* or an *ionization* of the atom. We can treat the phenomenon by the ordinary methods of quantum mechanics without direct reference to radiation. For these comparatively distant collisions, the magnetic moment of the particle is of secondary importance, because the forces associated with the magnetic moment decrease as the third power of the distance, whereas the Coulomb forces decrease as the square of the distance. Therefore we can consider the passing particle as a point charge.

If the distance of closest approach is of the order of the atomic dimensions, the interaction no longer involves the passing particle and the atom as a whole, but rather the passing particle and one of the atomic electrons.

As a consequence of the interaction, the electron is ejected from the atom with considerable energy. This phenomenon is often described as a knock-on process. If the energy acquired by the secondary electron is large compared with the binding energy, the phenomenon can be treated as an interaction between the passing particle and a free electron. Radiation phenomena can still be neglected, and the ordinary methods of quantum mechanics can be used. However, one can no longer neglect the magnetic moments or spins of the interacting particles. When the particles are identical (e.g., electron-electron collisions), exchange phenomena occur and acquire special importance when the minimum distance of approach becomes comparable with the deBroglie wavelength. The phenomena described above will be referred to as "non-radiative collision processes" or, more simply, "collision processes."

When the distance of closest approach becomes smaller than the atomic radius, the deflection of the trajectory of the passing particle in the electric field of the nucleus becomes the most important effect. Classically each deflection results in the emission of a weak electromagnetic radiation with a continuous frequency spectrum. Quantum-theoretically, a number of "soft" quanta, whose total energy is usually a very small fraction of the particle energy, accompany the deflection. In a few cases, however, one photon of energy comparable with that of the particle is emitted. Because of the comparatively small probability of this effect, we can treat the problem of the *scattering* of particles separately from that of *radiation* (or *bremsstrahlung*).

We treat the problem of scattering as a purely mechanical one, according to the methods of quantum mechanics. In this problem, we replace the actual atom by a fictitious, spherically symmetrical field of force, which coincides with the Coulomb field of the nucleus at small distances from the center of the atom, and falls off more rapidly than a Coulomb field at larger distances because of the partial shielding of the electric field of the nucleus by the planetary electrons.

The problem of computing the probability of photon emission by the passage of a charged particle through an atom (radiation probability) requires the application of quantum electrodynamics. As in the scattering problem, we still represent the atom schematically by a central field of force. However, the Hamiltonian of the system, which in the case of the scattering problem consisted of the Hamiltonian of the particle exclusively, now contains also the Hamiltonian of the electromagnetic field and a small interaction term that depends on the coordinates of both the particle and the field. This interaction term produces transitions corresponding to energy transfers between the particle and the electromagnetic field. As mentioned above, the probabilities of these transitions may be computed by the perturbation method.



If we now turn our attention to the interactions of photons with matter, we may again distinguish three cases, namely: interaction of a photon with an atom as a whole, interaction of a photon with a free electron, and interaction of a photon with the Coulomb field of the nucleus.

The interaction of a photon with an atom as a whole leads to the *photoelectric effect*. The importance of this effect in the field of high energies is negligible, so that we need not consider it in detail. The interaction of a photon with a free electron leads to the *Compton effect*. In this phenomenon the photon transfers part of its energy and momentum to the electron initially at rest. The interaction of a photon with the Coulomb field of the nucleus leads to the phenomenon of *pair production*, whereby the photon disappears and a positive and a negative electron simultaneously come into existence. For this phenomenon to occur, the energy of the photon must exceed the rest energy of the two electrons. The excess energy appears almost completely as kinetic energy of the two electrons, while the recoil of the nucleus takes care of the momentum balance.

Both Compton effect and pair production are typical quantum phenomena without classical counterpart. Their description requires the use of quantum electrodynamics along with quantum mechanics. In addition to the pair production of electrons one may envisage the possibility of pair production of heavier particles, for instance, $\mu$-mesons. The existence of this phenomenon has not yet been established experimentally, although it appears likely on theoretical grounds.

2.2. ***Application of the conservation laws to the collision of a particle with a free electron.*** As indicated above, a close collision between a charged particle and an atomic electron is not essentially different from a collision between a charged particle and a free electron. The application of the principles of conservation of energy and momentum leads to some useful relations.

Consider the vector diagram of Fig. 1. Let $m$ be the mass of the incident particle, $p$ its momentum before the collision, and $p''$ its momentum after the collision. Let $m_e$ be the mass of the electron, assumed to be initially at rest, $p'$ the momentum of the electron after the collision. The corresponding kinetic energy is $E' = \sqrt{c^2p'^2 + m_e^2c^4} - m_ec^2$, where $c$ represents the velocity of light (Appendix 2b). Let $\theta$ be the angle be-

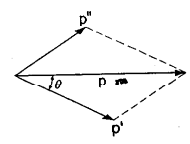


**Fig. 2.2.1.** Collision between a charged particle and a free electron.

tween the initial trajectory of the primary particle and the direction of motion of the electron after the collision. The principle of the *conservation of energy* gives:

$$\sqrt{p^2c^2 + m^2c^4} + m_ec^2 = \sqrt{p''^2c^2 + m^2c^4} + E' + m_ec^2. \tag{1}$$

The conservation of momentum gives:

$$p''^2 = p'^2 + p^2 - 2pp' \cos\theta. \tag{2}$$

Elimination of $p''$ between Eqs. (1) and (2) yields:

$$E' = 2m_ec^2 \frac{p^2c^2\cos^2\theta}{[m_ec^2 + (p^2c^2 + m^2c^4)^{1/2}]^2 - p^2c^2\cos^2\theta}. \tag{3}$$

The kinetic energy, $E'$, of the recoil electron increases with decreasing $\theta$. The maximum transferable energy corresponds to a "head-on" collision and has the value:

$$E'_m = 2m_ec^2 \frac{p^2c^2}{m_e^2c^4 + m^2c^4 + 2m_ec^2(p^2c^2 + m^2c^4)^{1/2}}. \tag{4}$$

For mesons and protons $m \gg m_e$, and one can neglect the term $m_e^2c^4$ in the denominator. For very large momenta ($p \gg m^2c/m_e$) Eq. (4) then becomes:

$$E'_m \approx pc \approx E. \tag{5}$$

This result, unlike that of nonrelativistic mechanics, indicates that a particle of very high energy can transfer almost all of its kinetic energy to an electron even if the mass of the particle is much larger than the electron mass. Thus a very-high-energy meson or proton can be practically "stopped" by a head-on collision with an electron.

On the other hand, if $m \gg m_e$ and if the condition:

$$p \ll \frac{m^2c}{m_e} \tag{6}$$

is satisfied, Eq. (4) becomes:

$$E'_m \approx 2m_ec^2\left(\frac{p}{mc}\right)^2 = 2m_ec^2\frac{\beta^2}{1-\beta^2}, \tag{7}$$

where $\beta$ is the velocity of the incident particle in terms of the light velocity (Appendix 2b). One sees that for heavy particles of sufficiently small momenta the maximum transferable energy depends only on the velocity.

**2.3. *Theoretical expressions for the collision probabilities of charged particles with free electrons (knock-on probabilities).*** Let $\Phi_{col}(E,E')\,dE'\,dx$ represent the probability for a charged particle of kinetic energy $E$, traversing a thickness of $dx$ g cm$^{-2}$, to transfer an energy between $E'$ and $E' + dE'$ to an atomic electron. The function $\Phi_{col}$ will be called the *differential collision probability*. In this section we shall list



the theoretical expressions of $\Phi_{col}$ for electrons and for heavier particles with charge equal, in absolute value, to the electron charge, $e$. We shall assume that $E'$ is sufficiently large so that the atomic electrons may be regarded as free.*

It is convenient to measure the thickness, $x$, in g cm$^{-2}$ and to introduce the constant

$$C = \pi N \frac{Z}{A} r_e^2 = 0.150 \frac{Z}{A} \quad \text{g}^{-1}\,\text{cm}^2, \tag{1}$$

where $Z$ and $A$ are the charge and mass numbers of the material, $N$ is Avogadro's number and $r_e = e^2/m_ec^2$ is the classical radius of the electron. $C$ represents the total "area" covered by the electrons contained in one gram, each considered as a sphere or radius $r_e$.

The parameters that enter in the computation of the collision probability are (§ 4.1): the *mass, m,* of the particle, its *spin* (measured in units of $\hbar$), and its *magnetic moment* (measured in units of $e\hbar/2mc$). We shall assume, however, that the magnetic moment has in all cases the "normal" value; namely, 0 for particles of spin 0, and 1 for charged particles of spin $\frac{1}{2}$ or 1.† In what follows, $\beta$ represents the velocity of the incident particle in terms of the velocity of light.

(a) *Negative Electrons (Negatons).* The collision probability for negatons with negatons has been calculated by Møller (MC32) on the basis of the Dirac theory. When the energy $E$ of the primary particle is large compared with $m_ec^2$ (and therefore $\beta \approx 1$), $\Phi_{col}$ is given by the following expression:

$$\Phi_{col}(E,E')\,dE' = 2Cm_ec^2\,dE'\left[\frac{E}{E'(E-E')} - \frac{1}{E}\right]^2, \tag{2}$$

or

$$\Phi_{col}(E,E')\,dE' = 2C\frac{m_ec^2E^2\,dE'}{(E-E')^2(E')^2}\left[1 - \frac{E'}{E} + \left(\frac{E'}{E}\right)^2\right]^2. \tag{2a}$$

Since one cannot distinguish between the primary and the secondary particle after the collision, Eq. (2) must be interpreted as giving the probability of a collision that leaves one negaton in the energy state $E'$ and the other in the energy state $E - E'$. Thus one takes into account all possible cases by letting $E'$ vary from 0 to $E/2$ (not $E$). Equation (2) is symmetrical in $E'$ and $E - E'$.

(b) *Positive Electrons (Positons).* Bhabha (BHI36) has calculated the collision probability for positons with negatons. For $E \gg m_ec^2$:

$$\Phi'_{col}(E,E')\,dE' = 2C\frac{m_ec^2\,dE'}{(E')^2}\left[1 - \frac{E'}{E} + \left(\frac{E'}{E}\right)^2\right]^2. \tag{3}$$

* Notice that the probability $\Phi$ of a certain interaction, measured in cm$^2$ g$^{-1}$, is related to the atomic cross-section, $\sigma$, for the same interaction measured in cm$^2$ by the equation: $\Phi = N\sigma/A$.

† In this sense protons and neutrons have anomalous magnetic moments (see § 4.4).

This expression represents the probability of a collision that gives rise to a secondary negaton of energy in $dE'$ at $E'$. The probability for a collision out of which the colliding positon comes with an energy in $dE'$ at $E'$ is:

$$\Phi''_{col}(E,E')\,dE' = 2C\,\frac{m_ec^2\,dE'}{(E-E')^2}\left[1-\frac{E'}{E}+\left(\frac{E'}{E}\right)^2\right]^2, \qquad (4)$$

as one can easily see by substituting $E - E'$ for $E'$ in Eq. (3). Thus the total probability for a positon-negaton collision after which *either* the negaton *or* the positon has an energy in $dE'$ at $E'$ is:

$$\Phi_{col}(E,E')\,dE' = [\Phi'(E,E') + \Phi''(E,E')]dE',$$

or

$$\Phi_{col}(E,E')\,dE' = 2C\,\frac{m_ec^2E^2\,dE'}{(E-E')^2E'^2}\left[1-\frac{E'}{E}+\left(\frac{E'}{E}\right)^2\right]^2\left[1-2\frac{E'}{E}+2\left(\frac{E'}{E}\right)^2\right]. \qquad (5)$$

This expression is analogous to that of Eq. (2a), which gives the collision probability between two negatons. Here again, as in Eq. (2a) one takes into account all possible cases by letting $E'$ vary from 0 to $E/2$ (FFL49). The difference between Eq. (2a) and Eq. (5), expressed by the additional factor:

$$\left[1-2\frac{E'}{E}+2\left(\frac{E'}{E}\right)^2\right],$$

in Eq. (5) arises from the fact that exchange phenomena have different effects in a negaton-negaton and in a positon-negaton collision.

(c) *Particles of Mass m and Spin* 0. Bhabha (BHJ38) has calculated the collision probability for particles of mass $m$ and spin 0:

$$\Phi_{col}(E,E')\,dE' = \frac{2Cm_ec^2}{\beta^2}\frac{dE'}{(E')^2}\left(1-\beta^2\frac{E'}{E'_m}\right). \qquad (6)$$

(d) *Particles of Mass m and Spin* $\frac{1}{2}$. The collision probability for particles of mass $m$ and spin $\frac{1}{2}$ has been calculated by Bhabha (BHJ38) and by Massey and Corben (MHJ39). It is:

$$\Phi_{col}(E,E')\,dE' = \frac{2Cm_ec^2}{\beta^2}\frac{dE'}{(E')^2}\left[1-\beta^2\frac{E'}{E'_m}+\frac{1}{2}\left(\frac{E'}{E+mc^2}\right)^2\right]. \qquad (7)$$

(e) *Particles of Mass m and Spin* 1. The collision probability for particles of mass $m$ and spin 1 has been calculated by Massey and Corben (MHJ39) and by Oppenheimer, Snyder, and Serber (OJR40). It is:

$$\Phi_{col}(E,E')\,dE' = \frac{2Cm_ec^2}{\beta^2}\frac{dE'}{(E')^2}\left[\left(1-\beta^2\frac{E'}{E'_m}\right)\left(1+\frac{1}{3}\frac{E'}{E_c}\right) + \frac{1}{3}\left(\frac{E'}{E+mc^2}\right)^2\left(1+\frac{1}{2}\frac{E'}{E_c}\right)\right]. \qquad (8)$$



where:

$$E_c = \frac{m^2c^2}{m_e}. \qquad (9)$$

Note that when $E'$ is very small compared with the maximum transferable energy and with $E_c$, Eqs. (2), (5), (6), (7), and (8) reduce to the following expression, known as the *Rutherford formula*:

$$\Phi_{col}(E,E')\,dE' = \frac{2Cm_ec^2}{\beta^2}\frac{dE'}{(E')^2}. \qquad (10)$$

Thus, at the limit for small values of $E'$, the collision probabilities of different kinds of particles become identical and depend only on the energy, $E'$, of the secondary electron and on the velocity, $\beta$, of the primary particle.

As long as $E'$ is small compared with both $E$ and $E_c$, Eqs. (7) and (8) reduce to (6) which means that, in this case, the collision probability of a heavy particle is independent of the spin. The difference between the collision probabilities of particles of different spin becomes appreciable when $E'$ is comparable with $E_c$ or with $E$, a condition that can occur only when $E$ itself is larger than $E_c$ (see Eq. 2.2.4). For these large values of $E'$, the collision probability is an increasing function of the spin. However, the difference between spin $\frac{1}{2}$ and spin 1 is much larger than the difference between spin 0 and spin $\frac{1}{2}$. Let us consider, in particular, the case $E' \ll E'_m$. The collision probabilities for spin 0 and spin $\frac{1}{2}$ follow from the Rutherford formula (10), while the collision probability for spin 1 becomes:

$$\Phi_{col}(E,E')\,dE' = \frac{2Cm_ec^2}{\beta^2}\frac{dE'}{(E')^2}\left(1+\frac{1}{3}\frac{E'}{E_c}\right). \qquad (11)$$

This expression contains an additional term that decreases with increasing energy as $1/E'$, whereas the Rutherford term decreases as $(1/E')^2$. When $E' > 3E_c$ the additional term, which represents the interaction due to the spin, becomes larger than the Rutherford term, which represents the Coulomb interaction.

Note that the influence of the spin on the collision probability manifests itself only for very close collisions. The theoretical predictions depend essentially on the hypothesis that the electromagnetic field of the particle can be described in the ordinary way even at distances smaller than $10^{-13}$ cm from the "center" of the particle. So far, this hypothesis lacks any experimental support, so that the validity of the formulae expressing the probabilities of large energy transfers cannot yet be considered as soundly established.

**2.4.** ***Classical derivation of Rutherford's formula.*** We have pointed out in the preceding section that at the limit for small values of $E'$, the collision probabilities of all particles with unit charge approach the expression given by Rutherford's formula (2.3.10). In order to illustrate the physical significance of this formula, we shall present, in this section, a derivation based on classical mechanics.

We shall begin by considering a problem of a more general nature than the one discussed so far; namely the problem of a particle of mass $m$, charge $ze$ and velocity $\beta c$ interacting electrically with a particle of mass $m'$ and charge $z'e$ at rest. We shall restrict our considerations to the case of small momentum transfers between the two particles, so that, in particular, we may neglect the motion of the target particle during the interaction.

Let $b$ represent the *impact parameter*, i.e., the distance of the line of motion of the incident particle from the target particle before the encounter. Under the assumptions made, $b$ also represents the minimum distance of approach of the two particles. The force between the two particles reaches its maximum value at the moment of closest approach. If we ignore the relativistic deformation of the field (Appendix 2d) for the present, the maximum value of this force is:

$$f = \frac{zz'e^2}{b^2}. \tag{1}$$

Let us first carry out the computation of the momentum transfer in a semi-quantitative way, which, however, brings out the significant physical features of the phenomenon. The "collision time" during which the value of the force is of the same order of magnitude as the maximum value given by Eq. (1) (say greater than $f/2$) is:

$$\tau = \frac{2b}{\beta c}. \tag{2}$$

Hence, the target particles acquires a momentum of the order of:

$$p' = f\tau = \frac{2zz'e^2}{b\beta c}. \tag{3}$$

For reasons of symmetry, this momentum is perpendicular to the trajectory of the incident particle.

In the case of relativistic velocities, the maximum value, $f$, of the force exerted by the particle on the electron is increased by a factor $1/\sqrt{1-\beta^2}$ over the value given by Eq. (1):

$$f = \frac{zz'e^2}{b^2}\frac{1}{\sqrt{1-\beta^2}}. \tag{4}$$

On the other hand, the "collision time" $\tau$, is decreased by a factor $\sqrt{1-\beta^2}$:

$$\tau = \frac{2b}{\beta c}\sqrt{1-\beta^2}. \tag{5}$$

Thus the product $f\tau$, which gives the momentum acquired by the electron, remains unchanged and Eq. (3) still holds.



A rigorous (classical) proof of Eq. (3) can be given as follows. Consider a cylinder with axis along the trajectory of the moving particle and radius equal to the impact parameter $b$ (Fig. 1). Assume, as before, that the trajectory of the moving particle is not appreciably affected by the collision and that the target particle does not move appreciably during the collision. Let the positive $X$-axis be in the direction of motion of the particle and let $\mathcal{E}_y$ be the component of the electric field of the moving particle normal to the surface of the cylinder. Since the particle is moving in the direction

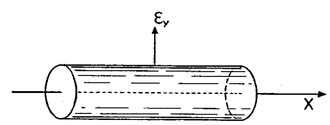


**Fig. 2.4.1.** Illustrating the derivation of the momentum transfer from a moving particle to a particle at rest.

of increasing $X$ with velocity $\beta c$, $\mathcal{E}_y$ depends on the coordinate $X$ and on the time $t$ through a function of the form:

$$\mathcal{E}_y = \mathcal{E}_y(X - \beta ct). \tag{6}$$

From the symmetry properties of the field of a moving charge, one concludes that the resultant momentum, $p'$, acquired by the target particle during the collision is perpendicular to the surface of the cylinder and has the magnitude:

$$p' = z'e\int_{-\infty}^{+\infty}\mathcal{E}_y(X - \beta ct)\,dt. \tag{7}$$

One may transform the integral with respect to $t$ for a fixed $X$ into an integral with respect to $X$ for a fixed $t$, as follows:

$$\int_{-\infty}^{+\infty}\mathcal{E}_y(X - \beta ct)\,dt = \frac{1}{\beta c}\int_{-\infty}^{+\infty}\mathcal{E}_y(X - \beta ct)\,dX. \tag{8}$$

Application of Gauss's theorem to the integral on the right hand side yields:

$$2\pi b\int_{-\infty}^{+\infty}\mathcal{E}_y(X - \beta ct)\,dX = 4\pi ze. \tag{9}$$

By combining Eqs. (7), (8) and (9) one obtains:

$$p' = \frac{2zz'e^2}{b\beta c}.$$

This expression for $p'$ is identical with that given by Eq. (3).

If one makes the assumption that the kinetic energy, $E'$, acquired by the target particle is small compared with its rest energy, one can compute $E'$ from the nonrelativistic relation between energy and momentum, obtaining:

$$E' = \frac{(p')^2}{2m'} = \frac{2z^2z'^2e^4}{m'c^2b^2\beta^2}. \tag{10}$$

A particle on traversing matter collides with electrons (for which $z' = 1$, $m' = m_e$) and with nuclei (for which $z' = Z$, $m' \approx AM$; $M$ represents here the proton mass). Since there are $Z$ electrons in each atom and since $A \approx 2Z$, Eq. (10) shows that the mean energy transfer to electrons is to the mean energy transfer to nuclei in the ratio of $(Z/m_e)/(Z/2M) = 2M/m_e \approx 4000$. Thus, as far as the energy loss is concerned, collisions with nuclei have a negligible effect compared with collisions with electrons and in this section we need to consider only the latter.

If the target particle is an electron, Eq. (10) may be rewritten as follows:

$$E' = 2m_ec^2 \frac{z^2r_e^2}{\beta^2b^2}, \tag{11}$$

where $r_e = e^2/m_ec^2$ is the *classical radius of the electron.*

The probability of an energy transfer in $dE'$ at $E'$ in a given thickness of material is equal to the probability of a collision with an impact parameter in $db$ at $b$, where $E'$ and $b$ are related by Eq. (11). The probability of a collision with impact parameter in $db$ at $b$ in a thickness of $dx$ g cm$^{-2}$ is given by the expression:

$$F(b)\, db\, dx = 2\pi b\, db\, N \frac{Z}{A}\, dx, \tag{12}$$

where $N$ is Avogadro's number, $Z$ is the charge number of the material through which the particle travels, and $A$ is the corresponding mass number. Differentiation of Eq. (11) yields in absolute value the relation:

$$2b\, db = 2m_ec^2 \frac{z^2}{\beta^2} r_e^2 \frac{dE'}{(E')^2}. \tag{13}$$

By combining Eqs. (12) and (13) one finds the following expression for the probability of an energy loss in $dE'$ at $E'$ on traversal of a thickness $dx$:

$$\Phi_{col}(E')\, dE'\, dx = \frac{2Cm_ec^2z^2}{\beta^2} \frac{dE'}{(E')^2}\, dx, \tag{14}$$

where $C$ is given by Eq. (2.3.1). Equation (14), with $z = 1$, is identical with Eq. (2.3.10).

The derivation of Rutherford's formula presented above brings out the physical basis for the dependence of $\Phi_{col}(E')$ on the various factors in Eq. (14). The factor $C$ expresses the proportionality of the collision probability to the electron density. The factor $1/\beta^2$ expresses the dependence of the energy transfer on the collision time, and the factor $z^2$ expresses the dependence of the energy transfer on the strength of the electric interaction between the particle and the electron. The factor $1/(E')^2$ expresses the fact that collisions with large impact parameters are more likely than collisions with small impact parameters. The collision probability does not contain any factor depending on the relativistic deformation of the electric field of the moving particle because this deformation produces two mutually compensating effects, namely, an increase in the field strength and a decrease in the collision time.



The restrictive assumptions underlying the computation of the collision probability and the use of classical mechanics instead of quantum mechanics place limits to the validity of the results obtained. One of the assumptions made is that the electrons are free. Actually they are bound to atoms and can be considered as free only if the collision time is short compared with their period of revolution. If, instead, the collision time is long compared with the period of revolution, the electrons react *adiabatically* to the slowly varying field of the passing particle and do not absorb energy from this field. Let $b_1$ represent the impact parameter corresponding to a collision time equal to the period of revolution, $T = 1/\nu$, of the atomic electrons. From Eq. (5) one obtains for $b_1$ the expression:

$$b_1 = \frac{\beta c}{2\nu\sqrt{1-\beta^2}}. \tag{15}$$

The arguments developed above show that the expression for the energy transfer $E'$ Eq. (11), loses its validity when the impact parameter is of the order of or greater than $b_1$.

Likewise it is clear that Eq. (11) must break down for very small impact parameters. According to this equation, $E'$ tends to infinity as $b$ tends to zero. Actually, of course, $E'$ cannot become larger than the maximum transferable energy $E'_m$ defined by Eq. (2.2.4). Moreover, Eq. (11) loses its validity when $E'$ becomes of the order of $m_ec^2$. This is so because the derivation of Eq. (11) is based upon non-relativistic mechanics; the relativistic correction, that becomes important as $E'$ approaches $m_ec^2$, causes $E'$ to increase with decreasing $b$ less rapidly than Eq. (11) would indicate. The condition $E' < m_ec^2$ is more restrictive than the condition $E' < E_{max}$, at least if the incident particle has relativistic velocity. It places the following approximate lower limit for the impact parameter:

$$b_2 = \frac{z}{\beta} r_e. \tag{16}$$

The condition $E' < m_ec^2$ is also more restrictive than the conditions implied in neglecting the deflection of the incident particle and the motion of the electron during the collision. The reader can easily prove that if the incident particle has relativistic velocity, these conditions set a lower limit for the impact parameter of the order of:

$$b_3 = 2zr_e\sqrt{1-\beta^2}.$$

If $1 - \beta \ll 1$, then $b_3 \ll b_2$. Therefore one may consider the inequalities:

$$b_1 \gg b \gg b_2 \tag{17}$$

as the classical conditions for the validity of Eq. (11). It is interesting to note that the lower limit of the impact parameter, $b_2$, is of the order of the classical electron radius.

Quantum-mechanical arguments introduce new limitations to the validity of Eq. (11). The uncertainty principle sets limits to the accuracy that can be achieved in "aiming" a projectile at a given target. Classical mechanics provides an adequate description of a collision process only if the impact parameter is large compared with the "aiming error." Let $b_q$ represent the minimum value of the aiming error. In order to evaluate $b_q$, consider the motion of the incident particle and of the electron in the center-of-mass system. In this frame of reference the two particles have equal and opposite momenta. If $p_0$ is the absolute value of the momenta and $b$ the impact parameter, the angular momentum in the center of mass system is $p_0b$. The angular momentum is conjugate to the angular coordinate. If no restriction is imposed upon the initial position of the incident particle, the angular coordinate has an uncertainty of the order of unity and the angular momentum has an uncertainty of the order of $\hbar$. The corresponding uncertainty, $b_q$, in the impact parameter is given by the equation:

$$b_q p_0 = \hbar. \tag{18}$$

If the incident particle is an electron, Eq. (18) together with Eq. (A.2.7) in the Appendix yields:

$$b_q \approx \frac{\sqrt{2}\hbar}{\sqrt{m_e c p}} \approx \frac{\sqrt{2}\hbar}{m_e c}(1-\beta^2)^{\frac{1}{4}}. \tag{19}$$

If the mass, $m$, of the incident particle is very large compared with the mass of the electron, the center of mass of the two particles coincides practically with the incident particle. In this case $p_0 = (m_e/m)p$ and Eq. (18) gives the following expression for $b_q$:

$$b_q = \left(\frac{\hbar}{m_e c}\right)\left(\frac{mc}{p}\right) = \frac{\hbar\sqrt{1-\beta^2}}{m_e c\beta}. \tag{20}$$

The length $\hbar/m_e c$ is $\hbar c/e^2 = 137$ times the classical radius of the electron. Therefore the limitation to the impact parameter imposed by the uncertainty principle is more strict than the limitation imposed by classical considerations, Eq. (16), unless the momentum $p$ of the incident particle is very large compared with $mc$.

**2.5. *Energy loss by collision (ionization loss).*** A charged particle moving through matter loses energy as a consequence of collisions with atomic electrons. In the computation of the collision loss, it is convenient to consider distant collisions and close collisions separately. We shall classify as a distant collision any collision that results in the ejection of an electron of energy smaller than a predetermined value, $\eta$. We shall classify as a close collision any collision that results in the ejection of an electron of energy larger than $\eta$. If the limiting energy $\eta$ is sufficiently small (and the corresponding impact parameter sufficiently large) we can treat all distant collisions by considering the primary particle as a point charge. If the limiting energy $\eta$ is sufficiently large (and the corresponding impact parameter sufficiently small) we can treat all close collisions by considering the atomic electrons as free particles. For practically all cases of importance in the field of high-energy phenomena, a limiting energy between $10^4$ and $10^5$ ev simultaneously satisfies both conditions specified above. In what follows we shall assume that the limiting energy lies within this range.

Let $k_{\mathrm{col}(<\eta)}(E)$ be the energy loss per g cm$^{-2}$ resulting from distant collisions. In the computation of $k_{\mathrm{col}(<\eta)}(E)$ it is essential to take into account the binding of the electrons to the atoms; i.e., one should consider the system formed by an atom and by the incident particle and then compute the probabilities for the various possible transitions leading to excitation or ionization of the atom. Bethe (BHA30; BHA32) developed a theory along these lines. With the help of Born's approximation, and for the case of particles with unit charge, he obtained the following result:



$$k_{\mathrm{col}(<\eta)}(E) = \frac{2Cm_e c^2}{\beta^2}\left[\ln\frac{2m_e c^2\beta^2\eta}{(1-\beta^2)I^2(Z)} - \beta^2\right], \tag{1}$$

where $I(Z)$ is the average ionization potential of an atom of atomic number $Z$.

The quantity $I(Z)$ can be evaluated theoretically, or it can be deduced from experimental data. Bloch (BF33) suggested the formula:

$$I(Z) = I_H Z, \tag{2}$$

where $I_H = 13.5$ is the energy corresponding to the Rydberg frequency. More accurate calculations were carried out by Wick (WGC41; WGC43) and by Halpern and Hall (HO48). Table 1 summarizes the various determinations of $I$. The discrepancies between these determinations reflect

**Table 1. Values of the average ionization potential of various substances**

| Substance | $Z$ | Author | Method | $I$ (ev) |
|---|---|---|---|---|
| Hydrogen | 1 | Bethe (BHA30) | Theoretical | 14.9 |
| Helium | 2 | Williams (WEJ37) | Theoretical | 35 |
| | | Halpern and Hall (HO48) | Theoretical | 40 |
| Carbon | 6 | Wick (WGC43) | Theoretical | 60 |
| | | Halpern and Hall (HO48) | Theoretical | 60 |
| Aluminum | 13 | Wilson (WRR41) | Experimental | 150 |
| Iron | 26 | Wick (GC43) | Theoretical | 243 |
| | | Halpern and Hall (HO48) | Theoretical | 430 |
| Gold | 79 | Livingston and Bethe (LMS37) | Experimental | 520 |
| Lead | 82 | Wick (WGC41) | Experimental | 1000 |
| | | Halpern and Hall (HO48) | Theoretical | 1200 |
| Air | | Livingston and Bethe (LMS37) | Experimental | 80.5 |
| | | Halpern and Hall (HO48) | Theoretical | 96 |
| Water | | Wick (WGC41) | Theoretical (hydrogen) Experimental (oxygen) | 63 |
| | | Halpern and Hall (HO48) | Theoretical | 80 |

the present uncertainty as to the actual values of the average ionization potential. This uncertainty, however, does not represent a serious source of error in the computations of $k_{col(<\eta)}(E)$ because $I$ enters only in the logarithm.

Equation (1) is valid for particles of any kind, with positive or negative charge equal to $e$ and *with velocity large compared with the velocity of atomic electrons.*

Consider next the energy loss per g cm$^{-2}$ resulting from close collisions, i.e., from collisions in which the energy transfer is greater than $\eta$. This quantity shall be called $k_{col(>\eta)}(E)$. In the computation of $k_{col(>\eta)}(E)$, one may consider the electrons as free. One thus obtains the following expression:

$$k_{col(>\eta)}(E) = \int_{\eta}^{E'_m} E'\Phi_{col}(E,E')\,dE', \tag{3}$$

where $E'_m$ is the maximum transferable energy [see Eq. (2.2.4)].

(a) *Heavy Particles.* For singly charged particles heavier than electrons and with energy small compared with $m^2c^2/m_e$ one can use Eq. (2.3.6) which gives (if $\eta \ll E'_m$):

$$k_{col(>\eta)}(E) = \frac{2Cm_ec^2}{\beta^2}\left[\ln\frac{E'_m}{\eta} - \beta^2\right]. \tag{4}$$

The total energy loss by collision per g cm$^{-2}$ (or *ionization loss*):

$$k_{col}(E) = -\frac{dE}{dx} \tag{5}$$

is the sum of $k_{col(<\eta)}$ and $k_{col(>\eta)}$ and has the expression:

$$k_{col}(E) = \frac{2Cm_ec^2}{\beta^2}\left[\ln\frac{2m_ec^2\beta^2E'_m}{(1-\beta^2)I^2(Z)} - 2\beta^2\right]. \tag{6}$$

This expression is independent of the arbitrary value chosen for the limiting energy $\eta$, as it should be. Substituting $E'_m$ from Eq. (2.2.7) transforms Eq. (6) into the following:

$$k_{col}(E) = \frac{2Cm_ec^2}{\beta^2}\left[\ln\frac{4m_e^2c^4\beta^4}{(1-\beta^2)^2I^2(Z)} - 2\beta^2\right]. \tag{7}$$

Within the limit of validity of Eq. (2.2.7), $k_{col}$ is only a function of $\beta$, i.e., of the velocity of the incident particle. Since $p/mc = \beta/\sqrt{1-\beta^2}$, one may also say that $k_{col}$ does not depend separately on the momentum and on the mass of the incident particle, but only on the ratio of these two quantities. Likewise, one may say that $k_{col}$ does not depend separately on the energy and the mass of the incident particle, but only on their ratio. The same is true of the quantity $k_{col(<\eta)}$ and, in this case, without the restrictive condition (2.2.6) that insures the validity of Eq.

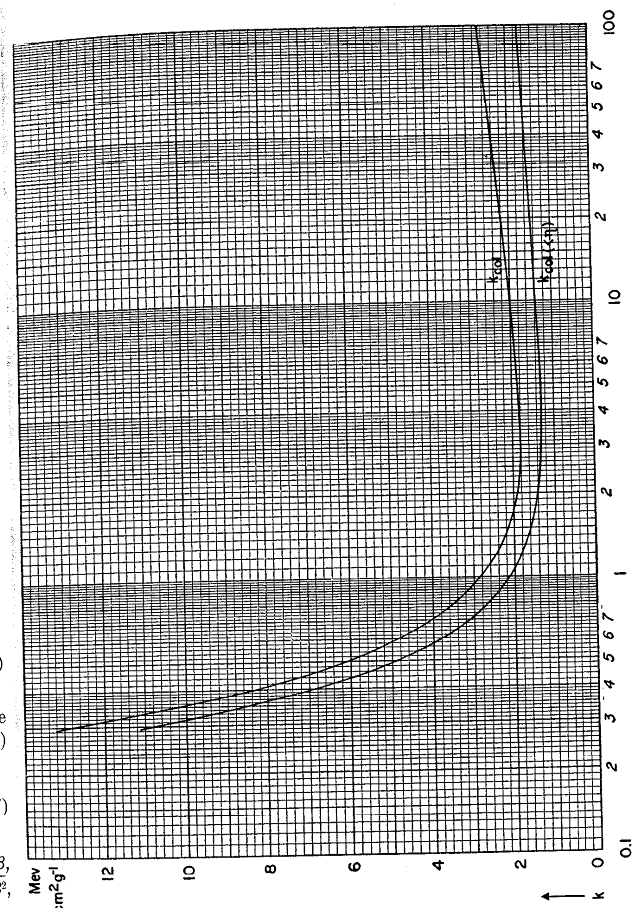


**Fig. 2.5.1.** The total collision loss, $k_{col}$, and the energy loss in distant collisions, $k_{col(<\eta)}$, for particles heavier than electrons in air, as functions of $p/mc$ ($\eta = 10^4$ ev). $k_{col(<\eta)}$ was computed from Eq. (2.5.1); $k_{col}$ was obtained from a paper by Smith (SJH47).

(2.2.7). The functional dependence of $k_{col(<\eta)}$ and $k_{col}$ on $p/mc$ for air is illustrated in Fig. 1.

In order to appreciate the physical significance of Eqs. (6) or (7) we shall derive an approximate expression for the collision loss of heavy particles by means of semi-classical considerations.

From Eq. (2.4.11) one finds, for a singly charged particle, that the energy loss per g cm$^{-2}$ due to collisions with impact parameter in $db$ at $b$ has the expression:

$$2\pi b\, db\, N \frac{Z}{A} E'(b) = \frac{4Cm_ec^2}{\beta^2} \frac{db}{b}. \tag{8}$$

It has been shown in § 2.4 that, if the energy of the incident particle is not very large compared with its rest energy, Eq. (2.4.11) is valid when $b_1 > b > b_q$, where $b_1$ and $b_q$ are given by Eqs. (2.4.15) and (2.4.20) respectively. For impact parameters larger than $b_1$ or smaller than $b_q$, Eq. (2.4.11) overestimates the energy transfer. Thus one may evaluate the total energy loss by integrating the expression (8) between $b_q$ and $b_1$:

$$k_{col}(E) \approx \frac{4Cm_ec^2}{\beta^2} \int_{b_q}^{b_1} \frac{db}{b} = \frac{4Cm_ec^2}{\beta^2} \ln \frac{b_1}{b_q}, \tag{9}$$

or, from Eqs. (2.4.15) and (2.4.20):

$$k_{col}(E) \approx \frac{2Cm_ec^2}{\beta^2} \ln \frac{\pi^2 m_e^2 c^4 \beta^4}{(1-\beta^2)^2 h^2 \nu^2}. \tag{10}$$

If one substitutes $I(Z)$ for $h\nu$ in Eq. (10), one obtains an expression for $k_{col}$ that does not differ significantly from Eq. (7).

Despite this agreement, one should not take the classical picture too literally. For example, the classical treatment does not give the correct number of energy transfers, nor their correct distribution in space. One can easily recognize this fact by considering that already for impact parameters considerably smaller than $b_1$ the "classical" energy transfer, $E'$, as given by Eq. (2.4.11), is smaller than the excitation energy of the atoms. Only when one computes the total energy loss by integrating the classical expression over all impact parameters (and neglects the impossibility of energy transfers smaller than the excitation energy) does one obtain a correct result.

Figure 1 shows that, for subrelativistic energies, the energy loss, $k_{col}$, decreases rapidly with increasing energy because of the term $\beta^2$ in the denominator. This term arises from the similar term in Eq. (2.4.14) and corresponds to the fact that, for a given impact parameter, the interaction between the passing particle and the atom becomes less effective as the time spent by the particle near the atom becomes shorter. When $\beta$ approaches its limiting value of 1, the factor $1/\beta^2$ becomes practically constant; $k_{col}$ goes through a flat minimum at a momentum equal to a small multiple of $mc$ and then begins to increase with increasing momentum because of the factor $1/(1-\beta^2)^2$ in the logarithm. The reason for this increase is twofold: (1) as the velocity increases, the relativistic deformation of the Coulomb field of the incident particle causes the effects of this particle to be felt at larger distances from its geometric path and therefore increases the upper limit of the impact parameter [see Eq. (2.4.15)]; (2) as the momentum increases, the quantum-theoretical uncertainty, which sets the lower limit of the impact parameter, decreases [see Eq. (2.4.20)].



The dependence on momentum of $k_{col(<\eta)}$ is very similar to that of $k_{col}$. In the relativistic region, however, $k_{col(<\eta)}$ increases with $p$ somewhat more slowly than $k_{col}$. The physical reason for this is that in the case of $k_{col(<\eta)}$, the lower limit of the impact parameter is determined by the limiting energy and does not vary with $p$. Thus the increase with momentum is caused exclusively by the effect of the relativistic deformation of the Coulomb field on the upper limit of the impact parameter.

(b) *Electrons.* The total energy loss of *negatons* and *positons* can be calculated easily from Eqs. (2.3.2), (2.3.3), (1), and (3). If one remembers that $\beta \approx 1$, one obtains:

$$k_{col} = 2Cm_ec^2 \left[ \ln \left( \frac{\pi^2 (m_ec^2)^2}{(1-\beta^2)^{3/2} I^2(Z)} \right) - a \right], \tag{11}$$

where $a = 2.9$ for negatons; $a = 3.6$ for positons.

Here again we may justify the theoretical expression for the energy loss by semi-classical considerations. Indeed Eq. (9) together with Eqs. (2.4.15) and (2.4.19) gives (since $\beta \approx 1$):

$$k_{col} \approx 2Cm_ec^2 \ln \frac{\pi^2 (m_ec^2)^2}{2(1-\beta^2)^{3/2} h^2 \nu^2}. \tag{12}$$

Equation (12) is very similar to Eq. (11). Note in both equations the term: $-\ln(1-\beta^2)^{3/2}$ that gives the dependence of the collision loss of electrons on velocity and compare it with the term: $-\ln(1-\beta^2)^2$ that gives the dependence on velocity of the collision loss of heavy particles. The derivation of Eqs. (10) and (12) shows that the difference arises from the different relation between the momenta of the incident particle in the center-of-mass system and in the laboratory system respectively.

The expression (2.4.11) for the energy transfer shows that the collision loss of a particle with multiple charge, $ze$, is $z^2$ times the collision loss of a particle with unit charge and the same velocity.

The *momentum* loss is easily obtained from the *energy* loss. Indeed, since $dp/dE = 1/\beta c$, the following simple relation holds:

$$-\frac{d(pc)}{dx} = -\frac{1}{\beta}\frac{dE}{dx} = \frac{k_{col}}{\beta}. \tag{13}$$

The momentum loss is a function of the velocity alone whenever this is true of the energy loss.

Some measurements of the collision loss of particles heavier than electrons will be discussed in § 6.4.

**2.6. *The density effect.*** So far, in investigating the interactions of charged particles with atoms, we have considered the latter as isolated. This is permissible to a large extent when the particle travels in a gas. When the particle travels in a condensed material we can still consider the atoms as isolated in the case of close collisions, but we cannot do so

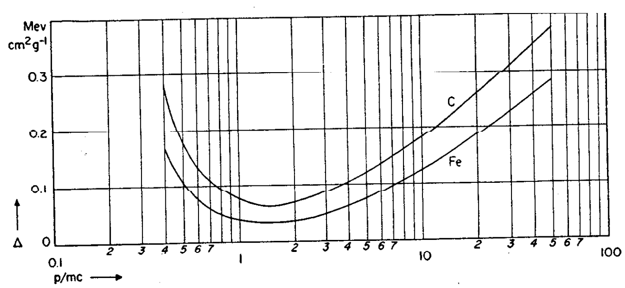


**Fig. 2.6.1.** The decrease in collision loss, Δ, due to density effect as a function of $p/mc$, for carbon and iron. From Wick (WGC43).

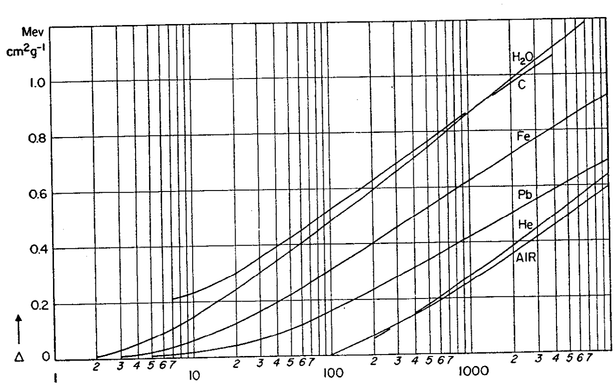


**Fig. 2.6.2.** The decrease in collision loss, Δ, due to density effect as a function of $p/mc$, for water, carbon, iron, lead, helium and air. From Halpern and Hall (HO48).

when the impact parameter is larger than the atomic distances. For such distant collisions one has to take into account the screening of the electric field of the passing particle by the atoms of the medium. The screening reduces the interaction and decreases, therefore, the energy loss. Since distant collisions become more and more important as the velocity increases, the correction to be applied to the expression for the energy loss is an increasing function of the velocity. The influence of the density on the collision loss was first suggested by Swann (SWF38) and quantitatively investigated by Fermi (FE39). According to Fermi, the quantity Δ to be subtracted from the energy loss, as calculated for isolated



atoms, is given by the following formulae, in the case of singly charged particles:

for $\beta < \epsilon^{-1/2}$, $\Delta(\beta) = \dfrac{2Cm_ec^2}{\beta^2}\ln\epsilon$;

$$\text{for}\quad \beta > \epsilon^{-1/2},\quad \Delta(\beta) = \frac{2Cm_ec^2}{\beta^2}\left[\ln\frac{\epsilon-1}{1-\beta^2} + \frac{1-\epsilon\beta^2}{\epsilon-1}\right]; \tag{1}$$

where $\epsilon$ is the dielectric constant of the medium relative to vacuum.

Halpern and Hall (HO40; HO48) and Wick (WGC41; WGC43) made a more refined analysis of the density effect by considering in detail the behavior of atomic electrons belonging to the different shells. Their computations confirmed the finding that the collision loss depends on the density of the absorbing material, but showed that the simplification made by Fermi in the development of the theory lead, in general, to an overestimate of the reduction in the collision loss.

Figure 1 represents the results of Wick's calculations for carbon and iron. Figure 2 represents the results of the calculations of Halpern and Hall for carbon, water, iron, lead, air, and helium. One sees that the agreement between the two sets of data, where they can be compared, leaves much to be desired.

The energy loss of charged particles in materials of finite density has been studied further by A. Bohr (BLA48), by Messel and Ritson (MH50.2), and by Schönberg (SbM51). These investigators called attention to the fact that part of the energy dissipated by high-energy particles in their interactions with atomic electrons goes into electromagnetic radiation (*Cerenkov* radiation) rather than into excitation or ionization of atoms. The intensity of the Cerenkov radiation (which, of course, must not be confused with the radiation that accompanies the deflection of the incident particle in the electric fields of nuclei) increases with increasing velocity. Indeed, it appears that the relativistic increase of the energy loss by distant collisions is mainly due to the increase of the Cerenkov radiation.

**2.7.** ***Statistical fluctuations in the energy loss by collision.*** The energy loss of a charged particle in matter is a statistical phenomenon because the collisions that are responsible for this loss are independent events. Thus particles of a given kind and of a given energy do not all lose exactly the same amount of energy in traversing a given thickness of material. The quantity $k_{col}(E)$ defined as "collision loss" in § 2.5 represents only an average value. The statistical fluctuations in the energy loss by collision are comparatively small because the average transfer of energy in each individual collision process is small and the number of collisions necessary to cause any appreciable energy change is correspondingly large.

For electrons, in general, collision processes are not the main cause of energy losses and especially not the main cause of fluctuations in the energy losses (§ 2.13). Therefore we shall confine ourselves here to the case of heavy particles. We shall take Eq. (2.3.6) as the expression of their collision probability for close collisions. The collision probability for distant collisions shall be such as to give the correct value of $k_{col}$ (Eq. 2.5.6), namely such that:

$$\int_0^E E'\Phi_{col}(E,E')\,dE' = k_{col}(E). \tag{1}$$

Let $w(E_0,E,x)\,dE$ represent the probability that a particle of initial energy $E_0$ has an energy between $E$ and $E + dE$ after traversing a thickness of $x$ g cm$^{-2}$ of matter. In order to find the equation that defines the function $w(E_0,E,x)$, it may be helpful to consider a large number of particles, all of the same energy $E_0$, incident upon the absorber; $w(E_0,E,x)$ is then the fractional number of particles that reach the depth $x$ with energy between $E$ and $E + dE$. As the particles traverse an additional thickness $dx$, the number of particles in the energy interval $dE$ at $E$ changes because of the two following phenomena: (1) some of the particles that have energies between $E$ and $E + dE$ at $x$ undergo a collision in $dx$ and are thereby removed from this energy interval; (2) some of the particles that arrive at the depth $x$ with energy $E + E'$ greater than $E$ undergo, in $dx$, a collision that brings them into the energy interval $dE$ at $E$. Thus the function $w(E_0,E,x)$ satisfies the following equation:

$$w(E_0,E,x + dx) - w(E_0,E,x) = -w(E_0,E,x)\,dx \int_0^\infty \Phi_{col}(E,E')\,dE' + dx \int_0^\infty w(E_0,E + E',x)\,\Phi_{col}(E + E',E')\,dE',$$

where, of course, $\Phi_{col}(E,E') = 0$ for $E' > E'_m$ and $w(E_0,E,x) = 0$ for $E > E_0$.

From the above equation one obtains:

$$\frac{\partial w(E_0,E,x)}{\partial x} = \int_0^\infty [w(E_0,E + E',x)\,\Phi_{col}(E + E',E') - w(E_0,E,x)\,\Phi_{col}(E,E')]\,dE'. \tag{2}$$

Assume, for the time being, that the thickness of matter traversed is sufficiently small so that the average energy loss is a small fraction of the initial energy. This case shall be described as the case of a "thin absorber." Since (except for velocities small compared with the velocity of light) the collision probability does not depend critically on the energy of the colliding particle, we may assume that $\Phi_{col}(E,E')$ is a function of $E'$ alone and, in Eq. (2), we may take $\Phi_{col}(E + E',E') = \Phi_{col}(E,E')$. We may also regard the collision loss, $k_{col}$, as constant and use the following expression for the average energy, $E_a(x)$, of the particles at the depth $x$:



$$E_a(x) = E_0 - xk_{col}(E_0). \tag{3}$$

Assume further that $w(E_0,E + E',x)$ varies only slightly while $\Phi_{col}(E,E')$ is appreciably different from zero. In this case one can expand $w(E_0,E + E',x)$ in a power series of $E'$ about $E$ and neglect terms beyond the second order. If one introduces the quantity:

$$\rho^2 = \int_0^\infty (E')^2\,\Phi_{col}(E,E')\,dE', \tag{4}$$

and one remembers Eq. (1), one obtains from Eq. (2):

$$\frac{\partial w(E_0,E,x)}{\partial x} = k_{col}\frac{\partial w(E_0,E,x)}{\partial E} + \frac{1}{2}\rho^2\frac{\partial^2 w(E_0,E,x)}{\partial E^2}. \tag{5}$$

A solution of this equation is:

$$w(E_0,E,x) = \frac{1}{(2\pi\rho^2 x)^{1/2}}\exp\left[\frac{-(E - E_a)^2}{2\rho^2 x}\right]. \tag{6}$$

For $x = 0$ the function defined by Eq. (6) reduces to the $\delta$-function* and therefore represents a single incident particle. If, moreover, $E_a \gg \rho\sqrt{x}$ and $E_0 - E_a \gg \rho\sqrt{x}$, this function satisfies the normalizing condition:

$$\int_0^{E_0} w(E_0,E,x) = 1, \tag{7}$$

and is thus the solution of our problem. We conclude that, when all of the conditions specified above are fulfilled, the distribution function $w$ at the depth $x$ is a Gaussian function of $E$ with maximum at $E_a(x)$ and width $\Delta(x)$ given by the following equation:

$$\Delta(x) = \rho x^{1/2}. \tag{8}$$

The expression for $\rho^2$, Eq. (4), contains the factor $(E')^2$, whereas the expression for $k_{col}$, Eq. (1), contains the factor $E'$. Therefore distant collisions are much less important in the computation of $\rho^2$ than they are in the computation of $k_{col}$, and one may evaluate the integral in Eq. (4) by assuming that $\Phi_{col}$ is given by Eq. (2.3.6) for all values of $E'$ down to $E' = 0$. One thus obtains:

$$\rho^2 = \frac{2Cm_ec^2E'_m}{\beta^2}\left(1 - \frac{\beta^2}{2}\right). \tag{9}$$

One can now express *a posteriori* the conditions for the validity of the solution just obtained by saying that the width, $\Delta$, of the distribution

* The $\delta$-function (or Dirac's improper function) is defined by the conditions:

$$\delta(x) = 0 \text{ for } x \neq 0; \qquad \int_{-\epsilon}^{+\epsilon} \delta(x)\,dx = 1 \text{ for any } \epsilon \neq 0.$$

curve must be large compared with the maximum transferable energy $E'_m$ yet small compared with $E_a$ and with $E_0 - E_a$. According to Eqs. (8) and (9), $\Delta \gg E'_m$ if the quantity:

$$G = \frac{2Cm_ec^2x}{\beta^2 E'_m} \tag{10}$$

is a large number.

When $G$ is not large, one cannot replace the integro-differential equation (2) by the differential equation (5), and the determination of $w$ becomes a difficult mathematical task.

By the method of the Laplace transformation, Landau (LLD44) has obtained a solution of Eq. (2) that is valid when $G$ is less than about 0.05.

A complete solution of the problem has been given by Symon (SKR48). In his treatment, Symon considers separately the case of a "thin absorber" $[E_a(x) \gtrsim 0.9E_0]$ and that of a "thick absorber" $[E_a(x) \lesssim 0.9E_0]$. In order to describe his results it is convenient to define the *most probable energy* at the depth $x$, $E_p(x)$, as the value of $E$ for which the function $w(E_0,E,x)$ is a maximum. Except in the limiting case of $G \gg 1$, where $w(E_0,E,x)$ is represented by a Gaussian function of $E$, the curve of $w$ vs. $E$ is not symmetric with respect to its maximum, and therefore the most probable energy $E_p(x)$ is different from the average energy $E_a(x)$. For *thin absorbers* the solution of the problem is contained in Figs. 1 and 2. In order to obtain the distribution function, one should use the following procedure:

(1) Determine the most probable energy loss $E_0 - E_p$ by means of the equation:

$$E_0 - E_p = \frac{2Cm_ec^2x}{\beta^2}\left[\ln\frac{4Cm_e^2c^4x}{(1-\beta^2)I^2(Z)} - \beta^2 + j\right], \tag{11}$$

where $j$ is a function of the parameter $G$ defined in Eq. (10) and of the particle velocity $\beta c$, and is given by the graphs in Fig. 1a.

(2) Determine the parameter $\Delta_0$, by means of the equation:

$$\Delta_0 = \frac{2Cm_ec^2x}{\beta^2}\,b, \tag{12}$$

where $b$ is a function of $G$ and $\beta$ and is given by the graphs in Fig. 1b. $\Delta_0$ has the dimensions of an energy and is related to the *width* of the distribution function (i.e., to the quantity $\Delta$ given by Eqs. (8) and (9) in the case of the Gaussian solution).

(3) Determine another parameter $\lambda$, which is related to the asymmetry of the distribution function, by means of the graphs in Fig. 2c.

When the quantities $E_0 - E_p$, $\Delta_0$ and $\lambda$ have been determined, Figs. 2a and 2b give the desired result. In both of these figures, the abscissa represents the quantity $(E_p - E)/\Delta_0$, namely the difference between the



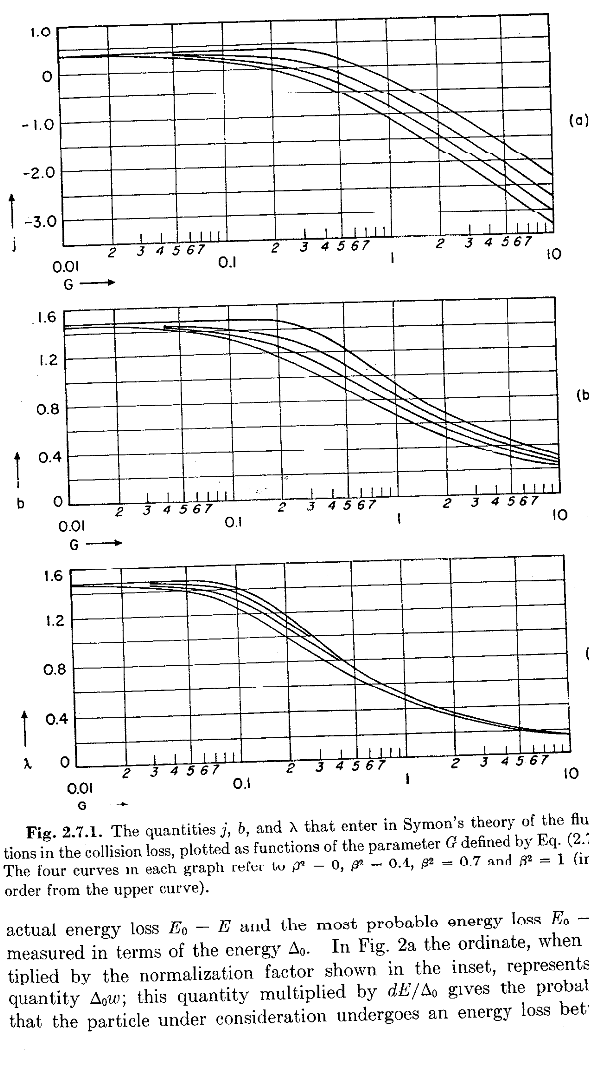


**Fig. 2.7.1.** The quantities $j$, $b$, and $\lambda$ that enter in Symon's theory of the fluctuations in the collision loss, plotted as functions of the parameter $G$ defined by Eq. (2.7.10). The four curves in each graph refer to $\beta^2 = 0$, $\beta^2 = 0.4$, $\beta^2 = 0.7$ and $\beta^2 = 1$ (in this order from the upper curve).

actual energy loss $E_0 - E$ and the most probable energy loss $E_0 - E_p$, measured in terms of the energy $\Delta_0$. In Fig. 2a the ordinate, when multiplied by the normalization factor shown in the inset, represents the quantity $\Delta_0 w$; this quantity multiplied by $dE/\Delta_0$ gives the probability that the particle under consideration undergoes an energy loss between

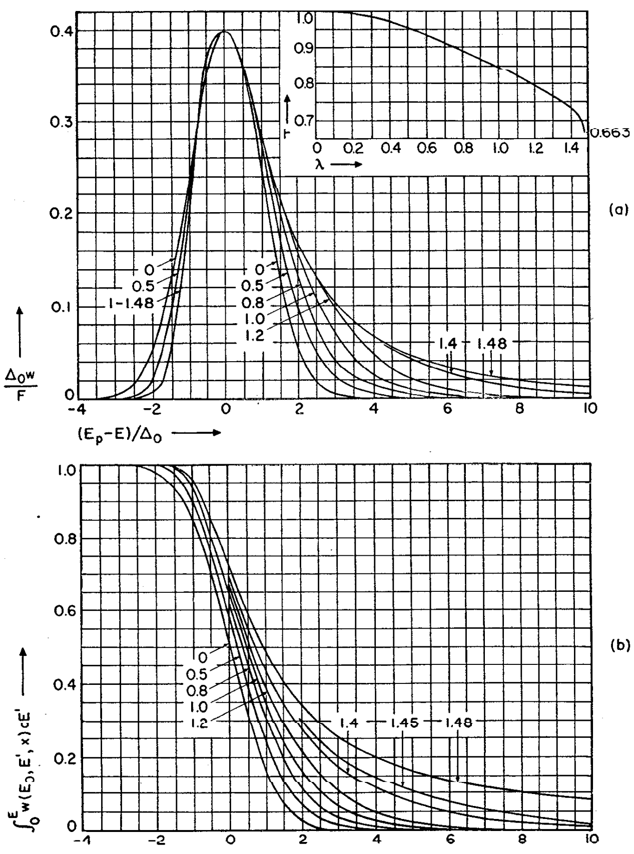


**Fig. 2.7.2.** Results of Symon's theory of the fluctuations in the collision loss. (a) The quantity $\Delta_0 w/F$ plotted as a function of $(E_p - E)/\Delta_0$. The various curves refer to different values of $\lambda$, as shown by the numbers attached to the curves. The normalization factor, $F$, is given as a function of $\lambda$ in the inset. (b) The quantity $\int_0^E w(E_0,E',x)dE'$ plotted as a function of $(E_p - E)/\Delta_0$. The various curves refer to different values of $\lambda$, as shown by the numbers attached to the curves.



$E_0 - E$ and $E_0 - E + dE$ in traversing the thickness $x$. In Fig. 2b the ordinate represents the quantity:

$$\int_0^E w(E_0,E',x)\, dE'$$

and thus gives the probability that the particle undergoes an energy loss greater than $E_0 - E$.

The different curves in Figs. 2a and 2b correspond to different values of the parameter $\lambda$, which, in turn, depends on $G$ (Fig. 1c). For $G \gg 1$, $\lambda \approx 0$ and $w$ becomes a Gaussian function of $E$, in agreement with Eq. (6). For $G \ll 1$, $\lambda \approx 1.48$ and the corresponding function $w$ becomes identical to Landau's solution. Notice that neither the treatment of Landau nor that of Symon applies to extremely thin absorbers since both treatments neglect fluctuations due to distant collisions; i.e., to collisions in which the atomic electrons cannot be treated as free.

One may also point out that the reason why Figs. 1 and 2 do not apply to cases where $E_a < 0.9E_0$ lies in the neglect of the dependence of $\Phi_{col}$ on the energy of the incident particle. The results of Symon concerning "thick absorbers" will not be described here.

**2.8. *The range of heavy particles.*** The collision process is only one of several mechanisms by which charged particles may lose energy. In the case of electrons it represents the most important source of energy loss only for comparatively small energies. At an energy of the order of 10 or 100 Mev, depending on the atomic number of the absorber, radiation losses become greater than collision losses (§§ 2.11, 2.12, 2.13 below). For $\mu$-mesons, collision losses remain the dominant factor up to energies of the order of $10^{11}$ or $10^{12}$ ev. For protons, radiation losses are never important, but the effect of nuclear interactions overshadows that of collision losses at energies of the order of 1000 Mev or greater. Thus, in general, collision processes represent the most important source of energy loss only for energies smaller than a certain value that depends on the nature of the particles.

When other types of energy losses are negligible compared with the collision loss, fluctuations in the energy loss are small (see preceding section) and, in a given material, all particles of a given energy travel *approximately* the same distance before being stopped. This distance is called the *range* or, more properly, the *mean range*. If one denotes by $R(E)$ the range of a particle of initial energy $E$, the function $R(E)$ satisfies the following differential equation:

$$\frac{dR}{dE} = \frac{1}{k_{col}(E)}. \qquad (1)$$

Note that the range thus defined represents the actual length of the trajectory in matter, *measured along the trajectory itself*. Sometimes the word "range" is used with a different meaning, namely to indicate the average thickness of absorber that a particle, incident perpendicularly upon it, is capable of traversing. To avoid confusion one may call this quantity the *effective* range. It is clear that the effective range will coincide with the actual range only when scattering is negligible.

We have shown in § 2.5 that, for energies small compared with $(m/m_e)mc^2$, the collision loss of a particle with unit charge is a universal function of the velocity, $\beta$, of the particle. For a particle with $z$ units of charge, $k_{col}$ is equal to $z^2$ times the same universal function. If one considers that $E = mc^2/\sqrt{1-\beta^2}$, one finds that Eq. (1) can be written in the form

$$\frac{d(R/m)}{d\beta} = \frac{g(\beta)}{z^2},$$

where $g$ is a function of $\beta$ alone. From this equation one obtains

$$\frac{R}{m} = \frac{G(\beta)}{z^2}, \tag{2}$$

where $G(\beta)$ is the integral of $g(\beta)$. Since $\beta$, in turn, is a function of $(p/m)$, or of $(E/m)$, Eq. (2) can also be written as follows:

$$\frac{R}{m} = \frac{1}{z^2}F_1\left(\frac{p}{m}\right), \tag{3}$$

or

$$\frac{R}{m} = \frac{1}{z^2}F_2\left(\frac{E}{m}\right). \tag{4}$$

Thus the quantity $z^2R/m$ is represented by the same function of the ratio $p/m$ (or $E/m$) for all kinds of particles of energy small compared with $(m/m_e)mc^2$.

Because of the statistical fluctuations in energy losses (see preceding section) particles of a given energy $E$ do not have *exactly* the same range in matter. This phenomenon is usually referred to as "straggling."* One may define a function, $W(E,r)\,dr$, representing the probability that a particle of energy $E$ travels a distance between $r$ and $r + dr$ before coming to rest. One may define the average range, $\bar{r}$, by means of the equation:

$$\bar{r} = \int_0^\infty W(E,r)r\,dr \tag{5}$$

(this quantity is practically equal, but not rigorously identical to the range, $R$, defined previously). One may also define the mean square fluctuation in range by means of the equation:

$$\langle(r-\bar{r})^2\rangle_{av} = \int_0^\infty W(E,r)(\bar{r}-r)^2\,dr. \tag{6}$$

* See, for example, Rutherford, Chadwick, and Ellis, *Radiation from Radioactive Substances*, The Macmillan Co., New York, 1930, p. 111.



This quantity has been computed by Symon (SKR48); for particles of mass $m$, his result can be written in the form:

$$\frac{\langle(r-\bar{r})^2\rangle_{av}^{1/2}}{\bar{r}} = \sqrt{\frac{200m_e}{m}}\,f\left(\frac{E}{mc^2}\right). \tag{7}$$

For the case of iron, the function $f$ is shown in Fig. 1. One sees that the relative values of the root mean square fluctuation in range are of the order of several per cent and decrease with increasing mass. The computations of Symon also show that the curves representing $W(E,r)$ vs. $r$ differ only

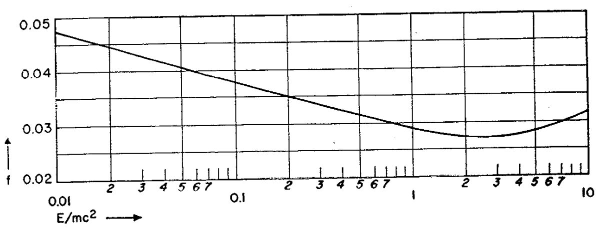


**Fig. 2.8.1.** The function $f(E/mc^2)$ that enters in the expression of the root mean-square fluctuation in range, Eq. (2.8.7), as computed for iron by Symon.

slightly from Gaussian curves. The most probable range is slightly larger than the average range. For $E < 10mc^2$, the difference between the two ranges is less than one per cent.

**2.9.** ***Numerical values for the collision loss and the range of charged particles.*** The curves in Fig. 1 give the total energy loss by collision in carbon, aluminum, iron, and lead for particles with unit charge and heavier than electrons. The values of $k_{col}$ for carbon and iron were computed from Eq. (2.5.7), corrected for the density effect according to the computations of Wick (Fig. 2.6.1). The values for aluminum were taken from Smith (SJH47); they are *not* corrected for density effect (which is unimportant for $p/mc < 4$). The curve for lead includes data of Livingston and Bethe (low momenta) (LMS37), of Wheeler and Ladenburg (medium momenta) (WJA41), and of Halpern and Hall (high momenta) (HO48). This curve is corrected for the density effect. A similar curve for air (obtained from the data of Livingston and Bethe (LMS37) and those of Smith (SJH47) has been plotted in Fig. (2.5.1).

The curves in Fig. 2 give the range-energy and the range-momentum relations in air for particles with unit charge and heavier than electrons. The data were taken from Wick (WGC43) and from Smith (SJH47). The density effect in air is small, except at very high energies, and has been neglected.

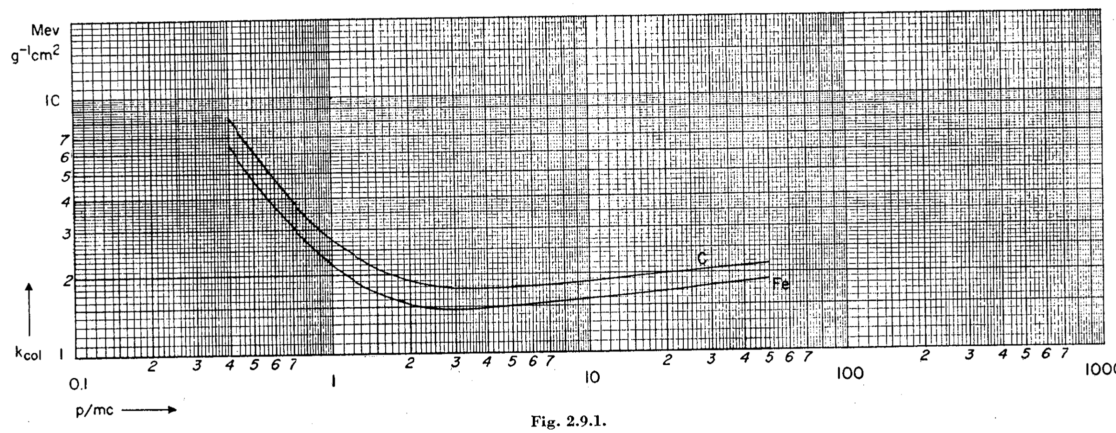


**Fig. 2.9.1.**

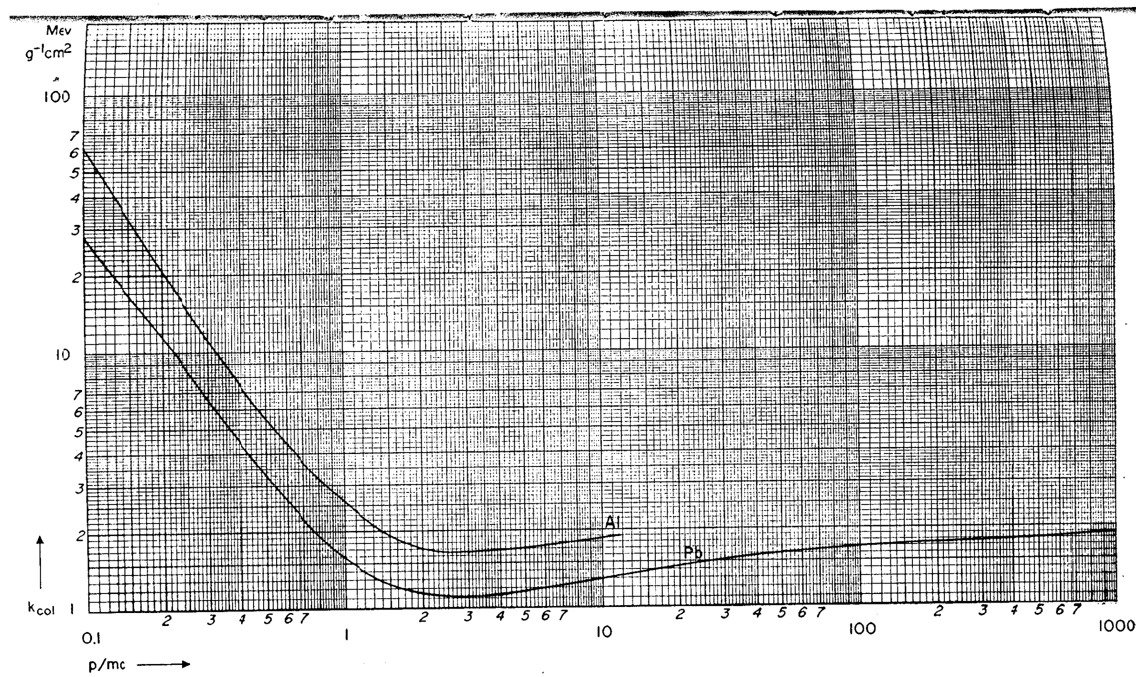


**Fig. 2.9.1.** Total energy loss by collision, $k_{col}$, as a function of $p/mc$ in carbon, aluminium, iron, and lead; $p$ is the momentum and $m$ the mass. These curves can be used for any particle with unit charge heavier than an electron.

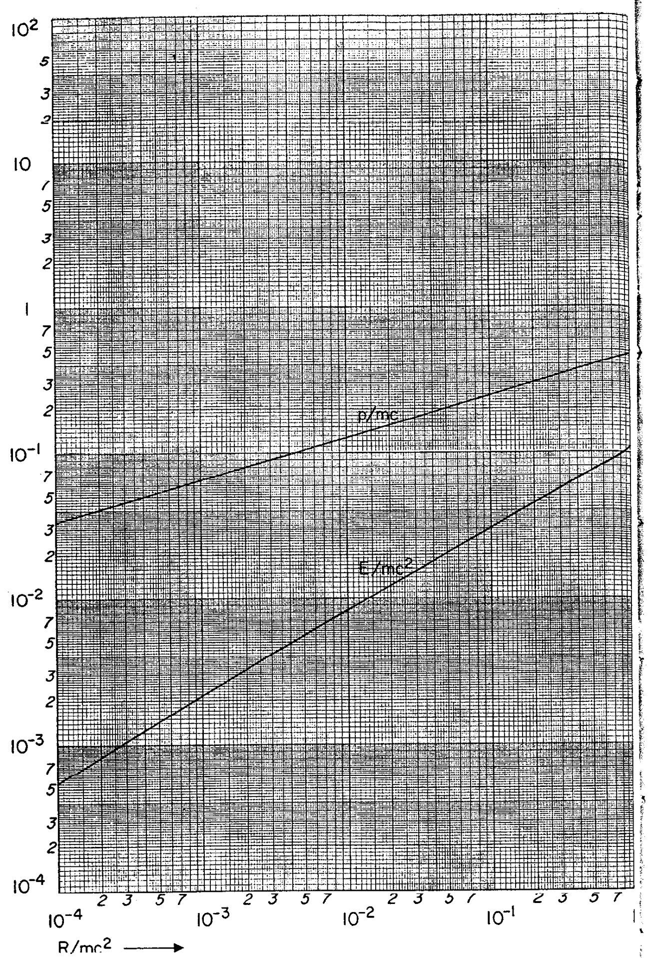


**Fig. 2.9.2.** $E/mc^2$ and $p/mc$ as functions of $R/mc^2$ in air; $p$ is the momentum, $E$ the kinetic energy, $R$ the range, and $m$ the mass. The rest energy $mc^2$ is measured in units of $10^8$ ev. The curves are valid for any particle of unit charge heavier than an electron, provided energy losses other than collision losses are negligible.



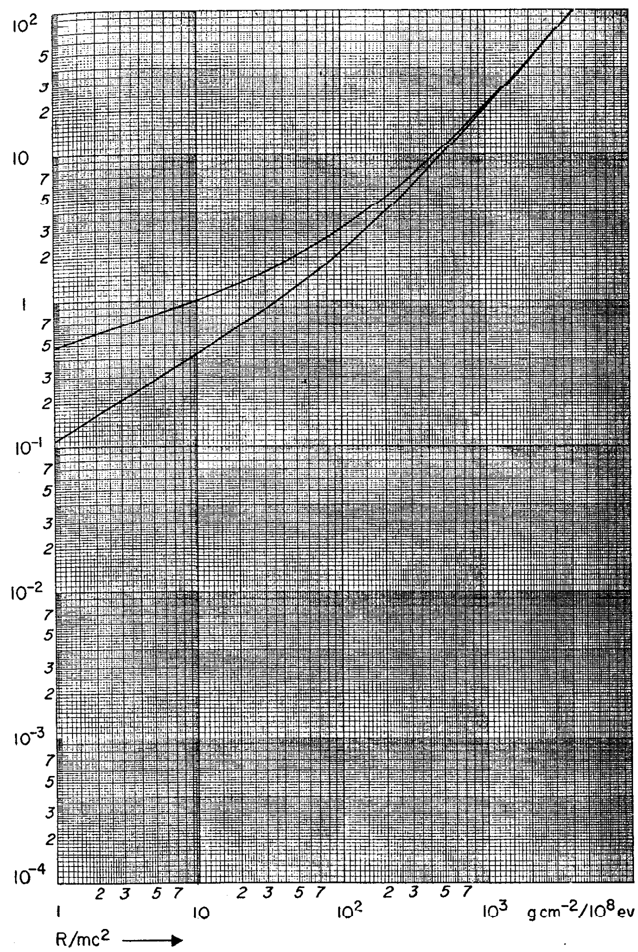


**Fig. 2.9.2.**

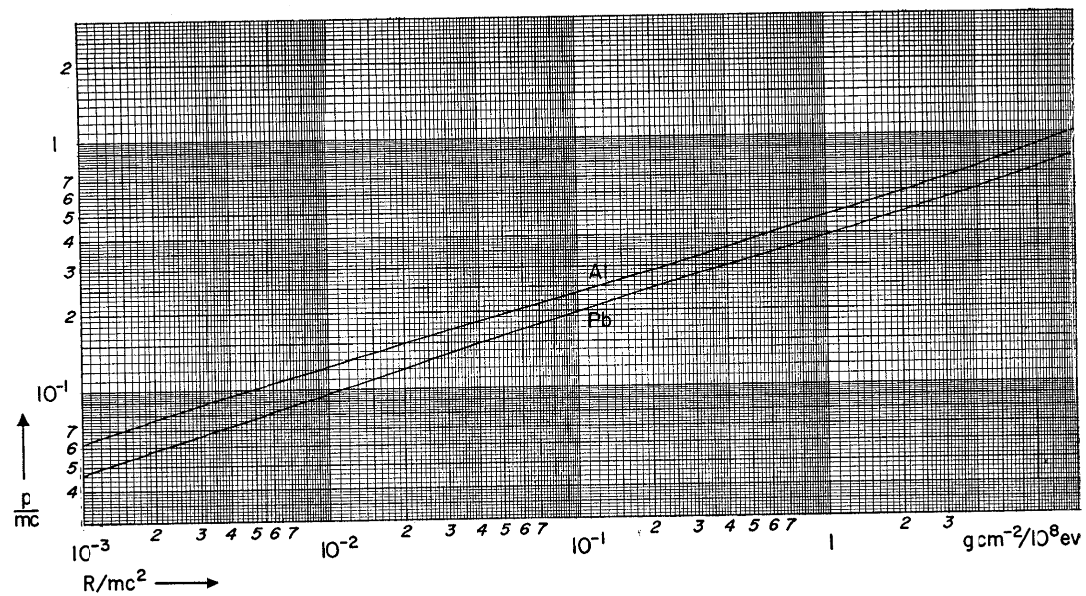


Fig. 2.9.3.



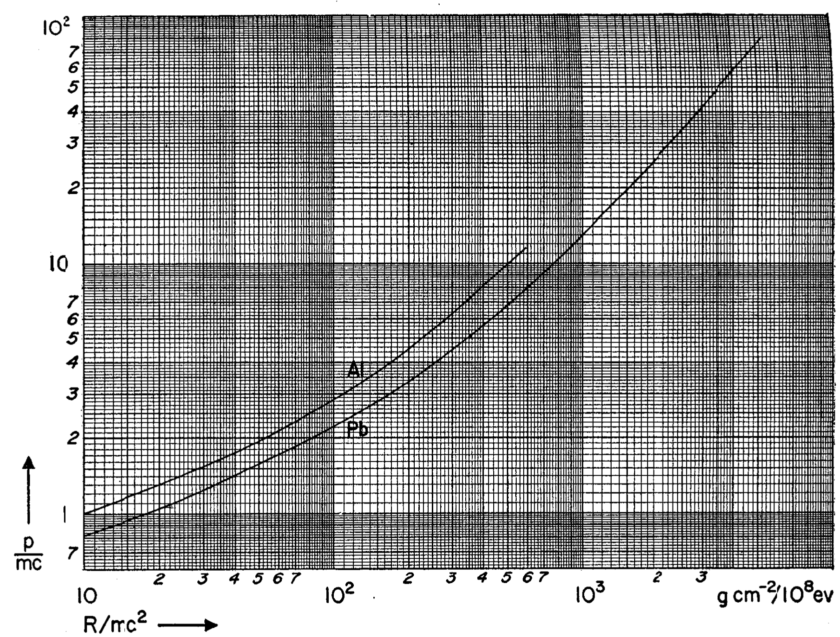


**Fig. 2.9.3.** $p/mc$ as a function of $R/mc^2$ in aluminum and lead; $p$ is the momentum, $R$ the range, and $m$ the mass. The rest energy $mc^2$ is measured in units of $10^8$ ev. The curves are valid for any singly charged particle heavier than an electron, provided energy losses other than collision losses are negligible.

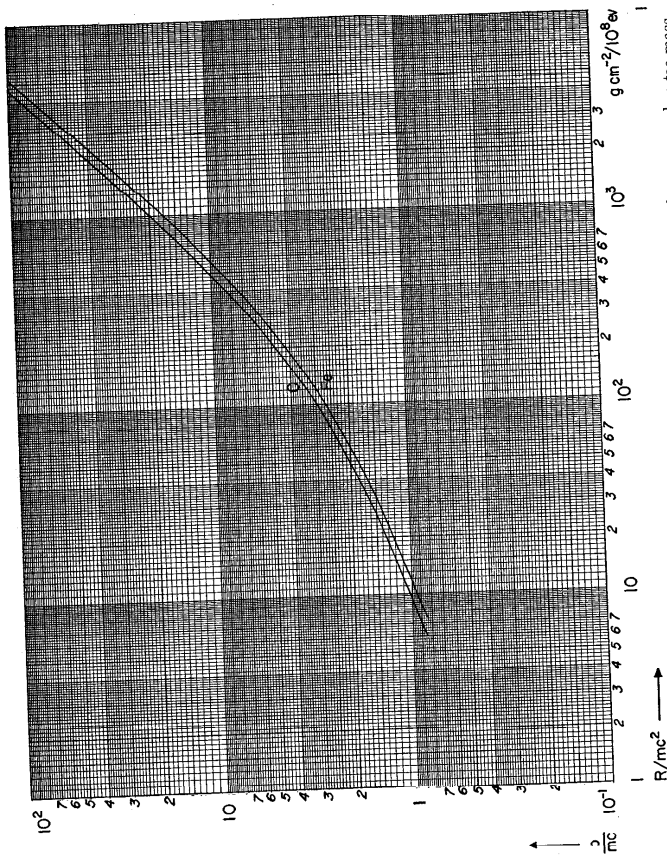


**Fig. 2.9.4.** $p/mc$ as a function of $R/mc^2$ in carbon and iron; $p$ is the momentum, $R$ the range, and $m$ the mass. The rest energy $mc^2$ is measured in units of $10^8$ ev. The curves are valid for any singly charged particle heavier than an electron, provided energy losses other than collision losses are negligible.





The curves in Figs. 3 and 4 give the range-momentum relation for carbon, aluminum, iron, and lead. The curve for aluminum was taken from Smith; it is not corrected for density effects. The curves for carbon and iron were taken from Wick (WGC43); they are corrected for density effect. The lead curve was also taken from Wick (WGC43) and extended to lower momenta with the results of Wheeler and Ladenburg. This curve is corrected for density effect.

**2.10.** ***The specific ionization.*** Collisions between charged particles and atoms of gases result partly in excitation, partly in ionization of the atoms.* Most of the electrons ejected in the ionization processes have energies very small compared with the energy of the primary particle, yet larger than the ionization energy of the atoms. They are able, therefore, to produce several ion pairs before coming to rest.

In the discussion of experimental data we shall encounter cases where the results of the observations are related to the number of ion pairs produced in the gas by the primary particle directly. We shall also encounter cases where the secondary ionization (or part of it) is measured along with the primary ionization. For a complete description of the ionizing effects of charged particles, it is thus appropriate to consider separately the following quantities:

(1) The *primary specific ionization,* $j_p$. This is the average number of collisions per g cm$^{-2}$ that result in the ejection of an electron from an atom.

(2) The *total specific ionization,* $j_t$. This is the total average number of ion pairs per g cm$^{-2}$ produced by the primary particle, by all of its secondary electrons, and by whatever tertiary rays may be produced by the secondary electrons.

(3) The *probable specific ionization,* $j_\eta$. This is the total average number of ion pairs per g cm$^{-2}$ produced by the primary particle and by all of its secondary electrons that are ejected from the atoms with an energy smaller than $\eta$.

The theory of the primary specific ionization was developed by Bethe.† In this theory one can consider the ionizing particle as a point charge and disregard its spin and magnetic moment.‡ The result, for a particle of charge $ze$, is expressed by the equation:

$$j_p = \frac{2Cm_ec^2}{\beta^2} z^2 \frac{r}{I_0}\left[\ln \frac{2m_ec^2\beta^2}{(1-\beta^2)I_0} + s - \beta^2\right], \qquad (1)$$

* The emission of Cerenkov radiation (§ 2.6) in gases is negligible, except at very high energies.

† See Bethe's article in H. B. der Phys., edited by Geiger and Sheel, Julius Springer Verlag, Berlin 1933, Vol. 24.1, p. 518.

‡ This is so because the great majority of ionizing events can be classified as "distant" collisions. Close collisions, in which the structure of the particle plays a part, can be disregarded in computing the *number* of ionizing events. The spin and the magnetic moment have also a negligible effect on the total collision loss. Only in phenomena where the number of high-energy secondaries is important can one distinguish the behavior of particles with different spins and different magnetic moments (see § 6.13).

where $I_0$ is the ionization potential of the *outer shell* of the atom, $r$ and $s$ are dimensionless constants that depend on the atomic structure, and the other symbols have the usual meanings. The constants $r$ and $s$ have been computed theoretically only for the case of hydrogen ($I_0 = 13.5$) and their values are: $r = 0.285$, $s = 3.04$.

There is reason to believe that $s$ does not change very much with atomic number, and, since it is small compared with the logarithmic term, it can be considered as independent of $Z$ in first approximation. Thus one can use Eq. (1) with $s = 3.04$ for computing the dependence of $j_p$ on $\beta$ for all substances. For substances different from hydrogen, the constant $r$, giving the absolute value of $j_p$, can be determined empirically from the measurement of the specific ionization at one known velocity.

Equation (1) indicates that, for a given velocity, the primary specific ionization is independent of the mass of the ionizing particle, but is proportional to the square of its electric charge $z$. In other words, the ratio $j_p/z^2$ is the same function of the velocity for all particles. Since the velocity of a particle is a function of $p/m$ (or $E/m$), the ratio $j_p/z^2$ depends only on the ratio of $p/m$ (or $E/m$).

The functional dependence of $j_p$ on $\beta$ (as well as its functional dependence on $p/m$, or $E/m$) is very similar to that of $k_{\text{col}(<\eta)}$ (see Eq. 2.5.1 and Fig. 2.5.1) and is explained by similar arguments. Methods for the measurement of the primary specific ionization will be described in §§ 3.8 and 3.11.

For the computation of the *total* specific ionization, $j_t$, and of the *probable* specific ionization, $j_\eta$, one assumes that the *total number of ion pairs produced when a high-energy particle is completely absorbed by a gas is proportional to the energy of the particle* and, for a given energy, is independent of the type of particle. Thus one can define a quantity $V_0$ which depends only on the nature of the gas, and which represents the *average energy expended per ion pair produced*. The quantities $j_t$ and $j_\eta$ are then directly related to the total and the probable energy losses, $k_{\text{col}}$ and $k_{\text{col}(<\eta)}$, as defined in § 2.5, and are given by the following equations:

$$j_t(E) = \frac{k_{\text{col}}(E)}{V_0}, \qquad j_\eta(E) = \frac{k_{\text{col}(>\eta)}(E)}{V_0}. \tag{2}$$

Because of the close relation between $j_t$ and $k_{\text{col}}$, the latter quantity is often called *ionization loss*, as already mentioned.

Table 1, taken from the volume on *Ionization and Counters* by Rossi and Staub* summarizes various experimental determinations of $V_0$ in different gases and for different kinds of particles. The data listed in this table bear out the approximate validity of the assumptions made. Indeed, the differences shown in Table 1 between the values of $V_0$ relative to any one substance are probably due, to a large extent, to experimental errors. In this connection one may recall the results of measurements by Jesse, Forstat, and Sanduskis (JWP50), made with $\alpha$-particles of accurately known energies, ranging from 5 to 9 Mev. In argon these experimenters found that the total number of ion pairs is proportional to the energy of the $\alpha$-particle to an accuracy better than 0.5 per cent. The same authors, analyzing data obtained by Stetter (SA43), found that $V_0$ for air changes by about 2 per cent over the same range of $\alpha$-particle energies.

* McGraw-Hill Book Co., Inc., New York, 1949, p. 227.



**Table 1. Energy $V_0$ spent in the formation of one ion pair**

| Gas | $V_0$ in ev | Particle and Energy | Reference |
|---|---|---|---|
| Air | 32.0 | electrons, 0.3 Mev | (GLH44) |
| Air | 36.0 | protons, 2.5–7.5 Mev | |
| Air | 35.1 | $\alpha$-particles, 7.8 Mev | |
| Air | 35.6 | $\alpha$-particles, 5.3 Mev | |
| $H_2$ | 36.0 | $\alpha$-particles, 5.3 Mev | |
| He | 31.0 | $\alpha$-particles, 5.3 Mev | |
| CO | 34.7 | $\alpha$-particles, 5.3 Mev | (ScK39) |
| $CO_2$ | 34.6 | $\alpha$-particles, 5.3 Mev | |
| $C_2H_4$ | 27.6 | $\alpha$-particles, 5.3 Mev | |
| Ne | 27.8 | $\alpha$-particles, 5.3 Mev | (ScK39) |
| A | 24.9 | $\alpha$-particles, 5.3 Mev | |
| A | 26.9 | electrons, 17.4 Kev | (NDB46) |
| Kr | 23.0 | $\alpha$-particles, 5.3 Mev | (ScK39) |
| Xe | 21.4 | $\alpha$-particles, 1.3 Mev | (GRW25) |

The approximate proportionality of the ionization to the energy dissipation may be theoretically justified with the following considerations. When the primary particle is absorbed by the gas, its energy is spent in exciting the atoms and in producing secondary rays partly by collisions and partly by radiation phenomena. The secondary rays will excite more atoms and produce tertiary electrons and photons, and so on. It is clear that an electron will continue to lose energy by inelastic collisions as long as its energy is larger than the lowest excitation potential of the atoms, and that a photon will readily be absorbed by photoelectric effect as long as its energy is larger than the minimum ionization potential. On the other hand, if an atom is brought to a highly excited state by inelastic collision of an electron or by absorption of a quantum, it promptly loses the excitation energy by emitting a photon or an Auger electron. It is seen that the degradation of the initial energy continues until there remain only a certain number of atoms in the lowest ionized level and a certain number of electrons and photons of a few ev energy. The fraction of the initial energy that is used in producing ionization depends essentially on the relative probability for excitation and ionization of the atoms. It is not appreciably affected by the nature of the primary particle nor by its energy, because most of the ionization and excitation processes are produced by secondary electrons of small energy. These considerations make us confident that the approximate proportionality between energy loss and specific ionization, which is experimentally established for energies up to a few Mev, will still hold for particles of much larger energy.

A method for the measurement of the probable specific ionization will be described in § 3.12.

2.11. ***Theoretical expressions for the radiation probability and for the average radiation loss of electrons.*** We have pointed out in § 2.1 that the emission of photons by charged particles (*bremsstrahlung*) is connected with the deflection of their trajectories in the electric fields of nuclei. The distance from the nucleus at which radiation phenomena occur plays an essential role in the development of the theory. If this distance is large compared with the nuclear radius and small compared with the atomic radius, the field acting on the particle during the radiation process can be considered as the Coulomb field of a point charge $Ze$ at the center of the nucleus. If the distance is of the order of the atomic radius, or larger, the screening of the electric field of the nucleus by the outer electrons must be taken into account. If, lastly, the distance is of the order of the nuclear radius, the electric field of the nucleus can no longer be considered as that of a point charge.

It turns out that radiation processes of electrons take place at distances from the nucleus that are large compared with the nuclear radius. Thus the nucleus can always be considered as a point charge. However, the screening effect of the outer electrons is often important. This effect has been calculated by Bethe and Heitler (BHA34) on the basis of the Fermi-Thomas model of the atom. The theory indicates that the influence of screening on a radiation process in which an electron of kinetic energy $E$ produces a photon of energy $E'$ is determined by the quantity:

$$\gamma = 100 \frac{m_e c^2}{U} \frac{v}{1 - v} Z^{-1/3}; \tag{1}$$

where, for short, we have introduced the following symbols for the total electron energy and the fractional photon energy:

$$U = E + m_e c^2, \qquad v = \frac{E'}{U}. \tag{2}$$

The screening effect is *greater* the *smaller* is $\gamma$. For $\gamma \gg 1$, screening can be practically neglected. The case of $\gamma \approx 0$ will be described as "complete screening." For a given value of $v$, $\gamma$ decreases with increasing $U$. Thus, if the primary energy is large enough, screening may be considered as "complete" for all energies of the emitted photons.

Let $\Phi_{\rm rad}(E,E')\,dE'\,dx$ be the probability for an electron of kinetic energy $E$ traversing a thickness of $dx$ g cm$^{-2}$ to emit a photon with energy in $dE'$ at $E'$. $\Phi_{\rm rad}(E,E')$ is called the differential radiation probability of electrons. The theoretical expression for $\Phi_{\rm rad}$ may be written in the following form:

$$\Phi_{\rm rad}(E,E')\,dE' = 4\alpha \frac{N}{A} Z^2 r_e^2 \frac{dE'}{E'} F(U,v), \tag{3}$$



where $\alpha = e^2/\hbar c = 1/137$, $F$ is a slowly varying function of $U$ and $v$, and the other symbols have the same meanings as before. For values of $U$ large compared with $m_e c^2$, the function $F$ is given by the following equations, each valid in a different range of $\gamma$:

*No screening* ($\gamma \gg 1$):

$$F(U,v) = \left[1 + (1-v)^2 - \frac{2}{3}(1-v)\right]\left[\ln\left(\frac{2U}{m_e c^2}\frac{1-v}{v}\right) - \frac{1}{2}\right]; \tag{4}$$

*Complete screening* ($\gamma \approx 0$):

$$F(U,v) = \left[1 + (1-v)^2 - \frac{2}{3}(1-v)\right] \ln 183 Z^{-1/3} + \frac{1}{9}(1-v); \tag{5}$$

*Intermediate cases*

($\gamma < 2$):

$$F(U,v) = [1 + (1-v)^2]\left[\frac{f_1(\gamma)}{4} - \frac{1}{3}\ln Z\right] - \frac{2}{3}(1-v)\left[\frac{f_2(\gamma)}{4} - \frac{1}{3}\ln Z\right]; \tag{6}$$

($2 < \gamma < 15$):

$$F(U,v) = \left[1 + (1-v)^2 - \frac{2}{3}(1-v)\right]\left[\ln\left(\frac{2U}{m_e c^2}\frac{1-v}{v}\right) - \frac{1}{2} - c(\gamma)^{-1/2}\right]. \tag{7}$$

The functions $f_1(\gamma)$, $f_2(\gamma)$, and $c(\gamma)$ are given in Fig. 1 and Table 1.

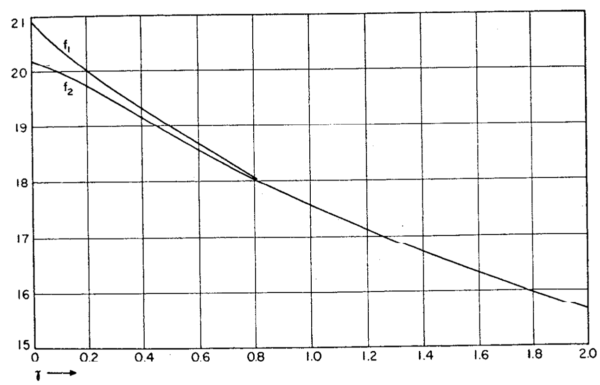


**Fig. 2.11.1.** The functions $f_1(\gamma)$ and $f_2(\gamma)$ in Eqs. (2.11.6) and (2.19.7). From Bethe and Heitler (BHA34).

**Table 1. Numerical values of the function $c(\gamma)$ in eqs. (7) and (2.19.8) from Bethe and Heitler (BHA34)**

| $\gamma$ | 2 | 2.5 | 3 | 4 | 5 | 6 | 8 | 10 | 15 |
|---|---|---|---|---|---|---|---|---|---|
| $c(\gamma)$ | 0.21 | 0.16 | 0.13 | 0.09 | 0.065 | 0.05 | 0.03 | 0.02 | 0.01 |

The average radiation loss of electrons per g cm$^{-2}$ is:

$$k_{\mathrm{rad}}(E) = \int_0^E E' \Phi_{\mathrm{rad}}(E,E')\, dE'. \tag{8}$$

If the energy of the electron is sufficiently small, $\gamma \gg 1$ for most photon energies and one may compute the radiation loss by means of Eq. (4) (no screening). On the other hand, if the electron energy is sufficiently large, $\gamma \ll 1$ for most photon energies and one may compute the radiation loss by means of Eq. (5) (complete screening). Thus one obtains:

for $mc^2 \ll U \ll 137 m_e c^2 Z^{-1/3}$:

$$k_{\mathrm{rad}}(E) = 4\alpha \frac{N}{A} Z^2 r_e^2 U \ln\left(\frac{2U}{m_e c^2} - \frac{1}{3}\right); \tag{9}$$

for $U \gg 137 m_e c^2 Z^{-1/3}$:

$$k_{\mathrm{rad}}(E) = 4\alpha \frac{N}{A} Z^2 r_e^2 E \left[\ln (183 Z^{-1/3}) + \frac{1}{18}\right]; \tag{10}$$

(Note that in this case the distinction between $E$ and $U$ is insignificant.)

For the intermediate cases the integral in Eq. (8) must be evaluated numerically.

The derivation of Eqs. (4) through (10) is based on Born's approximation. For electrons of relativistic velocities, this approximation is valid if $Z/137 \ll 1$, a condition well verified for light elements but not for heavy elements. It can be shown that the error introduced by the use of Born's approximation is proportional to $(Z/137)^2$. Its absolute value can be determined only by comparison with experimental data.

In dealing with radiation phenomena it is convenient to measure thickness in terms of a thickness, $X_0$, that is called *radiation length* and is defined by the equation:

$$\frac{1}{X_0} = 4\alpha \frac{N}{A} Z^2 r_e^2 \ln (183 Z^{-1/3}). \tag{11}$$

Let $t = x/X_0$. We shall introduce the *differential radiation probability per radiation length:*

$$\varphi_{\mathrm{rad}}(E,E') = X_0 \Phi_{\mathrm{rad}}(E,E'), \tag{12}$$

and the *average fractional energy loss per radiation length:*

$$-\frac{1}{U}\frac{dE}{dt} = \frac{X_0 k_{\mathrm{rad}}(E)}{U}. \tag{13}$$

The function $E'\varphi_{\mathrm{rad}}(E,E')$ is plotted against $v = E'/U$ for various values of $U$ and two substances (air and lead) in Figs. 2 and 3. The average fractional energy loss defined by Eq. (13) is given as a function of energy in Fig. 4. It appears that the description of radiation phenomena is only slightly dependent on atomic number when thicknesses are measured in radiation lengths. Moreover, the dependence on atomic number becomes less pronounced with increasing energy.



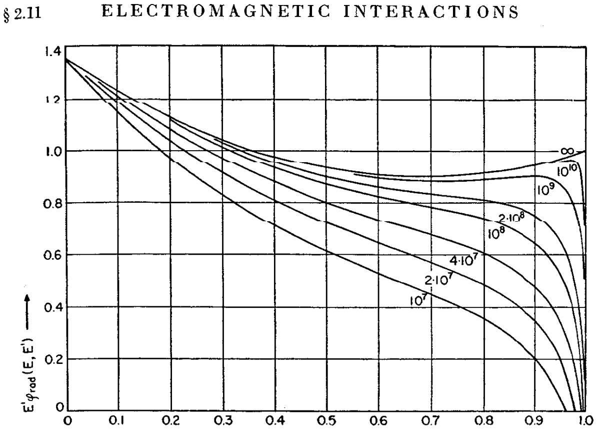


**Fig. 2.11.2.** Differential radiation probability per radiation length of *air* for electrons of various energies. Abscissa: $v = E'/U$. Ordinate: $E'\varphi_{\mathrm{rad}}(E,E')$. The numbers attached to the curves indicate the total energy, $U = E + m_e c^2$, of the primary electron. From Rossi and Greisen (RB41.1).

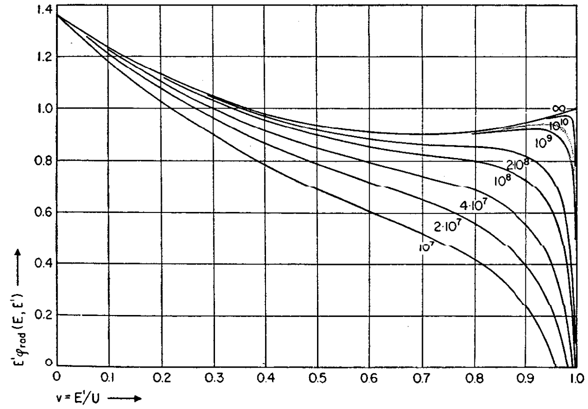


**Fig. 2.11.3.** Differential radiation probability per radiation length of *lead* for electrons of various energies. Abscissa: $v = E'/U$. Ordinate: $E'\varphi_{\mathrm{rad}}(E,E')$. The numbers attached to the curves indicate the total energy, $U$, of the primary electron. From Rossi and Greisen (RB41.1).

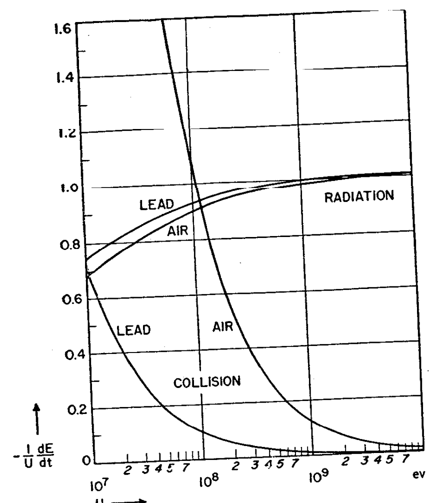


**Fig. 2.11.4.** Fractional energy loss by collision, $-\frac{1}{U}\left(\frac{dE}{dt}\right)_{col}$, and fractional energy loss by radiation, $-\frac{1}{U}\left(\frac{dE}{dt}\right)_{rad}$, for electrons, per radiation length of *air* or *lead*. From Rossi and Griesen (RB41.1).

In the limiting case of complete screening, Eq. (5) applies and the expression for $\varphi_{rad}$ may be written as follows:

$$\varphi_{rad}(E,E')\,dE' = \psi_{rad}(v)\,dv, \tag{14}$$

where $\psi_{rad}$ *is a function of the fractional photon energy, v, alone:*

$$\psi_{rad}(v) = \frac{1}{v}\left[1 + (1-v)^2 - (1-v)\left(\frac{2}{3} - 2b\right)\right]. \tag{15}$$

In this equation $b$ has the value:

$$b = \frac{1}{18 \ln (183Z^{-1/3})}. \tag{16}$$

$b$ is small compared with one and its value ranges only from 0.012 to 0.015 when $Z$ changes from 7.3 (air) to 82 (lead). One will not make any appreciable error by taking $b = 0.0135$ for all elements. Thus at the limit for complete screening the differential radiation probability per radiation length has almost exactly the same expression for all substances. In the same limiting case, the average fractional energy loss has the value:

$$-\frac{1}{U}\frac{dE}{dt} = 1 + b, \tag{17}$$



and is therefore independent of energy and almost identical to 1 (note that in this case $m_ec^2$ is negligible compared with $E$).

The above results are also valid for substances other than pure elements, provided one takes

$$\frac{1}{X_0} = \frac{p_1}{X_1} + \frac{p_2}{X_2}, \tag{18}$$

where $p_1, p_2 \ldots$, are the fractional weights of the various components and $X_1, X_2 \ldots$, the corresponding radiation lengths.

In a radiation process, the energy and the momentum of the incident particle are subdivided among three particles, namely the primary particle itself, the emitted photon, and the recoil nucleus. The nucleus, because of its large mass, does not acquire any large portion of the energy, but may acquire a transverse momentum comparable with the transverse momenta of the other two particles. Therefore the conservation laws of momentum and energy do not furnish a relation between energy and angle of emission of the photon in a radiation process, and in fact an electron of a given energy can emit photons of the same energy in different directions.

Stearns (StM49) has computed the root mean square angle of emission of photons in radiation processes of electrons. His result may be expressed by

$$\langle\Theta^2\rangle_{av}^{1/2} = q(U,E',Z)\,\frac{m_ec^2}{U}\ln\frac{U}{m_ec^2}, \tag{19}$$

where $q(U,E',Z)$ is a function of the atomic number $Z$ of the substance, the total energy $U$ of the electron and the energy $E'$ of the photons. The function $q$ is always of the order of magnitude of unity and depends primarily on the ratio $v = E'/U$. The curves in Fig. 5 giving $q$ as a function of $v = E'/U$ for $Z = 4$, $Z = 30$, and $Z = 90$ respectively are accurate within 3 per cent for any value of $E$ between 50 and 300 Mev. A point in the same figure shows the value of $q$ for $Z = 90$, $E'/U = \frac{1}{2}$, $U = 5000$ Mev.*

So far, in the computation of the radiation probability, we have only considered the effect of the electric field of nuclei and have disregarded that of the electric field of atomic electrons. The theory of this latter effect involves two separate problems: (1) computation of the radiation probability in the collisions of electrons with free electrons; (2) computation of the influence of the atomic structure on the radiative collisions with atomic electrons. The second problem has been studied in detail by Wheeler and Lamb (WJA39), under the assumption that the radiation probability of an electron in the field of a free electron is the same as the radiation probability in the field of a proton. The reason for this simple result is that the main contribution to radiative collisions comes from processes with small momentum transfers, and hence the incoming particle does not differentiate between heavy and light target particles.

* These results were obtained under the assumption that the maximum angle of photon emission is 20°. Stearns estimates that in most cases the error thus introduced does not exceed 10 per cent.

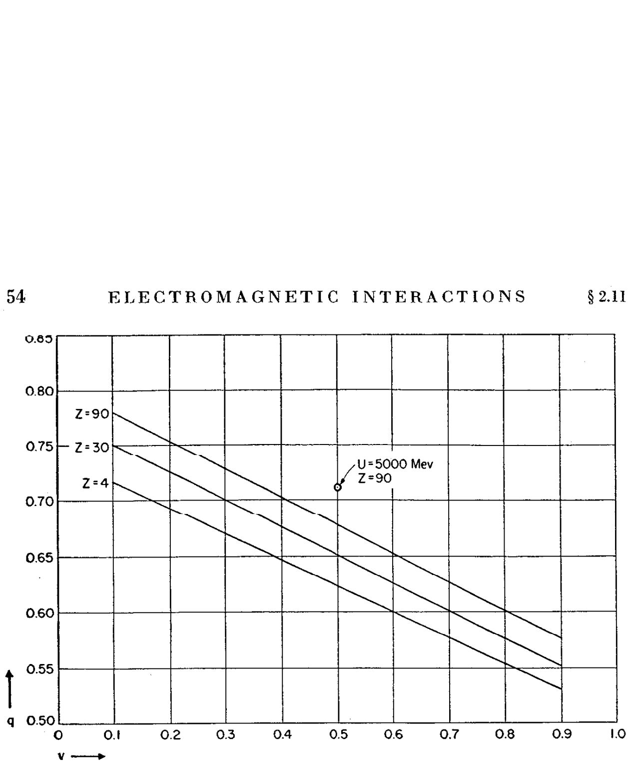


**Fig. 2.11.5.** The quantity $q$ in Eq. (2.11.19) plotted as a function of $v = E'/U$. The three curves refer to elements with atomic numbers 4, 30, and 90 respectively and are valid for 50 Mev $< U <$ 300 Mev. The circle represents the value of $q$ for $Z = 90$ and $U = 5000$ Mev. From Stearns (StM49).

Wheeler and Lamb have shown that the atomic structure affects the probability of radiative collisions with electrons and nuclei in a somewhat different manner. For our purposes, however, we may neglect this difference (which, incidentally, depends on the energy of the incident particle and on the atomic number of the substance). Under this approximation each atomic electron behaves like a singly charged nucleus in radiation phenomena, and one may take into account the effect of atomic electrons by simply changing $Z^2$ into $Z^2 + Z \equiv Z(Z + 1)$ in the expressions for the radiation probability and for the average radiation loss per g cm$^{-2}$. Alternately we may leave the expressions of the probabilities per radiation length unchanged and replace $Z^2$ with $Z(Z + 1)$ in the formula for the radiation length. This means replacing Eq. (11) with the following:

$$\frac{1}{X_0} = 4\alpha \frac{N}{A} Z(Z + 1) r_e^2 \ln (183 Z^{-1/3}). \qquad (20)$$

Numerical values of the radiation length in various substances, computed from Eq. (11) and from Eq. (20) are given in Table 2.

The radiation theory has been crudely tested by experiments on the energy loss of cosmic-ray electrons (§ 6.4) and on the development of cascade showers (§§ 6.7, 6.8, 6.10). Measurements on the $\gamma$-ray spectrum produced by artificially accelerated electrons (§ 6.2) have provided an



**Table 2. The radiation length, $X_0$, in various substances**

[Computed from Eq. (2.11.11)—which neglects the effect of atomic electrons, and according to Eq. (2.11.20)—which takes this effect into account.]

| Substance | Z | A | $X_0$ (g cm$^{-2}$) From Eq. (2.11.11) | From Eq. (2.11.20) |
|---|---|---|---|---|
| Carbon | 6 | 12 | 52 | 44.6 |
| Nitrogen | 7 | 14 | 45 | 39.4 |
| Oxygen | 8 | 16 | 39.7 | 35.3 |
| Aluminum | 13 | 27 | 26.3 | 24.4 |
| Argon | 18 | 39.9 | 20.8 | 19.7 |
| Iron | 26 | 55.84 | 14.4 | 13.9 |
| Copper | 29 | 63.57 | 13.3 | 12.9 |
| Lead | 82 | 207.2 | 5.90 | 5.83 |
| Air | 7.37 | 14.78 | 42.8 | 37.7 |
| Water | 7.23 | 14.3 | 42.2 | 37.1 |

accurate check of the dependence of the radiation probability on the photon energy.

**2.12.** ***Semi-quantitative justification of the theoretical results concerning radiation phenomena.*** In this section we shall present a semi-quantitative treatment of the radiation phenomenon by charged particles of arbitrary mass. The only purpose of this treatment is to provide a physical interpretation for the dependence of the radiation probability on the energy of the particles and the properties of the medium in which they move (see § 2.11) as well as for the difference in the behavior of electrons and heavier particles with regard to radiation phenomena (see § 2.14 below). Therefore we shall systematically disregard all purely numerical factors of the order of unity.

Consider a particle of charge $e$, mass $m$, and velocity $\beta c$ moving past a nucleus of charge $Ze$. Suppose $1 - \beta \ll 1$. Let $b$ be the impact parameter. Consider the nucleus as a point charge and assume that its mass is large compared with $m$, so that its motion during the collision may be neglected. In the proximity of the nucleus, the moving particle undergoes an acceleration and therefore radiates energy. According to classical electrodynamics, the energy radiated per unit time is given by the well known expression:

$$\frac{2e^2}{3c^3} a^2 \approx \frac{e^2}{c^3} a^2, \qquad (1)$$

where $a$ is the acceleration of the particle.

This formula, however, is valid only when the radiating particle has a velocity small compared with the velocity of light. Therefore, we shall consider the phenomenon in a frame of reference (frame $A'$) in which the particle is initially at rest and in which the nucleus moves with velocity $-\beta$. In this frame of reference, the velocity of the particle during the radiation process usually remains sufficiently small so that the energy radiated can be calculated by means of the expression (1). The maximum value of the acceleration is given by the following equation [see Eq. (2.4.4)]:

$$a'_{\max} = \frac{Ze^2}{mb^2}\frac{1}{\sqrt{1-\beta^2}}. \tag{2}$$

The acceleration has a magnitude of the order of $a'_{\max}$ for a time of the order of:

$$\tau' = \frac{2b}{\beta c}\sqrt{1-\beta^2}, \tag{3}$$

[see Eq. (2.4.5)] or, since we assume $1 - \beta \ll 1$ and neglect numerical factors of the order of unity,

$$\tau' = \frac{b}{c}\sqrt{1-\beta^2}. \tag{4}$$

Therefore, the energy radiated during the collision has, in the frame of reference $A'$, the approximate expression:

$$Q' = \frac{e^2}{c^3}(a'_{\max})^2\tau' = \frac{Z^2e^6}{m^2c^4}\frac{1}{\sqrt{1-\beta^2}}\frac{1}{b^3}. \tag{5}$$

Consider the Fourier analysis of the radiation field. Since the functions describing this field vary only slightly in time intervals small compared with the collision time, $\tau'$, the frequency spectrum will rapidly drop to zero for frequencies larger than:

$$\nu'_1 = \frac{1}{\tau'} = \frac{c}{b\sqrt{1-\beta^2}}. \tag{6}$$

Calculations show that in a crude approximation the frequency spectrum may be represented by a constant from zero to $\nu'_1$.

In order to determine the value, $Q$, of the energy radiated during the collision in the frame of reference in which the nucleus is at rest (frame $A$) consider that, in $A'$, the intensity of the radiation emitted at an angle $\psi$ to the direction of the acceleration is proportional to $\sin^2\psi$. Therefore the symmetry of the radiation field is such that its total momentum vanishes in $A'$. From Appendix 2b it then follows that:

$$Q = \frac{Q'}{\sqrt{1-\beta^2}} = \frac{Z^2e^6}{m^2c^4}\frac{1}{(1-\beta^2)}\frac{1}{b^3}. \tag{7}$$

One obtains the frequency distribution in $A$ from that in $A'$ by means of the formula for the relativistic Doppler effect:

$$\nu = \frac{1+\beta\cos\theta'}{\sqrt{1-\beta^2}}\,\nu', \tag{8}$$

where $\theta'$ is the angle of emission of the electromagnetic wave in $A'$. It follows that the maximum frequency, $\nu$, in the frame $A$ is of the order of:



$$\nu_1 \approx \frac{\nu'_1}{\sqrt{1-\beta^2}} \approx \frac{c}{b(1-\beta^2)} = \left(\frac{U}{mc^2}\right)^2\frac{c}{b}. \tag{9}$$

Since the frequency spectrum is roughly constant up to $\nu'_1$ in $A'$, it will be roughly constant up to $\nu_1$ in $A$. Thus the energy per unit frequency interval is the ratio of the total energy emitted to the maximum frequency:

$$\frac{dQ}{d\nu} = \frac{Q}{\nu_1} = \frac{Z^2e^6}{m^2c^5}\frac{1}{b^2}. \tag{10}$$

Equation (10) is the result of a classical computation. In the language of quantum theory, $(1/h)(dQ/d\nu)$ represents the number of photons per unit energy interval. Since a particle cannot radiate a photon of energy larger than its own kinetic energy, $E$, the classical expression (10) cannot hold for frequencies larger than:

$$\nu_2 = \frac{E}{h} \approx \frac{U}{h}. \tag{11}$$

On the other hand, it is reasonable to assume that Eq. (10) is approximately valid for frequencies small compared with $\nu_2$. These considerations lead to the conclusion that one can roughly approximate $dQ/d\nu$ with a function that has the constant value (10) for frequencies less than $\nu_1$ or $\nu_2$, whichever is smaller, and is zero for larger frequencies.

In order to determine the differential radiation probability, $\Phi_{\rm rad}(E, E')$, as defined at the beginning of § 11, note that $E'\Phi_{\rm rad}(E,E')\,dE'\,dx$ represents the average energy radiated into photons of energy between $E'$ and $E' + dE'$ by a particle traversing a thickness $dx$ of the absorber. In this thickness, the number of collisions with impact parameters between $b$ and $b + db$ is given by: $(N/A)\,dx\,2\pi b\,db$. Therefore Eq. (10) yields:

$$E'\Phi_{\rm rad}(E,E')\,dE' = \frac{dE'}{h}\int_{b_{\min}}^{b_{\max}}\frac{N}{A}\,2\pi b\,db\,\frac{Z^2e^6}{m^2c^5}\frac{1}{b^2},$$

or

$$E'\Phi_{\rm rad}(E,E') = \alpha\frac{N}{A}Z^2\left(\frac{m_e}{m}\right)^2 r_e^2\ln\frac{b_{\max}}{b_{\min}}, \tag{12}$$

where $b_{\min}$ and $b_{\max}$ represent the minimum and maximum value of the impact parameter, respectively, for which Eq. (10) is valid. In the case of electrons, Eq. (12) reduces to the following:

$$\Phi_{\rm rad}(E,E') = \alpha\frac{N}{A}Z^2r_e^2\frac{1}{E'}\ln\left(\frac{b_{\max}}{b_{\min}}\right) \tag{13}$$

This equation is similar to Eq. (2.11.3), the missing factor 4 being explained by the neglect of numerical coefficients in the derivation of Eq. (13).

Both Eqs. (2.11.3) and (13) show that $\Phi_{\rm rad}$ is proportional to $Z^2$. The reason for this is clearly indicated by our derivation of Eq. (13); in fact, the radiation loss is proportional to the square of the acceleration, which, in turn, is proportional to $Z$.

In both Eq. (2.11.3) and Eq. (13) $\Phi_{\rm rad}$ depends on the energy $E'$ of the secondary photon mainly through the factor $1/E'$. This corresponds to the fact that classically the Fourier spectrum of the electromagnetic radiation emitted in each collision is roughly constant.

The slowly varying function $F(U,v)$ in Eq. (2.11.3) is replaced by $\ln(b_{\max}/b_{\min})$ in Eq. (13). One may find a justification for the behavior of $F(U,v)$ by assuming that $b_{\min}$ is constant and investigating the dependence of $b_{\max}$ on $E$ and $E'$. For this purpose, note that according to Eq. (9) photons of energy $E'$ can only be produced in collisions with impact parameters less than:

$$b_1 = \frac{ch}{E'}\left(\frac{U}{m_e c^2}\right)^2. \tag{14}$$

On the other hand, it is clear that practically no electromagnetic radiation will be emitted if the impact parameter is greater than the atomic radius, $r_a$, for when $b > r_a$ the atomic electrons shield the particle from the electric field of the nucleus. For $r_a$ one may use the approximate expression derived from the Fermi-Thomas model:*

$$r_a = \frac{1}{\alpha^2} r_e Z^{-1/3}. \tag{15}$$

When $b_1 > r_a$, $b_{\max}$ coincides with $r_a$; in other words, the maximum impact parameter is determined by the screening. In this case $\ln(b_{\max}/b_{\min})$ has a constant value. In agreement with this result, Eq. (2.11.5) shows that, in the case of "complete screening," $F(U,v)$ is approximately a constant. When $b_1 < r_a$, $b_{\max}$ coincides with $b_1$, and $\ln(b_{\max}/b_{\min})$ contains the term $\ln(U^2/E') = \ln(U/v)$. Equation (2.11.4) indicates a similar dependence of $F(U,v)$ on $U$ and $v$, in the case of no screening.

Our considerations also justify the criterion for the influence of screening expressed by Eq. (2.11.1). Equations (14) and (15) indicate that the ratio $r_a/b_1$ is proportional to $(E'/U^2)Z^{-1/3} = (v/U)Z^{-1/3}$. The quantity $\gamma$ in Eq. (2.11.1) shows a similar dependence on $v$, $U$, and $Z$. Thus, when $\gamma$ is small, $r_a < b_1$ and the maximum effective parameter is determined by screening; when $\gamma$ is large, $r_a > b_1$ and the maximum impact parameter is not affected by screening. The physical explanation of the fact that screening becomes increasingly important as the electron energy increases lies in the relativistic increase of the electric field intensity, which causes the effective distance of collisions to increase in proportion to the electron energy. The reason why the influence of screening decreases with increasing photon energy is that distant collisions are more effective in producing low-energy photons than high-energy photons.

Equation (2.11.10) shows that, for large values of the electron energy $E$, the average radiation loss is proportional to this energy. To justify this result, note that, when the energy is sufficiently large, the maximum frequency of photons emitted in a given collision is determined by the

* See, for example, L. I. Schiff, *Quantum Mechanics*, McGraw-Hill Book Co., Inc., New York (1949), pp. 271–273.



quantum-theoretical condition (11) for all values of $b$ up to the atomic radius $r_a$. Thus the energy radiated in a collision with impact parameter $b$ is:

$$\left(\frac{dQ}{d\nu}\right)\nu_2 = \frac{Z^2 e^6}{m_e^2 c^5}\frac{1}{b^2}\frac{E}{h} \tag{16}$$

for $b < r_a$, and is zero for $b > r_a$. The total average energy loss per unit thickness is therefore:

$$k_{\rm rad}(E) = \frac{N}{A}\int_{b_{\min}}^{r_a} 2\pi b\, db\, \frac{Z^2 e^6}{m_e^2 c^5}\frac{E}{h}\frac{1}{b^2},$$

or

$$k_{\rm rad}(E) = \alpha \frac{N}{A} Z^2 r_e^2 E \ln\left(\frac{r_a}{b_{\min}}\right). \tag{17}$$

Equation (17) is very similar to Eq. (2.11.10). The method followed for its derivation clearly shows the origin of the factor $E$. The relativistic deformation of the electric field of the nucleus in the frame of reference in which the electron is initially at rest introduces a factor $1/\sqrt{1-\beta^2} \approx U/m_e c^2$ into the expression of $Q'$, Eq. (5). The Lorentz transformation introduces another factor $1/\sqrt{1-\beta^2}$ and makes $Q$ (the energy radiated in the frame of reference of the nucleus) proportional to $U^2$, Eq. (7). This classical result is modified by the condition that the energy of a photon cannot exceed the kinetic energy of the electron by which it is produced. Therefore the maximum frequency of the radiation is proportional to $E$ instead of being proportional to $U^2$ as in the classical approximation. This replaces the factor $U^2$ with a factor $E$ and makes the energy emitted in each collision proportional to the energy of the electron.

Note that the logarithmic term in Eq. (17) becomes practically identical to that in Eq. (2.11.10) if one takes $b_{\min} = r_e/\alpha$. In fact, this gives:

$$\frac{b_{\max}}{b_{\min}} = \frac{1}{\alpha^2} r_e Z^{-1/3} \frac{\alpha}{r_e} = 137 Z^{-1/3}.$$

Thus the proper value for the minimum impact parameter in radiation phenomena of electrons is of the order of 137 times the classical electron radius.

The semi-quantitative treatment of the radiation process given in this section accounts also for the theoretical result concerning the angle of emission of photons (Eq. 2.11.19). Consider a photon emitted at an angle $\Theta'$ in the frame of reference $A'$. Since $\beta \approx 1$, Eqs. (A2.3) in Appendix 2 yield the following value for the angle of emission $\Theta$ in the frame of reference $A$:

$$\Theta = \sqrt{1-\beta^2}\,\frac{\sin\Theta'}{\cos\Theta' + 1}. \tag{18}$$

The mean absolute value of the fraction is of the order of unity and therefore the mean absolute value of $\Theta$ is of the order of $\sqrt{1-\beta^2} = m_e c^2/U$. It is important to note that the mean angle of emission depends on the energy of the emitting particle but is almost independent of the energy of the emitted photons.

**2.13. *Comparison between radiation loss and collision loss. Fluctuations in the radiation loss.*** As already pointed out, the average loss by radiation increases rapidly with increasing energy, while the average energy loss by collision is practically a constant. Thus, at large energies radiation losses are much more important than collision losses, while at small energies the reverse is true. In Fig. 2.11.4 the curves giving the fractional energy loss by collision in one radiation length of air and lead are drawn for comparison with the corresponding radiation losses. One sees that the energy at which the radiation loss overtakes the collision loss decreases with increasing atomic number.

Another characteristic difference between radiation losses and collision losses lies in the fact that the energy loss by radiation occurs in fewer and larger steps than the energy loss by collision. Thus, whereas all electrons of a given energy traversing a given thickness lose practically the same energy by collision, they undergo considerable straggling in their energy loss by radiation.

Bethe and Heitler (BHA34) have computed the probability $w(E_0,E,t)\,dE$ that an electron of initial energy $E_0$ has energy between $E$ and $E + dE$ after traversing a thickness of $t$ radiation lengths. In order to simplify the calculations, they used an approximate expression for the differential radiation probability; i.e., they substituted the following equation for Eq. (2.11.15):

$$\psi_{\text{rad}}(v)\,dv = -\frac{1}{\ln 2}\frac{dv}{\ln(1-v)}. \tag{1}$$

Neglecting collision loss, they arrived at the following expression for $w(E_0,E,t)$:

$$w(E_0,E,t)\,dE = \frac{dE}{E_0}\frac{[\ln(E_0/E)]^{(t/\ln 2)-1}}{\Gamma(t/\ln 2)}. \tag{2}$$

Eyges (EL49) has shown how Eq. (2) may be derived by a straightforward method that also allows the use of a more accurate expression for the radiation probability. This derivation will be given in § 5.8. The same author has also computed the function $w(E_0,E,t)$ with the inclusion of collision loss.

**2.14. *Theoretical expressions for the differential radiation probability of heavy particles.*** The radiation processes of heavy particles (mesons, protons) differ in several respects from those of electrons. On account of the larger mass, heavy particles undergo smaller accelerations in passing near atomic nuclei and therefore suffer smaller radiation losses. This is shown by the factor $(m_e/m)^2$ in Eq. (2.12.12). Moreover the radiative processes of heavy particles take place, on the average, at a much smaller distance from the nucleus than radiative processes of electrons. In fact, according to Eq. (2.12.9) the maximum impact parameter of a collision capable of producing photons of a given energy is proportional to $(U/mc^2)^2$, where $U$ is the energy and $m$ the mass of the radiating particle.



From this it follows that screening can be neglected to a much greater extent in the theory of radiation processes of heavy particles than in the corresponding theory for electrons. On the other hand, it can be shown that, if the nucleus were a point charge, the minimum value for the impact parameter in radiative collisions of heavy particles would be smaller than the actual nuclear radius. Therefore one cannot neglect the fact that, for distances smaller than the nuclear radius, the electric field of the nucleus differs from that of a point charge. The important role that close collisions play in radiation phenomena of heavy particles also manifests itself in the strong dependence of the radiation probability on the spin of the radiating particle.

Christy and Kusaka (CRF41) computed the differential radiation probabilities for particles of arbitrary mass, $m$, of spin 0, $\frac{1}{2}$, or 1, and of "normal" magnetic moment (§ 2.3). In their computations they assumed that the kinetic energy of the particle is large compared with the rest energy, that the screening of the outer electrons is negligible, and that the potential of a nucleus is that of a point charge for distances larger than the nuclear radius, $r_n$, and is constant for distances smaller than $r_n$. We shall here give the results obtained by Christy and Kusaka in a form where the nuclear radius appears explicitly. For the numerical evaluation of the formulae one may take (see § 7.6):

$$r_n = 1.38 \cdot 10^{-13} A^{1/3}\ \text{cm} = 0.49 r_e A^{1/3}, \tag{1}$$

where $A$ is the atomic mass number.

The general expression for the radiation probability is:

$$\Phi_{\text{rad}}(E,E')\,dE' = \alpha \frac{N}{A} Z^2 r_e^2 \left(\frac{m_e}{m}\right)^2 \frac{dE'}{E'} F(U,v), \tag{2}$$

where $E$ is the kinetic energy of the primary particle, $U = E + mc^2$ is its total energy, $E'$ is the energy of the secondary photon, and $v = E'/U$. The quantity $F(U,v)$ is a slowly varying function of $U$ and $v$ and has the following expressions:

*Spin* 0:

$$F(U,v) = \frac{16}{3}(1-v)\left[\ln\left(\frac{2U}{mc^2}\frac{\hbar}{mcr_n}\frac{1-v}{v}\right) - \frac{1}{2}\right]. \tag{3}$$

*Spin* $\frac{1}{2}$:

$$\begin{aligned} F(U,v) &= \frac{16}{3}\left(\frac{3}{4}v^2 + 1 - v\right)\left[\ln\left(\frac{2U}{mc^2}\frac{\hbar}{mcr_n}\frac{1-v}{v}\right) - \frac{1}{2}\right] \\ &= 4\left[1 + (1-v)^2 - \frac{2}{3}(1-v)\right]\left[\ln\left(\frac{2U}{mc^2}\frac{\hbar}{mcr_n}\frac{1-v}{v}\right) - \frac{1}{2}\right]. \end{aligned} \tag{4}$$

*Spin* 1:

$$\begin{aligned} F(U,v) &= \left[\frac{16}{3}(1-v) + \frac{13}{12}v^2 - \frac{5}{24}\frac{v^4}{1-v}\right]\ln\left(\frac{\pi U}{3mc^2}\frac{\hbar}{mcr_n}\frac{1-v}{v}\right) \\ &\quad - \frac{v^2}{1-v}\frac{10 - 10v + 3v^2}{8} - \frac{52}{9}(1-v) \end{aligned}$$

$$+\frac{v^2}{1-v}\frac{34-34v+7v^2}{24}\ln^2\left(\frac{\pi U}{3mc^2}\frac{\hbar}{mcr_n}\frac{1-v}{v}\right)$$
$$+\frac{\pi U}{6mc^2}\frac{\hbar}{mcr_n}\frac{2v-2v^2+7v^3}{12}. \tag{5}$$

The expression for the radiation probability of particles of spin $\frac{1}{2}$ is very similar to that of electrons [see Eq. (2.11.4)]. The factor $(m_e/m)^2$ is due to the different mass. The form of the logarithmic term depends on the cut-off of the electric field at $r_n$. One can see this clearly by recalling the approximate expression for the radiation probability obtained in § 2.12, Eq. (2.12.12), and noting that, in the case under consideration, the appropriate values of $b_{\min}$ and $b_{\max}$ are: $b_{\min} = r_n$ and $b_{\max} = b_1 = (ch/E')(U/mc^2)^2$ [see Eq. (2.12.14)]. With these values the argument of the logarithm in Eq. (2.12.12) becomes:

$$\frac{b_{\max}}{b_{\min}} = \frac{ch}{E'}\left(\frac{U}{mc^2}\right)^2\frac{1}{r_n} = \frac{U}{mc^2}\frac{h}{mcr_n}\frac{1}{v}.$$

This expression is very similar to the argument of the logarithm in Eq. (4).

The theoretical results relative to particles of spin 0 and $\frac{1}{2}$ are probably correct up to primary energies of the order of 137 $(m^2c^2/m_e)Z^{-1/3}$, where the screening of the nuclear field by the atomic electrons becomes important. The radiation probability of particles of spin 1 is less affected by screening than the radiation probability of particles of spin 0 or $\frac{1}{2}$. However, Eq. (5) includes terms that cannot legitimately be computed by the methods used when $U$ is larger than $m^2c^2/m_e$. Thus Eq. (5) is valid only for primary energies lower than this limit. According to Christy and Kusaka one can obtain a minimum estimate for the radiation probability when $U > m^2c^2/m_e$ by neglecting the doubtful phenomena altogether. Equation (5) then becomes:

$$F(U,v) = \left[\frac{16}{3}(1-v) + \frac{13}{12}v^2 - \frac{5}{24}\frac{v^4}{1-v}\right]\ln\left(\frac{\pi U}{3mc^2}\frac{\hbar}{mcr_n}\frac{1-v}{v}\right)$$
$$-\frac{v^2}{1-v}\frac{10-10v+3v^2}{8} - \frac{52}{9}(1-v)$$
$$+\frac{v^2}{1-v}\frac{34-34v+7v^2}{24}\left[\ln^2\left(\frac{\pi U}{3mc^2}\frac{\hbar}{mcr_n}\frac{1-v}{v}\right)\right.$$
$$\left.-\ln^2\left(\frac{\pi U}{6Bmc^2}\frac{\hbar}{mcr_n}\right)\right] + \left(B + B\ln\frac{\pi U}{6Bmc^2}\frac{\hbar}{mcr_n}\right)\frac{2v-2v^2+7v^3}{12}, \tag{6}$$

where $B$ is a constant of the order of $1/\alpha = 137$.

Comparison between Eqs. (3), (4), and (5) or (6) indicates that the probability of large radiation losses is much greater for particles of spin 1 than for particles of spin $\frac{1}{2}$, and somewhat greater for particles of spin $\frac{1}{2}$ than for particles of spin 0. Hence the probability of large radiation losses by heavy particles varies with spin in the same way as the probability of large collision losses.



Numerical evaluation of the formulae shows that, for any value of the spin, large energy transfers are more likely to occur by radiation than by collision. The *total* energy loss, however, is determined mainly by collision processes up to much larger energies than in the case of electrons (indeed, the energy at which radiation loss overtakes collision is approximately $(m/m_e)^2$ times greater for particles of mass $m$ than for electrons, except for the case of particles of spin 1 or of anomalous magnetic moments).

Protons have spin $\frac{1}{2}$, but their magnetic moment has an anomalous value (see § 4.4). Therefore their radiation probability cannot be computed by means of Eq. (4). If one introduces an anomalous magnetic moment into the theory of radiation by protons, one obtains an extra term in the expression for the radiation probability. This term increases rapidly with increasing energy and eventually becomes the dominant term (PW41). It seems unlikely, however, that this result has any physical significance. Indeed, the anomalous magnetic moment of protons is probably due to the fact that a proton exists for part of the time as a neutron and a virtual $\pi$-meson, and it is not an intrinsic property of the proton. If this interpretation is correct, the theoretical expression for the radiation probability loses its validity when the proton energy reaches the value at which the effect of the anomalous magnetic moment should become noticeable (BrS49).

One should notice that in the close collisions of protons (or neutrons) with nuclei short-range nuclear forces come into play. The phenomena resulting from such nuclear interactions (scattering of the incident particle, charge exchange, meson production) may represent a more important source of electromagnetic radiation than the acceleration of the proton in the Coulomb field of nuclei.

**2.15.** ***Elastic scattering of charged particles. Expressions for the differential scattering probability.*** Consider now the second process that takes place when a charged particle passes in the neighborhood of a nucleus, namely the change in the direction of motion or *scattering*. As pointed out in § 2.1, one can neglect the radiation emitted during the process. Since the nucleus is very heavy with respect to the incident particle, the energy of the latter does not change appreciably and the collision is an *elastic* one.

Let $\Xi(\Theta)\,d\omega\,dx$ represent the probability that a particle of momentum $p$ and velocity $\beta c$, traversing a thickness of $dx$ g cm$^{-2}$, undergoes a collision which deflects the trajectory of the particle into the solid angle $d\omega$ at an angle $\Theta$ to its original motion. The quantity $\Xi(\Theta)$ shall be referred to as the differential scattering probability.

The theoretical expression for $\Xi(\Theta)$ depends on the nature of the scattering material as well as on the charge and the spin of the incident particle. If one assumes that the electric field of a nucleus is that of a point charge $Ze$ (i.e., if one neglects both the finite dimensions of the nucleus and the shielding of its field by the atomic electrons) and if one uses Born's approximation, one obtains the following expressions:

*Spin* 0, from Williams (WEJ39):

$$\Xi(\Theta)d\omega = \frac{1}{4}N\frac{Z^2}{A}r_e^2\left(\frac{m_ec}{\beta p}\right)^2\frac{d\omega}{\sin^4(\Theta/2)}; \tag{1}$$

*Spin* ½, from Mott (MNF29):

$$\Xi(\Theta)d\omega = \frac{1}{4}N\frac{Z^2}{A}r_e^2\left(\frac{m_ec}{\beta p}\right)^2\left(1 - \beta^2\sin^2\frac{\Theta}{2}\right)\frac{d\omega}{\sin^4(\Theta/2)}; \tag{2}$$

*Spin* 1, from Massey and Corben (MHJ39):

$$\Xi(\Theta)d\omega = \frac{1}{4}N\frac{Z^2}{A}r_e^2\left(\frac{m_ec}{\beta p}\right)^2\left[1 + \frac{1}{6}\left(\frac{p\beta}{mc}\right)^2\sin^2\Theta\right]\frac{d\omega}{\sin^4(\Theta/2)}. \tag{3}$$

All of the equations above are valid for singly charged particles of arbitrary mass.

For small deflections, the terms depending on spin become negligible and Eqs. (1), (2), and (3) reduce to the following formula, known as the *Rutherford scattering formula:*

$$\Xi(\Theta)d\omega = 4N\frac{Z^2}{A}r_e^2\left(\frac{m_ec}{\beta p}\right)^2\frac{d\omega}{\Theta^4}. \tag{4}$$

Equation (4) can be obtained by a classical computation. Consider a particle of mass $m$ and charge $e$ moving toward a nucleus of charge $Ze$ with an impact parameter $b$ and with a velocity $\beta c$.

According to Eq. (2.4.3) the nucleus acquires a momentum:

$$p' = \frac{2Ze^2}{b\beta c} \tag{5}$$

in a direction perpendicular to the initial trajectory of the incident particle. From Newton's third law, it follows that the incident particle acquires a transverse momentum of the same magnitude in the opposite direction. Therefore, if the particle has an initial momentum $p$, its trajectory is deflected by an angle $\Theta$ given by the equation:

$$\Theta = \frac{p'}{p} = \frac{2Ze^2}{b\beta cp}. \tag{6}$$

The probability of a deflection in $d\Theta$ at $\Theta$ equals the probability of a collision with impact parameter in $db$ at $b$, where $\Theta$ and $b$ are related by Eq. (6). The probability of a collision with impact parameter in $db$ at $b$ on traversal of 1g cm$^{-2}$ is:

$$\frac{N}{A}2\pi b\,db = \pi\frac{N}{A}d(b^2). \tag{7}$$

Eq. (6) gives, in absolute value:

$$d(b^2) = \left(\frac{2Ze^2}{\beta cp}\right)^2\frac{2\,d\Theta}{\Theta^3}. \tag{8}$$



Hence the probability of a deflection in $d\Theta$ at $\Theta$ has the expression:

$$\pi\frac{N}{A}\left(\frac{2Ze^2}{\beta cp}\right)^2\frac{2\,d\Theta}{\Theta^3}. \tag{9}$$

If one considers angles sufficiently small that $d\omega$ may be written as $2\pi\Theta\,d\Theta$, one obtains the following expression for the differential scattering probability:

$$\Xi(\Theta)\,d\omega = \frac{N}{A}\left(\frac{2Ze^2}{\beta cp}\right)^2\frac{d\omega}{\Theta^4}. \tag{10}$$

With the substitution: $r_e = e^2/m_ec^2$, Eq. (10) becomes identical to Eq. (4).

A similar computation yields the following expression for the scattering probability in the field of the electrons:

$$\Xi'(\Theta)\,d\omega = \frac{NZ}{A}\left(\frac{2e^2}{\beta cp}\right)^2\frac{d\omega}{\Theta^4}. \tag{11}$$

$\Xi'$ is smaller than $\Xi$ by a factor of $Z$. Thus, whereas collisions with atomic electrons are responsible for practically all of the energy loss, their contribution to scattering is relatively small.

As already pointed out, the expressions for the scattering probability given above were derived under the assumption that the electric field of a nucleus is that of a point charge $Ze$. The finite size of the nucleus on the one hand, and the screening of its field by the outer electrons on the other, limit the validity of the results to a certain range of angular deflections.

One can show that if $r_a$ is the radius of the atom [see Eq. (2.12.15)] and $\lambda\!\!\!^-$ the deBroglie wave length of the incident particle divided by $2\pi$, the screening of the electric field of the nucleus by the outer electrons does not appreciably affect the scattering probability when $\Theta \gg \lambda\!\!\!^-/r_a$, whereas it reduces $\Xi$ to a small fraction of the value given by Eqs. (1) to (4) when $\Theta$ becomes smaller than $\lambda\!\!\!^-/r_a$. Thus the range of validity of Eqs. (1) to (4) is approximately limited, on the lower end, by the angle:

$$\Theta_1 = Z^{1/3}\alpha^2\frac{\lambda\!\!\!^-}{r_e} = \frac{Z^{1/3}}{137}\frac{m_ec}{p}. \tag{12}$$

The effect of screening on the scattering probability was evaluated quantitatively by Goudsmit and Saunderson (GSA40.1; GSA40.2) and computed in a more rigorous manner by Molière (MG47). For instance, Goudsmit and Saunderson, using a potential of the form: $V = (Ze^2/r)\exp(-r/r_a)$ to represent the electric field of an atom, obtained the following expression of the scattering probability (valid for small values of $\Theta$):

$$\Xi(\Theta)\,d\omega = 4N\frac{Z^2}{A}r_e^2\left(\frac{m_ec}{\beta p}\right)^2\frac{d\omega}{(\Theta^2 + \Theta_1^2)^2}, \tag{13}$$

where $\Theta_1$ is given by Eq. (12). Equation (13) is practically identical to Eq. (4) for $\Theta \gg \Theta_1$. As $\Theta$ approaches $\Theta_1$, however, the function $\Xi$ defined by Eq. (13) deviates from the function $\Xi$ defined by Eq. (4). At the limit for $\Theta = 0$, the former tends to a finite value whereas the latter tends to infinity.

In order to take into account the finite size of the nucleus, we may assume with Williams (WEJ39) that the electric charge of the nucleus, instead of being concentrated in a point, is distributed in a sphere of radius $r_n$. It can be shown that this assumption does not materially affect the calculated value of the scattering probability $\Xi(\Theta)$ for $\Theta \ll \lambdabar/r_n$, whereas it causes $\Xi(\Theta)$ to go rapidly to zero for $\Theta > \lambdabar/r_n$. If one takes $r_n = 0.49 r_e A^{1/3}$ [see Eq. (2.14.1)] one finds that the range of validity of Eqs. (1) to (4) is limited approximately at the upper end by the angle:

$$\Theta_2 = \frac{\lambdabar}{0.49 r_e A^{1/3}} = 280 A^{-1/3} \frac{m_e c}{p}. \tag{14}$$

This equation, of course, is only significant when $\Theta_2$ is small compared with one. The fact that Eq. (14) gives values of $\Theta_2$ large compared with one when $p$ is small compared with $280 A^{-1/3} m_e c$, means that the finite size of the nucleus does not play any important role in the scattering of particles with sufficiently small momenta. In this case the expressions for the scattering probability computed under the assumption of a point nucleus, Eqs. (1), (2), and (3), are valid for angles up to $\Theta = \pi$.

In order to improve upon Williams' theory of electromagnetic scattering it would be necessary to determine the charge distribution in the nucleus. This problem has not yet been successfully solved. For a crude approximation one may assume that the charge is carried by the protons, represented as rigid spheres. In an improved model one may consider that the protons exist, for part of the time, as neutrons surrounded by a cloud of positive mesons, and the neutrons exist, for part of the time, as protons surrounded by a cloud of negative mesons. Another important point that should be taken into consideration is the possibility that after the collision the nucleus may be left in an excited state or that mesons may be produced. For a detailed discussion of these questions, in their relation to the theory of electromagnetic scattering, the reader may consult ref. (AE51).

**2.16. *The mean square angle of scattering.*** When a charged particle traverses a plate of finite thickness, it undergoes a large number of collisions, most of which produce very small angular deflections. One may want to compute the probability that, as a result of these successive collisions, the particle emerges from the plate with a given lateral displacement and with a given angular deflection.

As a first step toward the solution of this problem, we shall compute the mean square angle of deflection, $\langle\Theta^2\rangle_{av}$, as a function of the thickness, $x$, of matter traversed.

According to a general rule on the superposition of small and independent deviations, the value of $\langle\Theta^2\rangle_{av}$ at $x + dx$ equals the value of $\langle\Theta^2\rangle_{av}$ at $x$ plus the mean square angle of scattering in $dx$:

$$d\langle\Theta^2\rangle_{av} = dx \int \Theta^2 \Xi(\Theta)\, d\omega.$$

This equation may be written as follows:

$$\frac{d\langle\Theta^2\rangle_{av}}{dx} = \Theta_s^2, \tag{1}$$

where

$$\Theta_s^2 = \int \Theta^2 \Xi(\Theta)\, d\omega = \int \Theta^2 \Xi(\Theta) 2\pi\Theta\, d\Theta. \tag{2}$$



If one assumes that $\Xi$ is given by Eq. (2.15.4) for $\Theta_1 < \Theta < \Theta_2$ and is zero for $\Theta < \Theta_1$ or $\Theta > \Theta_2$, Eq. (2) yields:

$$\Theta_s^2 = 8\pi N \frac{Z^2}{A} r_e^2 \left(\frac{m_e c}{\beta p}\right)^2 \ln \frac{\Theta_2}{\Theta_1}, \tag{3}^*$$

or, with the values of $\Theta_1$ and $\Theta_2$ given by Eqs. (2.15.12) and (2.15.14):

$$\Theta_s^2 = 16\pi N \frac{Z^2}{A} r_e^2 \left(\frac{m_e c}{\beta p}\right)^2 \ln \left[196 Z^{-1/3} \left(\frac{Z}{A}\right)^{1/6}\right]. \tag{4}$$

Other theories, in which the screening effect is taken into consideration in a more accurate fashion, yield somewhat different results. For instance, if one assumes that $\Xi(\Theta)$ is given by Eq. (2.15.13) for $\Theta < \Theta_2$ and is zero for $\Theta > \Theta_2$, one obtains:

$$\Theta_s^2 = 4\pi N \frac{Z^2}{A} r_e^2 \left(\frac{m_e c}{\beta p}\right)^2 \left\{\ln \left[\left(\frac{\Theta_2}{\Theta_1}\right)^2 + 1\right] - 1\right\}.$$

For air, the value of $\Theta_s^2$ given by this equation differs about six per cent from that given by Eq. (3).

When the value of $\Theta_2$ given by Eq. (2.15.14) is greater than 1, it is more appropriate to take 1 rather than $\Theta_2$ as the upper limit of integration in Eq. (2). In this case one obtains in place of Eq. (4):

$$\Theta_s^2 = 16\pi N \frac{Z^2}{A} r_e^2 \left(\frac{m_e c}{\beta p}\right)^2 \ln \left(\frac{137 p}{Z^{1/3} m_e c}\right)^{1/2}. \tag{5}$$

One will use for the computation of $\Theta_s^2$ either Eq. (4) or Eq. (5), whichever gives the smaller result.

One should notice that in many experiments on scattering one disregards deflections exceeding an arbitrarily chosen angle, $\Theta_2'$, which is often smaller than the angle, $\Theta_2$, defined by Eq. (2.15.14) (see, for example, § 3.15). The root mean square angle of scattering measured in such experiments is smaller than that given by Eq. (4). One can obtain an approximate value for it by replacing $\Theta_2$ with $\Theta_2'$ in Eq. (3).

Note that $\Theta_s^2$ as given by Eq. (4) depends on the atomic number $Z$ of scattering material much in the same way as the radiation loss of high-energy electrons [see Eq. (2.11.10)]. The coefficient that multiplies $Z^{-1/3}$ in the logarithm varies from a value of 175 in the light elements (where $A = 2Z$) to a value of 169 in the heavy elements (where $A = 2.5Z$). The similar coefficient in Eq. (2.11.10) is 183. The difference between these coefficients is not theoretically significant and has a negligible effect on the result. If we recall the definition of radiation length [see Eq. (2.11.11)] and introduce the constant $E_s$ with the dimensions of an energy:

$$E_s = \left(\frac{4\pi}{\alpha}\right)^{1/2} m_e c^2 = 21 \cdot 10^6 \text{ ev}, \tag{6}$$

* This equation, of course, applies to singly charged particles. From the classical derivation of Rutherford's formula given in the preceding section, one can easily recognize that the expression of $\Theta_s^2$ for a particle with $z$ electronic charges contains an additional factor $z^2$.

we may write Eq. (4) as follows:

$$\Theta_s^2 = \left(\frac{E_s}{\beta cp}\right)^2 \frac{1}{X_0}. \tag{7}$$

If the scattering layer is sufficiently thin that the energy loss of the particle may be neglected, $\Theta_s^2$ is a constant and Eq. (1) yields:

$$\langle\Theta^2\rangle_{av} = \Theta_s^2 x, \tag{8}$$

or, if Eq. (7) is valid:

$$\langle\Theta^2\rangle_{av} = \left(\frac{E_s}{\beta cp}\right)^2 \frac{x}{X_0}. \tag{9}$$

Instead of considering the total deflection $\Theta$, it is often more convenient to consider the projection, $\theta$, of the deflection on a plane containing the initial trajectory. One can easily show that, for small angles, the mean square value of $\theta$ is one-half the mean square value of $\Theta$ [see next section, Eq. (2.17.3)]; thus if the energy loss is negligible:

$$\langle\Theta^2\rangle_{av} = \tfrac{1}{2}\Theta_s^2 x. \tag{10}$$

When energy loss cannot be neglected, Eq. (1) yields:

$$\langle\Theta^2\rangle_{av} = \int_0^x \Theta_s^2\, dx'. \tag{11}$$

$\Theta_s^2$ is a function of the momentum $p$ of the particle. If one takes $p$ as the variable of integration, Eq. (11) becomes

$$\langle\Theta^2\rangle_{av} = \int_{p_1}^{p_2} \frac{\Theta_s^2}{(-dp/dx)}\, dp, \tag{12}$$

where $p_2$ is the momentum at $x' = 0$ and $p_1$ the momentum at $x' = x$. This equation gives the mean square angle of scattering as a function of the momenta of the particle before and after traversal of the plate.

When Eq. (7) applies, Eq. (12) may be written as follows:

$$\langle\Theta^2\rangle_{av} = \frac{E_s^2}{c^2}\frac{1}{X_0}\int_{p_1}^{p_2} \frac{1}{\beta^2(-dp/dx)}\frac{dp}{p^2}. \tag{13}$$

For sufficiently high energies one may take $\beta = 1$ and $(-dp/dx) =$ constant (indeed, the product $\beta^2(-dp/dx)$ remains approximately constant down to lower energies than either of the two terms). Eq. (13) may then be integrated immediately and yields:

$$\langle\Theta^2\rangle_{av} = \frac{E_s^2}{c^2 p_1 p_2}\frac{p_2 - p_1}{(-dp/dx)}\frac{1}{X_0}, \tag{14}$$

or

$$\langle\Theta^2\rangle_{av} = \frac{E_s^2}{c^2 p_1 p_2}\frac{x}{X_0}. \tag{15}$$

Notice that this equation is the same as Eq. (9) with $p_1p_2$ substituted for $p^2$. Notice also that when $p_2 \gg p_1$, Eq. (14) reduces to the following:

$$\langle\Theta^2\rangle_{av} - \frac{E_s^2}{c^2 p_1}\frac{1}{(-dp/dx)}\frac{1}{X_0}. \tag{16}$$

We have so far neglected scattering in the field of atomic electrons. Presumably one can take this effect into consideration in an approximate manner with a procedure similar to that followed in the case of radiation phenomena; i.e., by replacing $Z^2$ with $Z(Z+1)$ in the expressions for the scattering probability and for the root mean square angle of scattering. This means that, when $p < 280A^{1/3} m_e c$, we may still use Eq. (7), provided we take Eq. (2.11.20) instead of Eq. (2.11.11) as a definition of the radiation length.



Before closing this section we wish to remind the reader that the theory developed above may not be relied upon to give very accurate results, because of the approximate manner in which we have taken into account the finite size of nuclei (which influences the probability of scattering at large angles) and the screening of the electric field of nuclei by the outer electrons (which influences the probability of scattering at small angles). The value of the mean square angle of scattering is strongly affected both by large-angle scattering and by small-angle scattering, as is clearly shown by the fact that the expression of $\Theta_s^2$ given by Eq. (3) diverges for both $\Theta_2 = \infty$ and $\Theta_1 = 0$. However, the arithmetic mean of the absolute value of the projected angle does not depend critically upon the behavior of the scattering probability at large angles. If one uses for the scattering probability the function computed by Moliere (MG47), which takes exact account of screening and is therefore accurate at small angles, one obtains the following expression for $|\theta|_{av}$ (GCY50):

$$|\theta|_{av} = 2zZ\sqrt{\frac{N}{A}x}\;\frac{m_e c r_e}{p\beta}\left[1.45 + 0.8\sqrt{\ln 0.2\pi Z^{-2/3}\frac{N}{A}r_e^2\frac{x}{\alpha}\frac{1}{(0.3\beta^2/\alpha^2 z Z^2)}}\right].$$

**2.17. *The distribution function.*** Consider a parallel and infinitely narrow beam of particles incident upon a plate of some scattering substance. Assume that these particles are all of the same kind and all have the same energy. Assume further that the plate is sufficiently thin so that the energy loss of the particles may be neglected. We wish to compute the spatial and angular distribution of the beam after traversal of a thickness $x$ of the scattering substance.

To simplify the problem, we shall consider only *small angular deflections.* Let us take a system of cartesian coordinates with the origin at the point of incidence and one of the axes in the direction of the motion of the incident particles. This axis will be denoted as the $x$ axis, while the other two will be the $y$ and $z$ axes respectively. All lengths will be measured in g cm$^{-2}$. Let us consider the projection of the motion of the particles on the $(x,y)$ plane and let $P(x,y,\theta_y)\,dy\,d\theta_y$ be the number of particles at the thickness $x$ having a lateral displacement in $dy$ at $y$ and traveling at an angle in $d\theta_y$ at $\theta_y$ with the $x$-axis. For reasons of symmetry, the same function $P$ describes also the space and angular distribution in the $(x,z)$ plane.

Let $\xi(\theta_y)\,d\theta_y$ represent *projected scattering probability,* i.e., the probability per unit thickness of an event in which the *projection* of the scattering angle in the $x$-$y$ plane lies between $\theta_y$ and $\theta_y + d\theta_y$. Since, for small angles,

$\Theta^2 = \theta_y^2 + \theta_z^2$, the function $\xi(\theta_y)$ is related to the scattering probability $\Xi(\Theta)$ by the following equation:

$$\xi(\theta_y) = \int \Xi[(\theta_y^2 + \theta_z^2)^{1/2}]\, d\theta_z. \tag{1}$$

In this and in the subsequent equations of this section it will be understood that the integrations extend to all values of the variables for which the integrands are different from zero.

Since

$$\xi(\theta_y) = \xi(-\theta_y),$$
$$\int \theta_y \xi(\theta_y)\, d\theta_y = 0. \tag{2}$$

Moreover, from Eq. (2.16.2) one obtains:

$$\Theta_s^2 = \int \Theta^2 \Xi(\Theta)\, d\omega = \int\int (\theta_y^2 + \theta_z^2)\Xi[(\theta_y^2 + \theta_z^2)^{1/2}]\, d\theta_y\, d\theta_z$$
$$= \int \theta_y^2\, d\theta_y \int \Xi[(\theta_y^2 + \theta_z^2)^{1/2}]\, d\theta_z + \int \theta_z^2\, d\theta_z \int \Xi[(\theta_y^2 + \theta_z^2)^{1/2}]\, d\theta_y,$$

or, using Eq. (1):

$$\int \theta_y^2 \xi(\theta_y)\, d\theta_y = \tfrac{1}{2}\Theta_s^2. \tag{3}$$

We now wish to compute the change that the function $P$ undergoes in the layer from $x$ to $x + dx$. At the depth $x$, there are $P(x,y,\theta_y)\, dy\, d\theta_y$ particles in the angular interval $d\theta_y$ at $\theta_y$ and with a lateral displacement in $dy$ at $y$. On traversing the additional thickness $dx$, some of these particles undergo a scattering collision and are thereby removed from the interval $d\theta_y$. Their number is:

$$dy\, d\theta_y\, dx\, P(x,y,\theta_y) \int \xi(\theta_y')\, d\theta_y'.$$

On the other hand, some of the particles that, at $x$, had a lateral displacement in $dy$ at $y$ but were not in the angular interval $d\theta_y$ at $\theta_y$, undergo, in $dx$, a scattering collision that brings them into this angular interval. Their number is:

$$dy\, d\theta_y\, dx \int P(x,y,\theta_y + \theta_y')\xi(\theta_y')\, d\theta_y'.$$

Thus scattering in $dx$ produces the following net change in the number of particles in $d\theta_y$ at $\theta_y$:

$$dy\, d\theta_y\, dx\left[\int P(x,y,\theta_y + \theta_y')\xi(\theta_y')\, d\theta_y' - P(x,y,\theta_y)\int \xi(\theta_y')\, d\theta_y'\right]$$
$$= dy\, d\theta_y\, dx \int [P(x,y,\theta_y + \theta_y') - P(x,y,\theta_y)]\xi(\theta_y')\, d\theta_y'.$$

The change in the spatial distribution due to scattering in the infinitesimal layer $dx$ (as contrasted to the change in the angular distribution) is only a second-order effect and can be disregarded. The spatial distribution, however, is modified because in the layer $dx$, particles traveling at an angle $\theta_y$ undergo a lateral displacement given by $dy = \theta_y\, dx$ (in g cm$^2$). It follows that the particles having a lateral displacement $y$ at the thickness $x + dx$ are those which had a lateral displacement $y - \theta_y\, dx$ at the thickness $x$. Thus the "drift" in $dx$ produces the following change in the number of particles in $dy$ at $y$:

$$P(x,y - \theta_y\, dx, \theta_y)\, dy\, d\theta_y - P(x,y,\theta_y)\, dy\, d\theta_y = -\theta_y\, dx\left(\frac{\partial P}{\partial y}\right) dy\, d\theta_y.$$



By adding the effects of scattering and drift, one obtains the following equation for $P(x,y,\theta_y)$:

$$\frac{\partial P}{\partial x} = -\theta_y \frac{\partial P}{\partial y} + \int [P(x,y,\theta_y + \theta_y') - P(x,y,\theta_y)]\xi(\theta_y')\, d\theta_y'. \tag{4}$$

Note that $\xi(\theta_y')$ is a rapidly decreasing function of $\theta_y'$. We shall now introduce the assumption that for all values of $\theta_y'$ for which $\xi(\theta_y')$ is appreciably different from zero, $P(x,y,\theta_y + \theta_y')$ may be developed in a power series of $\theta_y'$ and that the terms beyond the second order may be neglected. With this assumption, and remembering Eqs. (2) and (3), one obtains from Eq. (4):

$$\frac{\partial P}{\partial x} = -\theta_y \frac{\partial P}{\partial y} + \frac{\Theta_s^2}{4}\frac{\partial^2 P}{\partial \theta_y^2}. \tag{5}$$

We look for the solution of Eq. (5) that corresponds to a single incident particle. It can easily be shown that this solution is [see Fermi, as quoted in (RB41.1)]:*

$$P(x,y,\theta_y)\, dy\, d\theta_y = \frac{2\sqrt{3}}{\pi}\frac{1}{\Theta_s^2 x^2} \exp\left[-\frac{4}{\Theta_s^2}\left(\frac{\theta_y^2}{x} - \frac{3y\theta_y}{x^2} + \frac{3y^2}{x^3}\right)\right]. \tag{6}$$

Indeed, one sees upon substitution that (6) satisfies (5). That the boundary conditions are also fulfilled will become apparent from the following considerations.

By integrating the distribution function $P$ over $y$, one obtains a function, $Q(x,\theta_y)$, that represents the angular distribution irrespective of lateral displacement:

$$Q(x,\theta_y) = \int_{-\infty}^{+\infty} P(x,y,\theta_y)\, dy = \frac{1}{\sqrt{\pi}}\frac{1}{\Theta_s x^{1/2}} \exp\left(-\frac{\theta_y^2}{\Theta_s^2 x}\right). \tag{7}$$

Similarly, by integrating the function $P$ over $\theta_y$ one obtains a function, $S(x,y)$, that represents the distribution in space, irrespective of angle:

$$S(x,y) = \int_{-\infty}^{+\infty} P(x,y,\theta_y)\, d\theta_y = \sqrt{\frac{3}{\pi}}\frac{1}{\Theta_s x^{3/2}} \exp\left(-\frac{3y^2}{\Theta_s^2 x^3}\right). \tag{8}$$

It follows from Eqs. (7) and (8) that, for all values of $x$:

$$\int_{-\infty}^{+\infty} Q(x,\theta_y)\, d\theta_y = \int_{-\infty}^{+\infty} S(x,y)\, dy = 1. \tag{9}$$

Moreover, at the limit for $x = 0$, $Q$ is zero for all values of $\theta_y$ except $\theta_y = 0$, and $S$ is zero for all values of $y$ except $y = 0$; thus:

$$Q(0,\theta_y) = \delta(\theta_y), \qquad S(0,y) = \delta(y), \tag{10}$$

where $\delta$ is Dirac's improper function.

* The reader will find a direct proof of this equation in a paper by Eyges (EL48).

This argument proves that the solution (6) actually corresponds to a single particle incident at $x = 0$, $y = 0$ in the direction of the $x$ axis.

Equation (7) shows that, at every thickness, the projected-angle distribution irrespective of position is Gaussian. The mean square angle of scattering is given by:

$$\langle \theta_y^2 \rangle_{\text{av}} = \tfrac{1}{2}\Theta_s^2 x \tag{11}$$

in agreement with Eq. (2.16.10).

Similarly, Eq. (8) shows that at every thickness the distribution in $y$, irrespective of angle, is Gaussian. The mean square value of $y$ is

$$\langle y^2 \rangle_{\text{av}} = \frac{\Theta_s^2 x^3}{6}. \tag{12}$$

Note that, if one considers only those particles having a certain lateral displacement at a given thickness, their angular distribution is *not* Gaussian. The same remark applies to the space distribution of particles that have a certain angular deflection at a given thickness.

The distribution function (6) can be used to solve various problems arising in the discussion of experimental results. One of these problems is the following. Suppose one knows that a particle passes through two points $A$ and $B$ separated by a distance $x$. What is the probability that, halfway between $A$ and $B$, the trajectory has a lateral displacement between $y'$ and $y' + dy'$? One can show (RB41.1) that this probability is represented by a Gaussian function and that the mean square value of $y'$ is given by the equation

$$\langle y'^2 \rangle_{\text{av}} = \frac{\Theta_s^2 x^3}{96}. \tag{13}$$

Other consequences of Eq. (6) are discussed in a paper by Scott (SWT49).

The foregoing conclusions are based upon the assumption that $P(x,y,\ \vartheta_y + \theta_y')$ differs only slightly from $P(x,y,\theta_y)$ for all values of $\theta_y'$ for which $\xi(\theta_y')$ is appreciably different from zero. This is certainly the case when the maximum angle of deflection in a single scattering process is small compared with the root mean square angle of scattering. Comparison between Eqs. (2.15.14) and (2.16.9) shows that, for high-energy particles, this condition is equivalent to the following:

$$\frac{x}{X_0} \gg 46A^{-2/3}. \tag{14}$$

When the maximum angle of single scattering is large compared with $\Theta_s$ (which is always true for thin absorbers) the condition for the validity of Eq. (5) is not fulfilled. One can easily recognize that, in this case, the expression for $Q(x,\theta_y)$ derived from Eq. (5) cannot be correct for all angles. In fact, if $\Theta_2 \gg \Theta_s$, the function $Q(\theta_y)$ given by Eq. (7) becomes smaller than the function $x\xi(\theta_y)$ for large values of $\theta_y$. This is absurd because the total probability for a particle to suffer a deflection in $d\theta_y$ at $\theta_y$ on traversing a thickness $x$ of matter—a probability represented by $Q(x,\theta_y)\,d\theta_y$ cannot be smaller than the probability for this deflection to occur in a single scattering process—a probability represented by $x\xi(\theta_y)\,d\theta_y$.



From the above discussion it appears that the exact determination of the distribution function when the condition (14) is not satisfied requires the solution of the integro-differential equation (4). This difficult mathematical task was undertaken by Moliere (MG48) and (with a different method) by Snyder and Scott (SHS49.1). The following considerations are based upon the work of the latter authors.

Either by direct arguments, or by integration of Eq. (4) with respect to $y$, one obtains the following equation for $Q(x,\theta_y)$:

$$\frac{\partial Q}{\partial x} = \int [Q(x,\ \theta_y + \theta_y') - Q(x,\theta_y)]\xi(\theta_y')\,d\theta_y'. \tag{15}$$

If one uses Eq. (2.15.13) as the expression for the scattering probability $\Xi(\Theta)$ and neglects the fact that this equation is valid only for angles $\Theta$ smaller than a certain value, Eq. (1) yields the following expression for the projected scattering probability:

$$\xi(\theta_y) = 2\pi N \frac{Z^2}{A} r_e^2 \left(\frac{m_e c}{\beta p}\right)^2 \frac{1}{(\theta_y^2 + \Theta_1^2)^{3/2}}. \tag{16}$$

Neglect of the cut-off in the expression for $\Xi(\Theta)$ limits the validity of Eq. (16) to angles $\theta_y$ small compared with maximum angle of single scattering.

Consider the thickness $X_s$ ("scattering length") defined by the equation:

$$\frac{1}{X_s} = 4\pi N \frac{Z^2}{A} r_e^2 \left(\frac{m_e c}{\beta p}\right)^2 \frac{1}{\Theta_1^2},$$

or

$$\frac{1}{X_s} = \frac{4\pi}{\alpha^2} N \frac{Z^{4/3} r_e^2}{A\beta^2}, \tag{17}$$

[see Eq. (2.15.12)] and define a new set of variables:

$$s = \frac{x}{X_s};\quad \eta = \frac{\theta_y}{\Theta_1};\quad q(s,\eta)\,d\eta = Q(x,\theta_y)\,d\theta_y. \tag{18}$$

With these new variables Eq. (15) becomes:

$$\frac{\partial q}{\partial s} = \frac{1}{2}\int_{-\infty}^{+\infty} [q(s,\eta + \eta') - q(s,\eta)]\,\frac{d\eta'}{[(\eta')^2 + 1]^{3/2}}. \tag{19}$$

Equation (19) is independent of the scattering material as well as of the mass and of the energy of the particle. Therefore, if one measures thickness in terms of $X_s$ and angles in terms of $\Theta_1$, the distribution function becomes a function only of thickness and angle.

Note that the Gaussian solution, Eq. (7), with the value of $\Theta_s$ given by Eq. (2.16.4) and with the new variables $s$ and $\eta$, becomes:

$$q(s,\eta) = \frac{1}{\sqrt{\pi}\, g\sqrt{s}} \exp\left(-\frac{\eta^2}{g^2 s}\right), \tag{20}$$

where
$$g^2 = 2 \ln \frac{\Theta_2}{\Theta_1} = 4 \ln \left[ 196 Z^{-1/3} \left(\frac{Z}{A}\right)^{1/6} \right]. \tag{21}$$

The quantity $g$ depends on $Z$ and therefore the function $q(s,\eta)$ defined by Eq. (20), unlike the function $q(s,\eta)$ defined by Eq. (19), contains the atomic number explicitly. This is due to the fact that in deriving Eq. (20) we have assumed $\Xi(\Theta)$ to be zero for $\Theta > \Theta_2$, whereas in deriving Eq. (19) we have postulated the formal validity of Eq. (2.15.13) for all values of $\Theta$ up to infinity. However $g$ varies with $Z$ very slowly and therefore the function $q(s,\eta)$ given by Eq. (20) is *almost* the same for all substances.

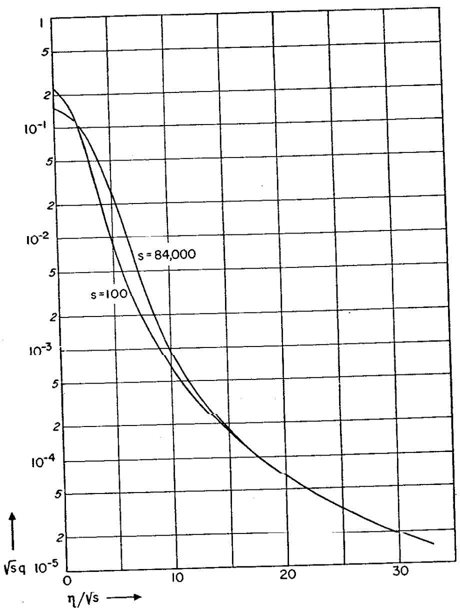


**Fig. 2.17.1.** The quantity $\sqrt{s}\, q$ as a function of $\eta/\sqrt{s}$ for $s = 100$ and $s = 84{,}000$. From Snyder and Scott (SHS49.1).



Snyder and Scott obtained a general solution of Eq. (19) with the method of the Fourier transforms, essentially by adding the contribution to $q$ of 0, 1, 2, . . . , $n$ scattering processes. They used numerical methods to determine $q(s,\eta)$ for values of $s$ from 100 to 84,000. Here we shall describe some of their results.*

Equation (20) shows that, when the Gaussian solution is valid, the quantity $\sqrt{s}q$ is represented by the same function of $\eta/\sqrt{s}$ for all values of $s$. This result holds approximately when one takes for $q$ the function determined by Snyder and Scott. As an illustration, the graphs in Fig. 1 give $\sqrt{s}q$ as a function of $\eta/\sqrt{s}$ for $s = 100$ and for $s = 84{,}000$ respectively. One sees that the two curves do not differ greatly from one another.

The four curves in Fig. 2 represent respectively: (1) $q$ vs. $\eta$ for $s = 100$ according to Snyder and Scott; (2) $q$ vs. $\eta$ for $s = 100$ computed according to the Gaussian approximation for air, Eq. (20); (3) $q$ vs. $\eta$ for $s = 100$ computed according to the Gaussian approximation for lead; (4) the probability for single scattering in the layer under consideration, i.e., the function $x\xi(\theta_y)$, expressed in terms of the new variables $s$ and $\eta$. One sees that the solution of Snyder and Scott approaches the Gaussian solution

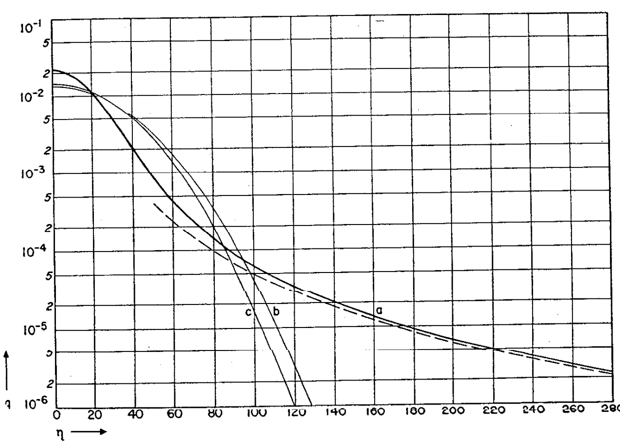


**Fig. 2.17.2.** The solid curves represent the distribution function $q(s,\eta)$ for $s = 100$: (a) according to the computations of Snyder and Scott (SHS49.1); (b) as given by Eq. (2.17.20) for air; (c) as given by Eq. (2.17.20) for lead. The dashed curve represents the probability of single scattering in a layer of $s = 100$ scattering lengths.

* Complete numerical tables are available from the Information and Publication Division, Brookhaven National Laboratory, Upton, N. Y.

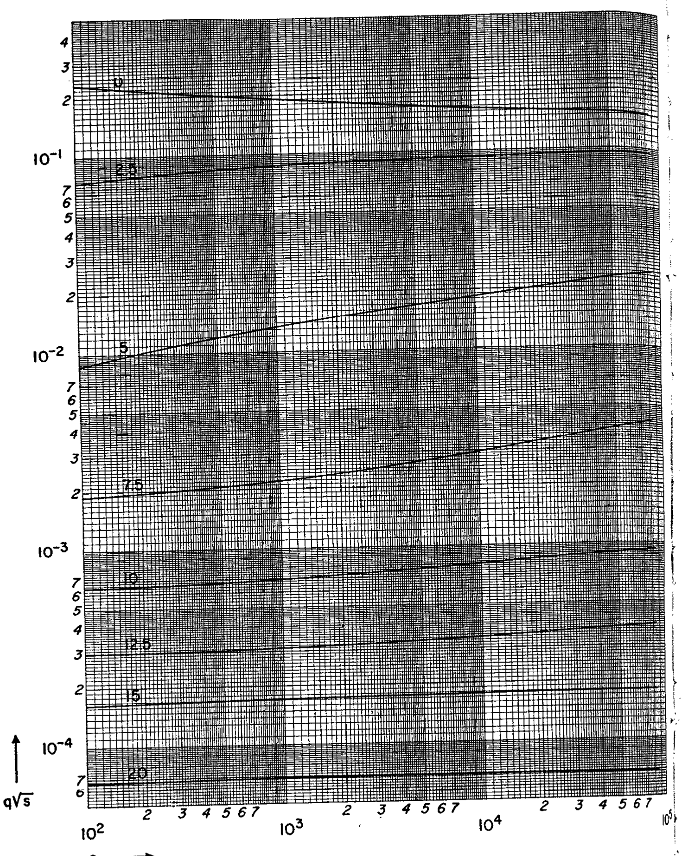


**Fig. 2.17.3.** Graphs of the quantity $q\sqrt{s}$ as a function of $s$ for eight values of $\eta/\sqrt{s}$ (0 to 20). From Snyder and Scott (SHS49.1).

at small angles and approaches the probability for single scattering at large angles. Physically this means that a small deflection is usually the result of a large number of small-angle scattering processes, whereas a large deflection is usually the result of a single large-angle scattering



process (plus a number of small-angle scattering processes having a comparatively minor effect). One often describes this situation by saying that small-angle deflections are due to *multiple* scattering, large-angle deflections to *single* scattering. The intermediate case, in which the observed deflections result from a small number of scattering processes, is usually referred to as the case of *plural* scattering.

In Fig. 3 the results of Snyder and Scott are plotted in a form more convenient for numerical applications.

In a later paper, Scott and Snyder (SWT50.1) applied their method to the determination of the distribution function for the lateral displacement; i.e., the function $S(x,y)$. Numerical tables of this function are also available.

In conclusion, it is appropriate to recall that the solution of Snyder and Scott is valid only for angles small compared with 1 and compared with the maximum angle of single scattering.

Some experimental results on scattering will be described in §§ 3.15 and 6.5.

**2.18. *Compton effect.*** The Compton effect can be described as an elastic collision between a photon and a free electron initially at rest.

One can use the principles of conservation of momentum and energy to establish a relation between the energy of the scattered photon and its

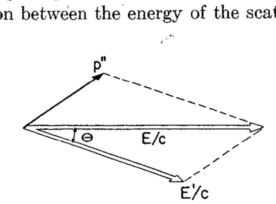


**Fig. 2.18.1.** The Compton effect.

angle of scattering. In Fig. 1, let $E$ be the energy of the incident photon, $E'$ the energy of the photon after the collision, $\theta$ the angle at which it is scattered, $E''$ the kinetic energy acquired by the electron, and $p''$ its momentum.

Conservation of energy gives:

$$E - E' = E''. \qquad (1)$$

Conservation of momentum gives:

$$(p'')^2 = \frac{E^2}{c^2} + \frac{(E')^2}{c^2} - 2\frac{EE'}{c^2}\cos\theta. \qquad (2)$$

If, in the second equation, we express $p''$ in terms of $E''$ and $m_e$, we find

$$(E'')^2 + 2m_ec^2E'' = E^2 + (E')^2 - 2EE'\cos\theta. \qquad (3)$$

Elimination of $E''$ by means of the energy equation yields:

$$E' = \frac{Em_ec^2}{m_ec^2 + E(1 - \cos\theta)}. \tag{4}$$

This equation, called the *Compton formula*, gives the energy of the scattered photon, $E'$, as a function of the energy of the incident photon, $E$, and the angle of scattering, $\theta$. Observe that $E'$ has its maximum value when the photon is scattered in the forward direction and its minimum value when the photon is scattered in the backward direction. For incident photons of energies large compared with $m_ec^2$:

$$E'_{\max} = E; \qquad E'_{\min} = \frac{m_ec^2}{2}. \tag{5}$$

A quantum-mechanical computation of the probability of Compton scattering carried out by Klein and Nishina (KO29) gave the following result:

$$\Phi_{\text{com}}(E,E')\,dE' = \frac{Cm_ec^2}{E}\frac{dE'}{E'}\left[1 + \left(\frac{E'}{E}\right)^2 - \frac{E'}{E}\sin^2\theta\right]. \tag{6}$$

In the above equation, which holds for $m_ec^2/2 < E' < E$,

$$\Phi_{\text{com}}(E,E')\,dE'\,dx$$

is the probability for a photon of energy $E$ traversing a thickness $dx$ g cm$^{-2}$ to undergo a Compton collision in which the scattered photon has an energy between $E'$ and $E' + dE'$, $C$ is the constant defined by Eq. (2.3.1) and $\theta$ is related to $E$ and $E'$ by Eq. (4).

When $E \gg m_ec^2$, $(E'/E)\sin^2\theta$ is negligible compared with 1 because $E'/E$ is much smaller than 1 except when $\theta$ is nearly 0. Thus Eq. (6) can be simplified into:

$$\Phi_{\text{com}}(E,E')\,dE' = \frac{Cm_ec^2}{E}\frac{dE'}{E'}\left[1 + \left(\frac{E'}{E}\right)^2\right]. \tag{7}$$

It appears from Eq. (7) that the scattering probability decreases rapidly with increasing $E'$; i.e., with decreasing $E''$.

Integration of $\Phi_{\text{com}}(E,E')$ from $E' = m_ec^2/2$ to $E' = E$ yields the total probability for a photon of energy $E$ to undergo Compton scattering in a thickness of $dx$ g cm$^{-2}$. When $E \gg m_ec^2$, one obtains from Eq. (7):

$$\int_{m_ec^2/2}^{E} \Phi_{\text{com}}(E,E')\,dE' = \frac{Cm_ec^2}{E}\left[\ln\frac{2E}{m_ec^2} + \frac{1}{2}\right]. \tag{8}$$

The expression (8) for the total scattering probability has been calculated under the assumption that Eq. (7) is valid for all values of $E'$. Actually Eq. (7) is valid only for those collisions in which the energy of the recoil electron is large compared with the binding energy, because otherwise the electron cannot be considered as free. The error, however, is negligible because of the small number of recoil electrons of low energy produced by Compton effect. One may recall that in the case of collision processes, instead, consideration of the binding forces is essential because most of the secondary electrons have small energies.



It is sometimes convenient to consider the total probability of Compton effect per radiation length. We shall call this quantity $\mu_{\text{com}}$:

$$\mu_{\text{com}} = X_0 \int_{m_ec^2/2}^{E} \Phi_{\text{com}}(E,E')\,dE'. \tag{9}$$

Figures 2.19.3 and 2.19.4 show plots of $\mu_{\text{com}}$ as a function of energy for air and lead respectively.

**2.19. *Pair production by photons.*** Pair production, like the Compton effect, is a typically quantum phenomenon that does not lend itself to classical description. From the point of view of Dirac's theory, it may be looked upon as a photoelectric effect whereby an electron is raised from a state of negative energy to a state of positive energy, leaving a "hole" in the infinite distribution of negative-energy electrons. The theory of pair production is closely related to that of radiation processes, and consequently the equations describing the two processes are very similar. Indeed, in the case of a radiation process an electron makes a transition between two states of positive energy, and a photon is emitted. In the case of pair production a photon is absorbed and causes an electron to make a transition from a state of negative energy to a state of positive energy.

The phenomenon of pair production is induced by the strong electric field that surrounds the nuclei. The nucleus in whose vicinity a photon undergoes materialization takes part of the momentum of the photon. However, because of its large mass, the nucleus does not acquire any appreciable energy. Therefore the sum of the total energies, $E' + m_ec^2$ and $E'' + m_ec^2$, of the two electrons of the pair is very nearly equal to the energy, $E$, of the original photon.

$$E' + E'' + 2m_ec^2 = E. \tag{1}$$

Let $\Phi_{\text{pair}}(E,E')\,dE'\,dx$ be the probability for a photon of energy $E$ traversing a thickness of $dx$ g cm$^{-2}$ to produce a pair, in which the positon has a kinetic energy between $E'$ and $E' + dE'$. We shall call $\Phi_{\text{pair}}$ the differential probability of pair production.

As in radiation phenomena, it is important to consider the distance from the nucleus at which the process occurs because at large distances the electric field of the nucleus is screened by the outer electrons. The influence of screening is determined by the quantity:

$$\gamma = 100\,\frac{m_ec^2}{E}\,\frac{1}{v(1 - v)}\,Z^{-1/3}, \tag{2}$$

where
$$v = \frac{E' + m_e c^2}{E}, \tag{3}$$

is the fractional energy of the positon. Screening is negligible when $\gamma \gg 1$. Screening is important when $\gamma \ll 1$. The case $\gamma \approx 0$ is referred to as "complete screening." For a given value of $v$, $\gamma$ decreases with increasing $E$. Thus, for large energies of the primary photon the screening can be considered as complete for all processes of pair production.

Under the assumption that $E \gg m_e c^2$, the theoretical expression for $\Phi_{\text{pair}}$ may be written in the following form (see BHA34):

$$\Phi_{\text{pair}}(E,E')\,dE' = 4\alpha N \frac{Z^2}{A} r_e^2 \frac{dE'}{E} G(E,v), \tag{4}$$

where $G(E,v)$ is a slowly varying function of $E$ and $v$. The following equations give the expressions of $G(E,v)$ corresponding to various ranges of $\gamma$:

*No screening* ($\gamma \gg 1$):

$$G(E,v) = \left[v^2 + (1-v)^2 + \frac{2}{3}v(1-v)\right]\left[\ln \frac{2E}{m_e c^2} v(1-v) - \frac{1}{2}\right]; \tag{5}$$

*Complete screening* ($\gamma \approx 0$):

$$G(E,v) = \left[v^2 + (1-v)^2 + \frac{2}{3}v(1-v)\right]\ln 183 Z^{-1/3} - \frac{1}{9}v(1-v); \tag{6}$$

*Intermediate cases*

($\gamma = 2$):

$$G(E,v) = [v^2 + (1-v)^2]\left[\frac{f_1(\gamma)}{4} - \frac{1}{3}\ln Z\right] + \frac{2}{3}v(1-v)\left[\frac{f_2(\gamma)}{4} - \frac{1}{3}\ln Z\right]; \tag{7}$$

($2 < \gamma < 15$):

$$G(E,v) = \left[v^2 + (1-v)^2 + \frac{2}{3}v(1-v)\right]\left[\ln \frac{2E}{m_e c^2} v(1-v) - \frac{1}{2} - c(\gamma)\right]. \tag{8}$$

The functions $f_1(\gamma)$, $f_2(\gamma)$, and $c(\gamma)$ are the same that enter in the expressions (2.11.6) and (2.11.7) for the radiation probabilities, and are given by Fig. 2.11.1 and Table 2.11.1. The functions $G$ are symmetrical with respect to $v$ and $(1-v)$; therefore $\Phi_{\text{pair}}$ is symmetric with respect to the energy of the negaton and that of the positon. In the case of complete screening $\Phi_{\text{pair}}$ is a function of the fractional energy $v$ only [see Eq. (6)].

Integration of $\Phi_{\text{pair}}(E,E')$ from $E' = 0$ to $E' = E - 2m_e c^2$ yields the total probability for a photon with energy $E$ to produce a pair in a thickness of $dx$ g cm$^{-2}$.



At the limit for small and large energies, respectively, Eqs. (5) and (6) give:

$m_e c^2 \ll E \ll 137 m_e c^2 Z^{-1/3}$:

$$\int_0^{E-2m_ec^2} \Phi_{\text{pair}}(E,E')\,dE' = 4\alpha N \frac{Z^2}{A} r_e^2 \left[\frac{7}{9}\ln\frac{2E}{m_e c^2} - \frac{109}{54}\right]; \tag{9}$$

$E \gg 137 m_e c^2 Z^{-1/3}$:

$$\int_0^{E-2m_ec^2} \Phi_{\text{pair}}(E,E')\,dE' = 4\alpha N \frac{Z^2}{A} r_e^2 \left[\frac{7}{9}\ln(183 Z^{-1/3}) - \frac{1}{54}\right]. \tag{10}$$

For the intermediate cases the integral must be evaluated numerically. Equation (10) shows that the total probability for pair production at large energies is a constant in a given material.

The derivation of Eqs. (5) to (10) is based upon Born's approximation and becomes inaccurate for large values of $Z$, like the derivation of the corresponding formulae relative to radiation processes (see § 2.11). Indeed, experiments to be described in § 6.2 have shown that the theoretical expressions have a systematic error, which increases gradually with increasing $Z$ and amounts to about 12 per cent for lead.

One can conveniently express the probabilities for pair production in terms of the radiation length defined in § 2.11. We shall call:

$$\varphi_{\text{pair}}(E,E') = X_0 \Phi_{\text{pair}}(E,E') \tag{11}$$

the differential probability for pair production per radiation length, and:

$$\mu_{\text{pair}}(E) = \int_0^{E-2m_ec^2} \varphi_{\text{pair}}(E,E')\,dE' \tag{12}$$

the total probability for pair production per radiation length. In the case of complete screening $\mu_{\text{pair}}$ has the constant value:

$$\mu_0 = \frac{7}{9} - \frac{b}{3}, \tag{13}$$

where $b$ is the same as in Eqs. (2.11.15) and (2.11.17). In the same limiting case, the expression of $\varphi_{\text{pair}}$ may be written as follows:

$$\varphi_{\text{pair}}(E,E')\,dE' = \psi_{\text{pair}}(v)\,dv, \tag{14}$$

where
$$\psi_{\text{pair}}(v) = v^2 + (1-v)^2 + (\tfrac{2}{3} - 2b)v(1-v). \tag{15}$$

Figures 1 and 2 show the quantity $E\varphi_{\text{pair}}(E,E')$ plotted as a function of $v$ for various values of $E$ and for two different substances (air and lead). From these figures and from Eqs. (14) and (15) one sees that, if one measures thicknesses in radiation lengths, the functions describing pair production, like those describing radiation processes, depend only slightly on atomic number, and are almost entirely independent of $Z$ at the limit for large energies.

The total probability for Compton scattering, Eq. (2.18.8), decreases rapidly with increasing photon energy, while the total probability for pair production is a slowly increasing function of the energy. Thus, at large energies most of the photons are absorbed by pair production, while at small energies most of the photons are absorbed by Compton effect. The absorption of photons by pair production and Compton effect in lead and air are compared in Figs. 3 and 4. One sees that the energy at which the pair production becomes dominant decreases with increasing atomic number.

As already pointed out, the nucleus in whose vicinity pair production occurs acquires some momentum. Therefore the angle of emission of the

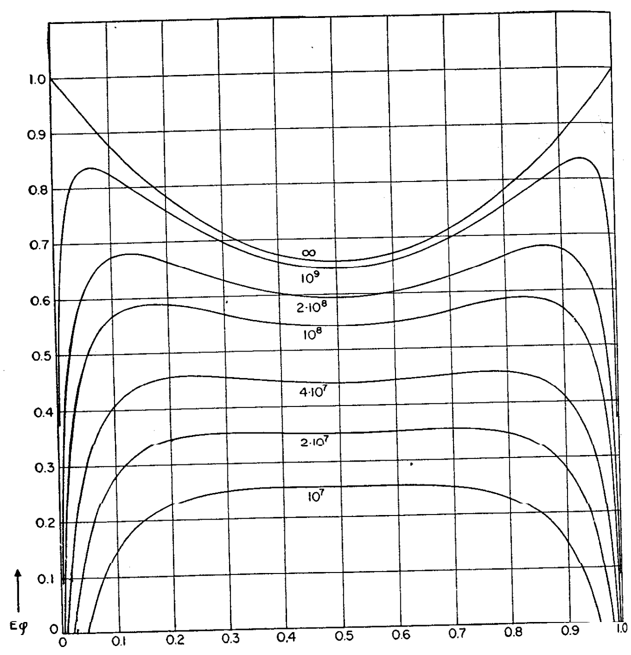


**Fig. 2.19.1.** Differential probability of pair production per radiation length of *air* for photons of various energies. Abscissa: $v = (E' + m_ec^2)/E$; ordinate $E\varphi_{\text{pair}}(E,E')$. The numbers attached to the curves indicate the energy $E$ of the primary photon. From Rossi and Greisen (RB41.1).



two electrons of a pair is not determined by their energy and by that of the primary photon. According to Stearns (StM49) the root mean square angle between the trajectory of a secondary electron of energy $E'$ and that of the primary photon of energy $E$ is given by an expression of the form:

$$\langle\theta^2\rangle_{\text{av}}^{1/2} = q'(E,E',Z)\,\frac{m_ec^2}{E}\ln\left(\frac{E}{m_ec^2}\right); \tag{16}$$

where $q'$ is a function of the atomic number $Z$, the energy $E$ of the primary photon, and the energy $E'$ of the secondary electron. The function $q'$ is always of the order of unity and depends primarily on the ratio

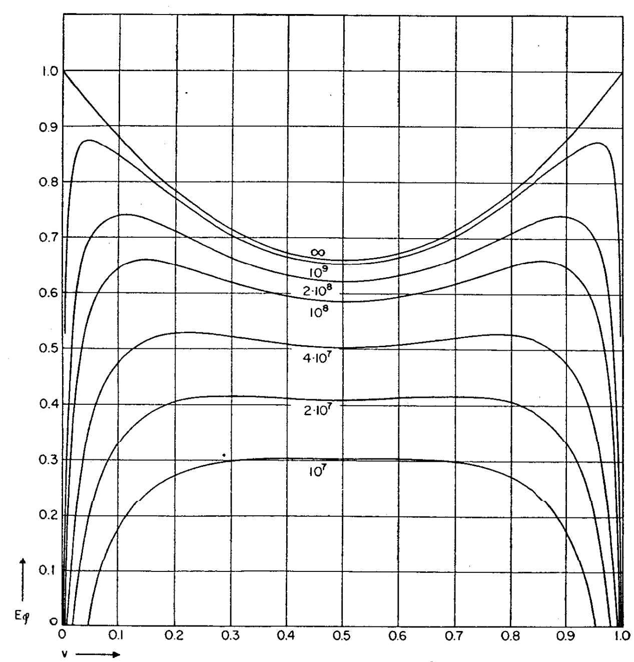


**Fig. 2.19.2.** Differential probability of pair production per radiation length of *lead* for photons of various energies. Abscissa: $v = (E' + m_ec^2)/E$; ordinate: $E\varphi_{\text{pair}}(E,E')$. The numbers attached to the curves indicate the energy $E$ of the primary photon. From Rossi and Greisen (RB41.1).

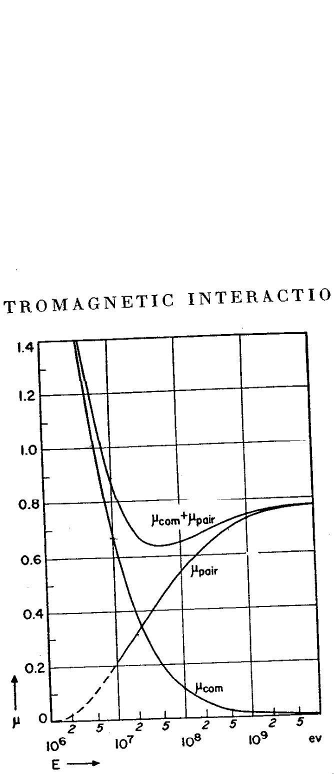


**Fig. 2.19.3.** The total probability per radiation length of *air* for Compton scattering ($\mu_{com}$), for pair production ($\mu_{pair}$), and for either effect ($\mu_{com} + \mu_{pair}$). For $E < 10^7$ ev $\mu_{pair}$ cannot be calculated with the formulas given in the text, which are only valid when $E \gg m_e < c^2$, and a more accurate equation must be used (BHA34). From Rossi and Greisen (RB41.1).

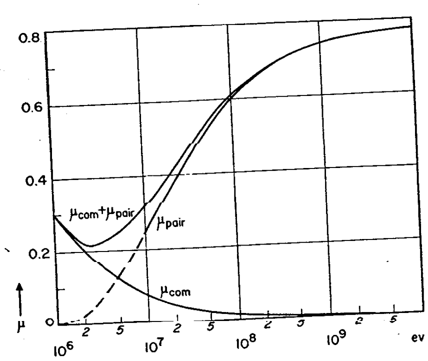


**Fig. 2.19.4.** The total probability for radiation length of *lead* for Compton scattering ($\mu_{com}$), for pair production ($\mu_{pair}$) and for either effect ($\mu_{com} + \mu_{pair}$). For $E < 10^7$ ev, $\mu_{pair}$ cannot be calculated with the formulas given in the text and a more accurate equation must be used (BHA34). From Rossi and Greisen (RB41.1).



$v = (E' + m_e c^2)/E$. The curves in Fig. 5 give $q'$ as a function of $v$ for various values of $Z$. These curves are accurate within 3 per cent for values of $E$ between 50 and 300 Mev. A point in the same figure shows the value of $q'$ for $Z = 90$, $v = \frac{1}{2}$, $E = 5000$ Mev.* Note the similarity

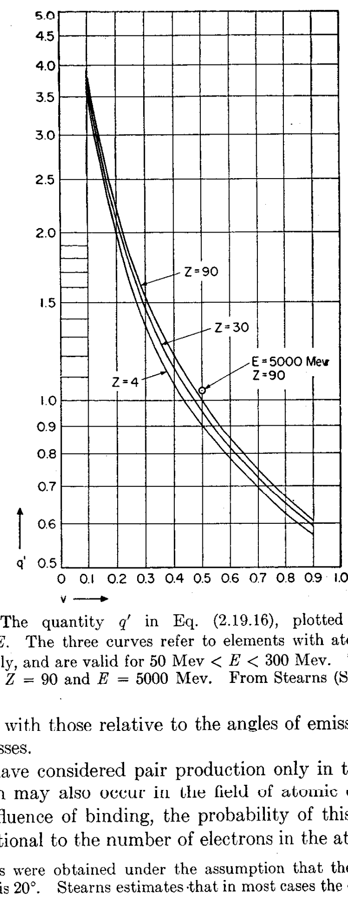


**Fig. 2.19.5.** The quantity $q'$ in Eq. (2.19.16), plotted as a function of $v = (E' + m_e c^2)/E$. The three curves refer to elements with atomic numbers 4, 30, and 90, respectively, and are valid for 50 Mev $< E <$ 300 Mev. The circle represents the value of $q$ for $Z = 90$ and $E = 5000$ Mev. From Stearns (StM49).

of these results with those relative to the angles of emission of photons in radiation processes.

So far we have considered pair production only in the field of nuclei. Pair production may also occur in the field of atomic electrons. If one neglects the influence of binding, the probability of this effect is approximately proportional to the number of electrons in the atom.

* These results were obtained under the assumption that the maximum angle of electron emission is 20°. Stearns estimates that in most cases the error thus introduced does not exceed 10 per cent.

Pair production in the field of electrons has been studied theoretically by various authors (WJA39; WKM47; BA47; VV48). For high-energy photons, and within the limits of the accuracy required for our purposes, we may regard the pair-production cross-section of an atomic electron as equal to that of a proton. In other words, we may take into account the effect of atomic electrons in pair production phenomena by changing $Z^2$ into $Z(Z+1)$ in the equation giving the differential or total probability of pair production per g cm$^{-2}$. Or we may leave the expressions for the probabilities of pair production per radiation length unchanged and use Eq. (2.11.20) instead of Eq. (2.11.11) as the definition of radiation length.

**2.20. *Direct pair production by charged particles.*** One can understand the mechanism of pair production by a fast-moving charged particle by considering that the electromagnetic field of such a particle is equivalent to a flux of photons. When the particle passes in the neighborhood of an atomic nucleus, each at its associated "virtual" photons has a certain probability of undergoing a materialization process, giving rise to an negaton-positon pair.* Here again as in the phenomena of radiation by electrons and pair production by photons, the screening of the Coulomb field of the nucleus by the outer electrons often plays an important role. On the other hand, the spin of the incident particle has a relatively minor effect.

The theory of pair production by charged particles has been worked out by Heitler and Nordheim (HEW34) for the case of particles with velocities small compared with the velocity of light, and by Bhabha (BHJ35) for the case of particles with relativistic velocities [see also (RG36)]. Here we shall closely follow a presentation of the theoretical results given by R. Davisson.†

Let $U$ be the total energy at the incident particle and $E$ the corresponding kinetic energy. Let $U_1'$ and $U_2'$ be the total energies of the negaton and positon arising from a process of pair production. Let $U' = U_1' + U_2'$ be the combined kinetic energy of the two particles and $E'$ the corresponding kinetic energy. Let

$$v = \frac{U'}{U} \tag{1}$$

be the energy of the pair, expressed as a fraction of the incident energy, and let

$$\mu = \frac{(U_1' - U_2')}{(U_1' + U_2')} \tag{2}$$

* A similar approach can be used in the investigation of radiation processes by charged particles, e.g., electrons. In the frame of reference in which the electron is at rest, a radiation process may be described as the result of a Compton collision between one of the virtual photons associated with the fast moving nucleus and the electron; see (WC34).

† Private communication.



be the difference between the energies of the two partners of the pair, expressed as a fraction of the total energy of the pair. From (1) and (2) one obtains:

$$U_1' = \frac{Uv(1+\mu)}{2}; \qquad U_2' = \frac{Uv(1-\mu)}{2} \tag{3}$$

Let $\chi(E,E',\mu)\,dE'\,d\mu\,dx$ represent the probability that a singly charged particle of mass $m$ and kinetic energy $E$, on traversing a thickness $dx$ of matter, produces an electron pair with energy between $E'$ and $E' + dE'$, and with a value of $\mu$ between $\mu$ and $\mu + d\mu$. The quantity $\chi$ may be expressed as follows:

$$\chi(E,E',\mu) = \frac{8}{\pi}\alpha^2 \frac{N}{A} Z^2 r_e^2 H(U,v,\mu), \tag{4}$$

where $H$ is a dimensionless function of $U$, $v$, and $\mu$, and the other symbols have the usual meanings.

Available theoretical results yield expressions for $H$ that are valid only within restricted and widely separated regions of the variables $v$ and $\mu$. One may tentatively bridge the gaps by using as a guide the criterion that the quantity $H$ must be a smoothly varying function of its variables.

From the formulae contained in Bhabha's paper (BHJ35) one obtains four expressions of $H$ that apply to the following extreme cases: (1) pair of low energy, no shielding; (2) pair of low energy, complete shielding; (3) pair of high energy, no shielding; (4) pair of high energy, complete shielding. The four corresponding regions will be denoted, respectively, as follows: *IN*, *IS*, *IIN*, and *IIS*. The four expressions are:

Region *IN*:

$$H(U,v,\mu) = \frac{1}{Uv}\frac{1}{3}\left(1+\frac{\mu^2}{2}\right)\ln\left[\frac{k_1}{(m/m_e)v}\right]\ln\left(k_1'\frac{U}{m_e c^2}v\frac{1-\mu^2}{4}\right); \tag{4a}$$

Region *IS*:

$$H(U,v,\mu) = \frac{1}{Uv}\frac{1}{3}\left(1+\frac{\mu^2}{2}\right)\ln\left[\frac{k_2}{(m/m_e)v}\right]\ln\left(\frac{k_2'}{\alpha Z^{1/3}}\right); \tag{4b}$$

Region *IIS*:

$$H(U,v,\mu) = \frac{1}{Uv^3}\frac{1}{2}\left(\frac{m_e}{m}\right)^2\ln\left(\frac{k_3}{\alpha Z^{1/3}}\frac{m}{m_e}v\right); \tag{4c}$$

Region *IIN*:

$$H(U,v,\mu) = \frac{1}{Uv^3}\frac{1}{2}\left(\frac{m_e}{m}\right)^2\ln\left(2k_4\frac{U}{mc^2}\right). \tag{4d}$$

In the above equations the symbols $k$ and $k'$ indicate constants of the order of magnitude of one, but otherwise undetermined. One sees that, in the regions *II*, $H$ is independent of $\mu$. In the regions *I* the dependence of $H$ on $\mu$ is mainly determined by the factor $(1 + \mu^2/2)$.

We shall use Eqs. (4) beyond their legitimate ranges of validity (explicitly given in Bhabha's paper) by defining the four regions as follows:

Region *IN*:
$$\frac{2m_ec^2}{U} < v < \frac{2m_e}{m}; \qquad v < \frac{(2m_ec^2/U)}{\alpha Z^{1/3}}.$$

Region *IS*:
$$\frac{2m_ec^2}{U} < v < \frac{2m_e}{m}; \qquad v > \frac{(2m_ec^2/U)}{\alpha Z^{1/3}}.$$

Region *IIS*:
$$\frac{2m_e}{m} < v < 1; \qquad v < \left(\frac{2m_e}{m}\right)\alpha Z^{1/3}\left(\frac{U}{mc^2}\right).$$

Region *IIN*:
$$\frac{2m_e}{m} < v < 1; \qquad v > \left(\frac{2m_e}{m}\right)\alpha Z^{1/3}\left(\frac{U}{mc^2}\right).$$

Notice that if $m = m_e$ the regions *II* disappear. Notice also that for $U/mc^2 < 1/\alpha Z^{1/3}$ only the "unshielded" regions exist, whereas for $U/mc^2 > (m/2m_e)(1/\alpha Z^{1/3})$ the region *IIN* disappears.

Integration of $\chi(E,E',\mu)\,dE'\,d\mu\,dx$ over $\mu$ yields the probability for the production in $dx$ of a pair with energy in $dE'$, irrespective of the division of the energy between the two electrons. This probability has an expression of the form:

$$\frac{8}{\pi}\alpha^2\frac{N}{A}Z^2r_e^2L(U,v)\,dE'\,dx,$$

where:
$$L(U,v) = \int_{-1}^{+1} H(U,v,\mu)\,d\mu. \tag{5}$$

If in Eq. (4a) we replace the expression $(1-\mu^2)/4$ in the logarithm by its average value $\frac{1}{6}$ we obtain:

Region *IN*:
$$L(U,v) = \frac{7}{9}\frac{1}{Uv}\ln\left[\frac{k_1}{(m/m_e)v}\right]\ln\left(\frac{k_1'}{6}\frac{U}{m_ec^2}v\right); \tag{6a}$$

Region *IS*:
$$L(U,v) = \frac{7}{9}\frac{1}{Uv}\ln\left[\frac{k_1}{(m/m_e)v}\right]\ln\left(\frac{k_2'}{\alpha Z^{1/3}}\right); \tag{6b}$$

Region *IIS*:
$$L(U,v) = \frac{1}{Uv^3}\left(\frac{m_e}{m}\right)^2\ln\left(\frac{k_3}{\alpha Z^{1/3}}\frac{m}{m_e}v\right); \tag{6c}$$

Region *IIN*:
$$L(U,v) = \frac{1}{Uv^3}\left(\frac{m_e}{m}\right)^2\ln\left(2k_4\frac{U}{mc^2}\right). \tag{6d}$$

One can make use of the freedom in the choice of the constants $k$ and $k'$ to match the various expressions for $L$ at the boundaries between



the different regions. Thus taking $k_1 = k_2$, $k_1' = 3k_2'$ insures continuity at the boundary between *IN* and *IS*, and taking $k_3 = k_4$ insures continuity at the boundary between *IIS* and *IIN*.

A quantity of considerable physical interest is the fractional average energy loss per g cm$^{-2}$ due to pair production. The expression for this quantity may be written as follows:

$$-\frac{1}{U}\frac{dE}{dx} = \frac{8}{\pi}\alpha^2\frac{N}{A}Z^2r_e^2\frac{m_e}{m}M(U), \tag{7}$$

where:
$$M(U) = \frac{m}{m_e}\int_{2mc^2/U}^{1} L(U,v)\,Uv\,dv \tag{8}$$

(the lower limit of this integral corresponds to the lower boundary of region *IN*).

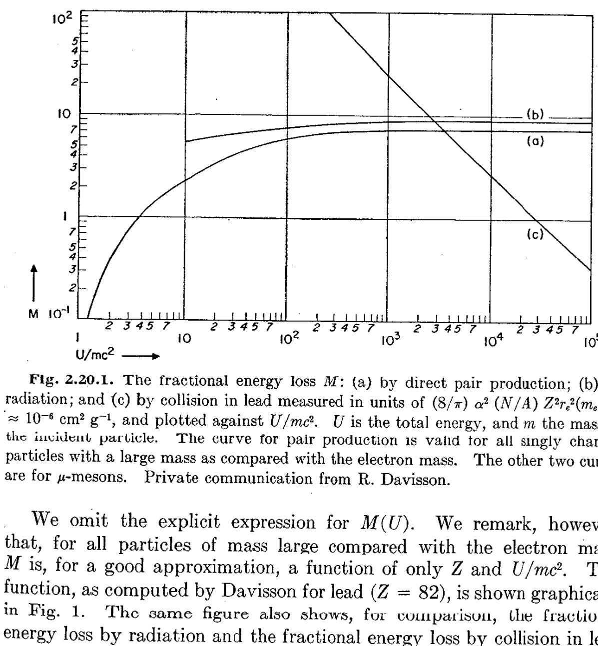


**Fig. 2.20.1.** The fractional energy loss $M$: (a) by direct pair production; (b) by radiation; and (c) by collision in lead measured in units of $(8/\pi)\,\alpha^2\,(N/A)\,Z^2r_e^2(m_e/m) \approx 10^{-6}$ cm² g$^{-1}$, and plotted against $U/mc^2$. $U$ is the total energy, and $m$ the mass of the incident particle. The curve for pair production is valid for all singly charged particles with a large mass as compared with the electron mass. The other two curves are for $\mu$-mesons. Private communication from R. Davisson.

We omit the explicit expression for $M(U)$. We remark, however, that, for all particles of mass large compared with the electron mass, $M$ is, for a good approximation, a function of only $Z$ and $U/mc^2$. This function, as computed by Davisson for lead ($Z = 82$), is shown graphically in Fig. 1. The same figure also shows, for comparison, the fractional energy loss by radiation and the fractional energy loss by collision in lead both computed for $\mu$-mesons and measured in units of