CHAPTER

5

Cascade Showers *

5.1. ***Formulation of the problem, and definitions.*** The theory of electromagnetic interactions developed in Chapter 2 shows that charged particles traversing matter lose energy by *collision* and by *radiation*. Most of the energy lost by collision is spent in exciting atoms or ejecting from the atoms electrons of small energy, and is thus dissipated. The energy lost by radiation, instead, is fairly uniformly distributed among secondary photons of all energies from zero up to the energy of the primary particle itself.

For electrons of small energy and for heavier particles of practically all energies, collision losses are more important than radiation losses. Thus the electromagnetic interactions of heavy particles and of low-energy electrons result mainly in an energy dissipation. Electrons of large energy, however, lose most of their energy by radiation. Hence in the interactions of high-energy electrons with matter only a small fraction of the energy is dissipated, while a large fraction is spent in the production of high-energy photons. The secondary photons, in turn, undergo materialization or Compton collision. Either process gives rise to electrons of energy comparable with that of the photons. These new electrons radiate more photons, which again materialize into electron pairs or produce Compton electrons. At each new step the number of particles increases and their average energy decreases. As the process goes on, more and more electrons fall into an energy range where radiation losses cannot compete with collision losses, until eventually the energy of the primary electron is completely dissipated in excitation and ionization of atoms.

The phenomenon outlined is called a *multiplicative shower*, or a *cascade shower*. It is clear that a shower can be initiated by a high-energy photon as well as by a high-energy electron. Occasionally a meson or a proton, too, can give rise to a shower by producing a secondary electron or photon of high energy. As a concrete example, Fig. 1 shows a cloud-chamber picture of a large shower as it develops through a number of lead plates

* Parts of this chapter are taken from the article "Cosmic-Ray Theory" by Rossi and Greisen (ref. RB41.1).

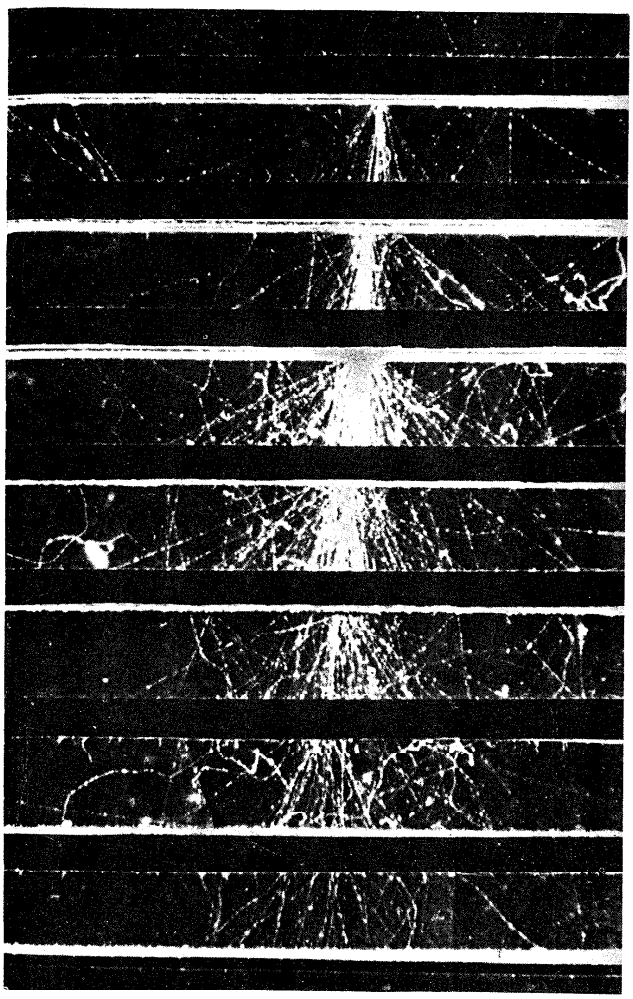

Fig. 5.1.1. Cloud-chamber picture of a large cascade shower. The plates across the chamber are lead, 1.27 cm thick. From C. Y. Chao.

each 1.27 cm thick. The energy of the initiating particle is roughly several times 10^{10} ev.

Stated in all generality, the theoretical problem of multiplicative showers is the following: assume that an electron or a photon of energy E_0 is incident upon a layer of matter and consider a plane perpendicular to the direction of the primary particle, at a depth t from its point of incidence. Divide this plane into infinitesimal elements $d\sigma_1$, $d\sigma_2$, . . . $d\sigma_m$; divide the solid angle into infinitesimal elements, $d\omega_1$, $d\omega_2$, . . . $d\omega_n$; divide the energy interval from 0 to E_0 into infinitesimal elements: dE_1, dE_2, . . . dE_l. What is the probability that M_π electrons and M_γ photons pass through the plane under consideration and that each of these particles has energy in a given dE_l, spatial coordinates in a given $d\sigma_m$ and angular coordinates in a given $d\omega_n$? This problem is one of almost hopeless mathematical complexity. It is usually possible, however, to introduce some simplifications that greatly facilitate the mathematical problem.

(a) *Separation of the Longitudinal Development from the Angular Spread.* At high energies the angles of emission of secondary electrons and photons are very small (see § 2.11, § 2.19); the scattering of electrons is also small, at least in substances of low atomic number (see § 2.16); so that the shower develops essentially in the direction of the incident particle. This enables one to treat the *longitudinal development* of showers and their *lateral spread* as two separate problems. More specifically, one first determines the dependence of the functions describing the shower on the thickness, t, of matter traversed, by neglecting the path increase due to angular spread of the shower particles. One then investigates the lateral spread around the axis and the angular distribution of the shower particles, making use of the information gained in the solution of the first problem. This procedure is justified in the case of substances of low atomic number. In air, for instance, electrons with energy below 10^8 ev lose energy mainly by collision and therefore do not participate effectively in the development of the showers. On the other hand, the scattering of 10^8-ev electrons in air is still very small. In heavy elements, however, radiation losses are comparatively more important, and therefore multiplication continues down to much smaller electron energies. In lead, for instance, multiplication practically ceases at about 10^7 ev, an energy at which scattering is very conspicuous. Thus, in heavy elements the assumption that the longitudinal shower development can be treated as separate from its angular spread is justified only if one restricts one's attention to the behavior of the more energetic shower particles.

In this volume we shall discuss in detail only the longitudinal development of showers. The theory of the lateral spread, which is important mainly in connection with the problem of air showers, will be discussed in a subsequent volume now in preparation.

(b) *The Average Behavior of Showers.* In many problems one is concerned only with the *average behavior* of showers; e.g., one may want to know the *average number* of electrons (or photons) with energy greater than E that emerge from a layer of thickness t. This quantity is much easier to compute than the *probability* for the number of these particles to have a predetermined value. In fact, most of the results of the shower theory obtained so far refer to the average behavior of the shower. The problem of computing the probability function has been treated only in a preliminary way, generally by first computing the average behavior of the shower and then by estimating the probability of a given deviation from this average behavior. The latter problem is known as the *fluctuation problem.*

It may be appropriate to define here some of the quantities that are used to describe the average longitudinal development of a shower.

We shall denote by $\pi(E,t)\,dE$ the average number of electrons with energy between E and $E + dE$ at the depth t.* Correspondingly we shall denote by $\gamma(E,t)\,dE$ the average number of photons with energy between E and $E + dE$ at t. We shall call the functions $\pi(E,t)$ and $\gamma(E,t)$ the *differential spectra* of electrons and photons respectively. We shall call integral spectra of electrons and photons the functions:

$$\Pi(E,t) = \int_E^\infty \pi(E',t)\,dE', \qquad \Gamma(E,t) = \int_E^\infty \gamma(E',t)\,dE', \tag{1}$$

representing, respectively, the average number of electrons and photons of energy greater than E at the depth t. We shall call *energy dissipation* and denote by the symbol $\rho(t)$ the energy per unit thickness expended by all shower particles in excitation and ionization of atoms. If the shower propagates in a gas, the number of ion pairs per unit thickness is given by $\rho(t)/V_0$ where V_0 is the energy per ion pair (see § 2.10). We shall use the notations:

$$\pi^{(\gamma)}(E_0,E,t), \quad \pi^{(\gamma)}(E_0,E,t), \quad \text{etc.}$$

* The exact meaning of this function depends on the specific problem under consideration. Suppose, for example, that we are interested in the properties of showers produced by individual electrons of a given energy in a thickness t of matter. We may then interpret the function $\pi(E,t)\,dE$ as the average value for the number of shower electrons with energy in dE at E that cross a plane perpendicular to the trajectory of the initiating electron every time one such electron arrives upon the absorber. Suppose, instead, that electrons or photons arrive upon the absorber at a constant rate and with an arbitrary energy distribution. We may then interpret the function $\pi(E,t)\,dE$ as the number per unit time of shower electrons with energy in dE at E that cross a plane perpendicular to the direction of the incident beam at a depth t within the absorber.

when we wish to refer explicitly to a shower initiated by an electron of energy E_0, or by a photon of energy E_0.

Let us consider the "average" shower produced by an electron or photon of energy large compared with the energy at which radiation losses overtake collision losses and let us restrict our attention to secondary shower particles of energy small compared with the initial energy. It is then clear, from the qualitative description of a shower given at the beginning of

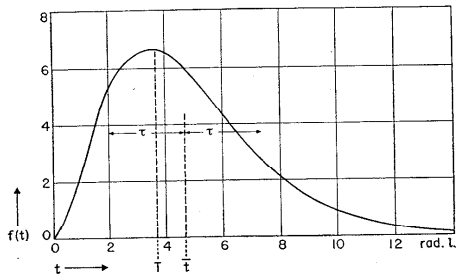

Fig. 5.1.2. Qualitative behavior of shower functions, $f(t)$. T, optimum thickness; $\bar{t}$, abscissa of center of gravity; τ, longitudinal spread.

this section, that each of the quantities π, γ, Π, ,Γ and ρ is a function of t, $f(t)$, of the type illustrated in Fig. 2; i.e., it increases rapidly at first with increasing t, goes through a maximum and then decreases again towards zero. The abscissa T at the maximum is often referred to as the *optimum thickness*.

Along with the functions π, γ, Π, Γ, ρ, we shall consider their *moments* with respect to t; p_n, g_n, P_n, G_n, r_n. Thus

$$p_n(E) = \int_0^\infty t^n \pi(E,t)\, dt, \qquad g_n(E) = \int_0^\infty t^n \gamma(E,t)\, dt, \quad \text{etc.} \tag{2}$$

These quantities do not depend on t, and, largely for this reason, are much easier to compute than the corresponding functions π, γ, etc.

Some of the moments have a simple physical significance and can be used directly in the solution of certain problems. Consider, first, the moments of zero order. These quantities are usually referred to as *track lengths*. The track length represents the area under the shower curve (see Fig. 2). Thus, for instance, r_0 is the total energy dissipated by the shower particles before complete absorption of the shower, and is therefore

equal to the energy of the electron or photon that has produced the shower; $p_0(E)\, dE$ is the total distance traveled by all shower electrons while their energy lies between E and $E + dE$; $P_0(E)$ is the total distance traveled by all shower electrons of energy greater than E, etc.

The following is an example of a problem whose solution requires only knowledge of the track length. Suppose that a primary radiation, which is not appreciably absorbed by matter, produces secondary electrons of energy E_0 at the rate of N electrons per unit thickness. What is the number, $n(E)\, dE$, of electrons with energy in dE at E in equilibrium with the primary radiation? For the solution of this problem consider that the electrons observed at any given depth arise from cascade multiplication of electrons produced at different distances, t, above the place of observation. The number of such electrons produced at distances between t and $t + dt$ is $N\, dt$. The average contribution of each of these electrons is $\pi^{(\pi)}(E_0,E,t)$. If one assumes that the thickness of material traversed by the primary radiation is greater than the range of the showers initiated by the secondary electrons, one obtains for $n(E)$ the expression:

$$n(E) = N \int_0^\infty \pi^{(\pi)}(E_0,E,t)\, dt = N p_0^{(\pi)}(E_0,E). \tag{3}$$

Higher moments are also directly connected with quantities of physical significance. Thus the *center of gravity* of the average distribution curve for electrons of energy E is given by:

$$\bar{t}_\pi(E) = \frac{\int_0^\infty t\pi(E,t)\, dt}{\int_0^\infty \pi(E,t)\, dt} = \frac{p_1(E)}{p_0(E)}, \tag{4}$$

and similar expressions hold for the centers of gravity of the other shower functions. Another example is the *longitudinal spread*, τ. For shower electrons of energy E, this quantity is defined by the following equation:

$$[\tau_\pi(E)]^2 = \frac{\int_0^\infty [t - t_\pi(E)]^2 \pi(E,t)\, dt}{\int_0^\infty \pi(E,t)\, dt}; \tag{5}$$

from which it follows that:

$$[\tau_\pi(E)]^2 = \frac{p_2(E) - [p_1(E)]^2/p_0(E)}{p_0(E)}. \tag{6}$$

Similar expressions give the longitudinal spread corresponding to the other shower functions.

The above considerations show that knowledge of the first few moments is already sufficient to establish the main features of a shower curve. It can also be proved that, in principle at least, one can compute the shower curve exactly if all of its moments are known.

5.2. ***Approximations.*** Even the discussion of the average behavior of showers represents a difficult mathematical problem, and it is thus very desirable to introduce simplifications wherever possible.

The phenomena mainly responsible for the development of showers are radiation and pair production processes in the electric fields of nuclei. Therefore the natural unit of thickness in shower theory is the *radiation length* [see Eq. (2.11.11)]. With this unit, the theoretical expressions for the probabilities of the above processes become practically independent of atomic number at the limit for high energies. Radiation and pair-production processes in the field of atomic electrons are of secondary importance (except in the lightest elements) and need to be taken into consideration only in an approximate manner. One may do this by simply changing Z^2 into $Z(Z+1)$ in the expression for the radiation length (see §§ 2.11, 2.19).

The derivation of the theoretical expressions for the probabilities of radiation and pair production is based on Born's approximation. As mentioned in §§ 2.11 and 2.19, there are reasons to believe that this approximation yields more reliable results for elements of low atomic number than for elements of high atomic number. This is confirmed by experiments on pair production (§ 6.2) showing that the probability of this effect is close to the theoretical value in elements of low atomic number but is appreciably lower in elements of high atomic number. As expected (see § 2.11), the deviations from the theoretical predictions appear to be proportional to Z^2. In order to correct for the error involved in the use of Born's approximation one may multiply the expression for the radiation length by the factor $1 + 0.12\ (Z/82)^2$, where the numerical coefficient is chosen so as to fit the experimental data on pair production in lead at 88 Mev.

With the corrections for the effect of atomic electrons and for the inaccuracy of Born's approximation, the equation for the radiation length becomes:

$$\frac{1}{X_0} = 4\alpha \frac{N}{A} Z(Z+1) r_e^2 \ln (183 Z^{-1/3}) / \left[1 + 0.12 \left(\frac{Z}{82}\right)^2\right]. \tag{1}$$

Table 5.24.1 lists the values of the radiation length in various substances computed according to Eq. (1).

We have seen in § 2.3 that the differential collision probabilities of positive and negative electrons have different mathematical expressions. However the difference is negligible except for those very rare cases in which the two secondary electrons come out of the collision with comparable energies.

In what follows we shall neglect the small difference in the collision probabilities of positive and negative electrons. For both kinds of particles, we shall represent the collision probability by a symmetric expression: $\varphi_{\text{col}}(E,E') = \varphi_{\text{col}}(E,E-E')$; it shall be understood that $\varphi_{\text{col}}(E,E')\,dE'$ gives the probability per radiation length of a collision from which *either one* of the two particles emerges with an energy in dE' at E'.

For φ_{col} one may use a symmetrized Rutherford formula [see Eq. (2.3.10)]:

$$\varphi_{\text{col}}(E,E') = 2CX_0 m_e c^2 \left[\frac{1}{(E')^2} + \frac{1}{(E-E')^2}\right]$$

$$= 2CX_0 \frac{m_e c^2 E^2}{(E')^2(E-E')^2}\left[1 - 2\frac{E'}{E} + 2\left(\frac{E'}{E}\right)^2\right]. \tag{2}$$

For greater accuracy, one may use the arithmetic mean of the symmetric collision probabilities for positive and negative electrons [see Eqs. (2.3.2a) and (2.3.5)]:

$$\varphi_{\text{col}}(E,E') = \frac{1}{2} 2CX_0 \frac{m_e c^2 E^2}{(E')^2(E-E')^2}$$

$$\left[1 - \frac{E'}{E} + \left(\frac{E'}{E}\right)^2\right]^2 \left[1 + 1 - 2\frac{E'}{E} + 2\left(\frac{E'}{E}\right)^2\right]$$

or

$$\varphi_{\text{col}}(E,E') = 2CX_0 \frac{m_e c^2 E^2}{(E')^2(E-E')^2}\left[1 - \frac{E'}{E} + \left(\frac{E'}{E}\right)^2\right]^3 \tag{3}$$

From the theory of showers we wish to obtain accurate data on the number and energy distribution of *electrons and photons with energy large compared with a given energy* η_0, as well as on the *total energy dissipation of all* shower particles. However, we do not ask of the shower theory detailed information on electrons and photons of energy near or below η_0. The value of η_0 is determined by the kind of experimental data that we wish to compare with the theory. Here, however, for the sake of convenience, we shall make the following choice:

$$\eta_0 = 5 \cdot 10^6 \text{ ev}. \tag{4}$$

Since η_0 is large compared with the rest energy of electrons, in the equations describing the behavior of electrons of energy greater than η_0 one can always take:

$$E = U = pc, \tag{5}$$

where E is the kinetic energy, U the total energy, and p the momentum.

One will recall that no simple expression exists for the probability of

collisions in which the secondary electron acquires an energy of the order of the binding energy of electrons in atoms. There is, however, a formula, Eq. (2.5.1), for the mean energy loss, $k_{\mathrm{col}(<\eta_0)}$, due to all collisions in which the secondary electrons acquire energies smaller than η_0. It is thus expedient to replace these collisions by a continuous energy dissipation and introduce a collision loss per radiation length:

$$\epsilon(E) = X_0 k_{\mathrm{col}(<\eta_0)}(E). \tag{6}$$

For consistency, electrons whose energy falls below the limiting value η_0 by collision loss should no longer be considered as shower particles and their energy should be added to $\epsilon(E)$. With this addition, the quantity $\epsilon(E)$ tends to infinity as E approaches η_0 because, under the assumption of a continuous energy loss, an electron of energy η_0 removes from the shower a finite amount of energy, namely its own energy, on traversing an infinitesimal thickness. The quantity $\epsilon(E)$ defined by Eq. (6), instead, is everywhere finite. One can see this easily by considering that for E equal to or smaller than η_0, $k_{\mathrm{col}(<\eta_0)}$ becomes identical to the total collision loss [see Eq. (2.5.11)]. This quantity is finite because in its computation one considers as "lost" only the energy of those electrons that come out of a collision with an energy of less than one-half the energy of the incident electron. The use of Eq. (6), however, does not invalidate any of the conclusions that are not already invalidated by the assumption of a continuous energy loss. Indeed, the two definitions of $\epsilon(E)$ differ only when $E < 2\eta_0$, and in this energy region the substitution of a continuous energy loss for the actual collision processes grossly distorts the energy distribution of electrons and photons.

For the computation of the energy dissipation, $\rho(t)$, we shall use the formula:

$$\rho(t) = \int_0^\infty \pi(E,t)\epsilon(E)\, dE. \tag{7}$$

This expression is clearly consistent with the conservation of energy. Indeed, we have systematically neglected energy transfers to atoms in the computation of the probability for radiation, pair production, and Compton effect as well as in the computation of the probability for collisions giving rise to secondary electrons of energy greater than η_0. Therefore collisions giving rise to secondary electrons with energy less than η_0 represent the only process in which energy is "lost" from the stream of shower particles and eventually transformed into excitation and ionization energy of atoms.

Equation (7) is equivalent to the assumption that the energy given to secondary electrons of energy less than η_0 is dissipated by these electrons immediately. Actually, of course, the secondary electrons travel a certain distance before losing all of their energy. However, the error introduced by neglecting the finite range of electrons of energy less than η_0 will be small as long as this range is small compared with distances in which the function $\rho(t)$ changes appreciably. This means small compared with one radiation length, if one considers showers initiated by particles of energy very large compared with η_0. Computations show that this condition is well fulfilled.

Note that photons of $5 \cdot 10^6$ ev are much more penetrating than electrons of the same energy. In heavy elements their mean free path is several radiation lengths, so that for the computation of $\rho(t)$ one cannot describe radiation losses giving rise to photons of energy less than $5 \cdot 10^6$ ev as a continuous energy loss. Therefore one should either avoid a separation of radiation losses into a continuous and discontinuous part, or take a value much smaller than $5 \cdot 10^6$ ev for the dividing energy. For the same reason, one should take correct values for the probabilities of photon absorption down to energies at which the photon mean free path becomes smaller than one radiation length. This implies that, in heavy elements, photoelectric absorption should not be disregarded altogether [for numerical values of the various absorption coefficients of photons at low energies, see (ERD48)].

In the shower theory one often replaces the variable collision loss with a constant collision loss, ϵ_0, called the *critical energy*, and one takes for ϵ_0 the collision loss per radiation length of electrons of energy ϵ_0. Therefore ϵ_0 is given by the following equation:

$$\epsilon_0 = \epsilon(\epsilon_0). \tag{8}$$

Table 5.24.1 lists the values of ϵ_0 for various substances, along with the values of Z, A, and X_0. For the computation of the figures given in column 5 we have used Eq. (2.5.1) with $\eta = 5 \cdot 10^6$ ev as the expression for the collision loss. The values given in column 6 include a correction for the density effect on collision losses (see § 2.6). Note that ϵ_0 coincides approximately with the energy at which radiation losses and collision losses of electrons are equal (it would be practically identical to this energy if the asymptotic expression for the radiation loss were valid at $E = \epsilon_0$). In first approximation, ϵ_0 is inversely proportional to the atomic number, since X_0 goes approximately as A/Z^2 and A/Z is approximately constant.

The substitution of a constant energy loss, ϵ_0, for the variable energy loss $\epsilon(E)$ may be justified by the following arguments. In all substances $\epsilon_0 > \eta_0$. For $E > \eta_0$ the function $\epsilon(E)$ varies slowly with energy. Therefore ϵ_0 represents correctly the energy loss of electrons in a wide energy region around ϵ_0. For $E \gg \epsilon_0$, collision losses are unimportant and, therefore, do not need to be taken into account accurately. The choice of $\epsilon(E)$ for E close to η_0 is also irrelevant because the functions π and γ become meaningless in this energy region and the computation of the function $\rho(t)$ is not affected appreciably by this choice.

As long as we confine our attention to energies large compared with the critical energy, we can develop the theory of showers by considering only radiation phenomena and pair production. Furthermore, if the energies under consideration are also large compared with $137 m_0 c^2 Z^{-1/3}$, radiation phenomena and pair production can be described by the asymptotic formulae for complete screening (see § 2.11 and § 2.19). In what follows we shall call "Approximation A" the approximation in which one neglects collision processes and Compton effect and uses the asymptotic formulae to describe radiation processes and pair production. Note that

the shower theory under Approximation A gives identical results for all substances, provided, of course, thicknesses are measured in radiation lengths (see § 5.7).

For energies in the neighborhood of the critical energy, Compton effect can still be disregarded, but collision processes must be taken into account. These, however, do not contribute appreciably to the production of secondary electrons with energies greater than η_0, hence it will be sufficient to consider only their influence on the energy loss of electrons. We shall call "Approximation B" the approximation in which the Compton effect is neglected, the collision loss is described as a constant energy dissipation, ϵ_0, and the asymptotic formulae for radiation processes and pair production are used. Approximation B yields identical results for all elements, provided one measures thicknesses in radiation lengths and energies in terms of the critical energy (see § 5.12). Note, however, that the basic assumptions of Approximation B are better justified in the case of light elements than in the case of heavy elements. Indeed, Figs. 2.11.4, 2.19.3, and 2.19.4 show that in air, for example, the asymptotic expressions for radiation loss and pair production remain approximately valid down to almost the critical energy, whereas in lead at the critical energy the asymptotic expression for the radiation loss is about 1.5 times too large and that for the probability of pair production is off by about a factor 3.

For energies small compared with the critical energy, both Compton effect and collision processes contribute considerably to the absorption, as well as to the production of shower particles. In air, for instance, (see Fig. 2.19.3), a photon of less than about $2 \cdot 10^7$ ev is more likely to undergo Compton scattering than materialization. Considering that electrons and photons are present in comparable numbers, one also finds that, in air, electrons with energies of the order of 10^7 ev are about as likely to be produced by collision processes of electrons as by materialization of photons.

It will be shown that the shower problem under Approximation A can be solved completely. When the energies under consideration are not too close to the initial energy, one can use an analytical procedure, on the basis of principles laid down by Carlson and Oppenheimer (COF37), and by Landau and Rumer (LLD38). For energies close to the initial energy one can apply a method of successive approximations, developed by Bhabha and Heitler (BHJ37).

The shower problem under Approximation B, too, has been partially solved by analytical means (SHS38; SbR38; TIE39; BHJ42; BHJ43; BHJ48; SHS49.2).

In the energy range in which Compton effect and collision processes need to be taken into account in detail and where the asymptotic expressions for radiation losses and pair production do not represent acceptable approximations, an analytical solution of the shower problem is still lacking. Most of the theoretical results that are applicable to this energy range concern the moments of the shower functions, i.e., quantities in which the variable t has been integrated out and which, therefore, are easier to compute by numerical methods than the corresponding shower functions.

Note that nuclear interactions of photons or electrons have been systematically disregarded in the theory of showers. Experimental evidence shows that the nuclear interactions have much smaller cross-sections than electromagnetic interactions and therefore cannot affect the development of showers appreciably.

Direct pair production by electrons, too, has been disregarded in all calculations known to the writer. The effect of this process on the development of showers is certainly small, but may not be negligible.

5.3. ***Some qualitative results on showers.*** It is possible to obtain a qualitative description for the development of a cascade shower by considering the following extremely crude model.* Assume that each electron of energy greater than the critical energy undergoes a radiation process at a distance of one radiation length from the place where it has been produced and that, in this process, it loses half of its energy to a secondary photon. Assume that each photon undergoes a materialization process at a distance of one radiation length from the place of production, splitting its energy equally between the two secondary electrons. Neglect Compton effect. Neglect collision loss of electrons of energy greater than the critical energy. Assume that electrons, when their energy becomes smaller than the critical energy, cease to radiate and are soon brought to rest by collision losses.

According to this model, an electron of energy $E_0 \gg \epsilon_0$ incident upon matter will give rise to one electron and one photon of energy $E_0/2$ after one radiation length. In the next radiation length the secondary electron will produce an electron-photon pair, the secondary photon will produce a negaton-positon pair. There will be now four particles, each with an energy of $E_0/4$. After three radiation lengths the shower will contain eight particles, of which five are electrons and three photons. After t radiation lengths the total number of electrons and photons will be:

$$N = 2^t, \tag{1}$$

and the energy of each particle:

$$E = E_0 2^{-t}. \tag{2}$$

Thus the total number of particles with energy greater than E (i.e., the quantity $N(E) = \Pi(E,t) + \Gamma(E,t)$, according to the notations intro-

* See W. Heitler, *Quantum Theory of Radiation*, second edition, Oxford University Press (1944), § 21.

duced in § 5.1), will increase exponentially with t from $t = 0$ to $t = T(E)$, where:

$$T(E) = \frac{\ln (E_0/E)}{\ln 2}, \tag{3}$$

and then drop abruptly to zero. The number of these particles at the maximum will be:

$$N_{\max} = E_0/E. \tag{4}$$

The discontinuous character of the function $N(E)$ is a consequence of our oversimplified assumptions. In the actual case we should expect a smooth dependence of N on t. However, we may also expect the maximum for the actual shower curve to occur approximately at the place indicated by our model, and the number of particles at the maximum to be approximately proportional, if not equal, to the ratio E_0/E. We shall see later that these predictions are confirmed by the exact theory.

According to our assumptions, multiplication continues only as long as electrons have energies greater than the critical energy. Therefore the curve for the total number of particles, irrespective of energy, will reach a maximum value of E_0/ϵ_0 at $t = \ln (E_0/\epsilon_0)/\ln 2$.

In the crude model discussed above, all shower particles at a given depth have the same energy, and this energy decreases with increasing depth. In the actual case the shower particles at a given depth have an energy distribution whose shape changes gradually as the shower progresses, so that low-energy particles become relatively more abundant. One may consider an *average spectrum*, namely the energy distribution averaged over the depth and weighted according to the number of particles present at each depth. This function is represented by the integral:

$$\int_0^\infty N(E,t)\, dt,$$

and is identical to the track length defined in § 5.1.

From Eq. (1) one obtains:

$$\int_0^\infty N(E,t)\, dt - N(0) + N(1) + N(2) + \ldots \sim \int_0^{T(E)} 2^t\, dt;$$

from which it follows, if $2^T \gg 1$:

$$\int_0^{T(E)} N(E,t)\, dt \approx \frac{e^{T \ln 2}}{\ln 2} = \frac{E_0/E}{\ln 2}. \tag{5}$$

Thus the average integral spectrum of shower particles obeys a $1/E$ law. This result, of course, holds only for energies greater than the critical energy. At these energies, according to our model, electrons and photons behave in an identical manner and have therefore the same energy spectrum.

We may summarize the results of the preceding analysis with the following statements, which apply to a shower initiated by either an electron or a photon of energy E_0 large compared with the critical energy.

(a) The number of electrons or photons in the initial stages of the shower development increases exponentially with depth.

(b) The number of electrons or photons of energy greater than E goes through a maximum at a thickness of about $\ln (E_0/E)$. Thus the optimum thickness increases slowly with increasing primary energy.

(c) The number of electrons or photons with energies greater than E at the maximum is approximately proportional to E_0/E.

(d) The shower curve for the total number of shower particles, irrespective of energy, has a maximum at a thickness of about $\ln (E_0/\epsilon_0)$.

(e) The total number of shower particles at the maximum is approximately proportional to E_0/ϵ_0.

(f) For energies large compared with the critical energy, the integral spectra of electrons and photons averaged over the shower have approximately the same shape and show a $1/E$ energy dependence.

5.4. *The method of successive collisions.* The theory of cascade showers had its origin in two independent papers by Bhabha and Heitler (BHJ37) and by Carlson and Oppenheimer (CJF37), respectively. Bhabha and Heitler introduced an iteration method that may be described as the *method of successive collisions*. The procedure is as follows. Given an electron of energy E_0 incident at $t = 0$, one calculates the probability, $f_0(E_0,E,t)$, of this electron having an energy larger than E at the depth t. Then one computes the number of photons with energy larger than E emitted by the electron at various points of its path, and the number, $f_1(E_0,E,t)$, of electrons produced by these photons (called electrons of the first generation) that reach the thickness t with energy larger than E. In a similar way one evaluates the electrons of the second and succeeding generations. The total number of electrons with energy greater than E at the thickness t is finally expressed as the sum of the numbers of electrons of the various generations:

$$\Pi(E_0,E,t) = f_0(E_0,E,t) + f_1(E_0,E,t) + f_2(E_0,E,t) + \ldots \tag{1}$$

The series converges fairly well only when E is not much smaller than E_0 and t not larger than a few radiation lengths.

The original work of Bhabha and Heitler was later extended by Arley (AN38; AN40). Bhabha and Heitler, as well as Arley, made the following approximations. They used Eq. (2.13.2) for computing the probability, f_0, that an electron has an energy greater than a certain fraction of its original energy after traversing a thickness t. For the differential radiation probability, φ_{rad}, and for the differential probability of pair production, φ_{pair}, they used the following simplified expressions:

$$\varphi_{\text{rad}}\left(\frac{E'}{E}\right)\frac{dE'}{E} = \frac{dE'}{E'}, \tag{2}$$

$$\varphi_{\text{pair}}\left(\frac{E'}{E}\right)\frac{dE'}{E} = \frac{4}{9}\frac{dE'}{E}. \tag{3}$$

Bhabha and Heitler neglected Compton effect and collision loss completely. Arley neglected both Compton effect and collision loss for energies greater than the critical energy and assumed that electrons of energy less than the critical energy lose energy *only* by collision processes. Tables of the numerical results may be found in the original papers and in a review article by Rossi and Greisen (RB41.1).

5.5. *The analytical method—diffusion equations.* Carlson and Oppenheimer approached the shower problem with a different method. By considering the various production and absorption phenomena that take place in an infinitesimal layer, dt, they obtained a set of equations describing the variation with thickness in the numbers of electrons and photons of each energy interval. Next they proceeded to solve these equations by making use of some drastic simplifications in the expressions for the probabilities of the elementary phenomena. This analytical method, later extended and perfected by Landau and Rumer (LLD38), Snyder (SHS38), Serber (SbR38), and others, has proved very fruitful. We shall discuss it in some detail and describe the most significant results that it has produced.

We first write the equations of the shower theory in a general form, *without the approximation of a continuous energy loss* discussed in § 5.2.

Consider the quantities $\pi(E,t)\,dE$ and $\gamma(E,t)\,dE$ giving the numbers of electrons and photons, respectively, with energies between E and $E + dE$ at the thickness t. As the radiation traverses the additional infinitesimal thickness dt, the number of electrons in the energy interval dE at E changes because of the following effects.

(a) Photons with energy E' greater than E produce a certain number of electrons in the energy interval (E,dE). This number is:

$$dE\,dt\int_E^\infty \gamma(E',t)\varphi_{\gamma\pi}(E',E)\,dE'.$$

Here $\varphi_{\gamma\pi}(E',E)\,dE$ represents the probability per radiation length for a photon of energy E' to produce an electron of energy E in dE. This may happen either by pair production or by Compton effect; thus:

$$\varphi_{\gamma\pi}(E',E) = 2\varphi_{\text{pair}}(E',E) + \varphi_{\text{com}}(E',E' - E). \tag{1}$$

In this equation, φ_{pair} is the differential probability for pair production; $\varphi_{\text{com}}(E',E' - E)\,dE$ is the probability per radiation length for a photon of energy E' to undergo a Compton collision in which the scattered photon has an energy $E' - E$ in dE. The factor 2 before φ_{pair} arises from the fact that each process of pair production gives rise to two electrons.

(b) Electrons with energy E' greater than E produce a certain number of electrons in the energy interval dE at E. This number is:

$$dE\,dt\int_E^\infty \pi(E',t)\varphi_{\pi\pi}(E',E)\,dE'.$$

Here $\varphi_{\pi\pi}(E',E)\,dE$ represents the probability per radiation length for an electron of energy E' to produce an electron of energy E in dE. This may happen either by a radiation process in which the electron loses an energy $E' - E$, or by a collision process in which one of the two colliding electrons comes out of the collision with an energy E; thus:

$$\varphi_{\pi\pi}(E',E) = \varphi_{\text{rad}}(E',E' - E) + \varphi_{\text{col}}(E',E). \tag{2}$$

In this equation φ_{rad} is the differential probability for radiation and φ_{col} is the symmetric differential collision probability discussed in § 5.2. If, in $\varphi_{\text{col}}(E',E)$ one lets E vary from 0 to E', one considers each collision process twice. Therefore one does not need to multiply the function $\varphi_{\text{col}}(E',E)$ by 2 in order to take into account the two electrons that come out of each collision.

(c) Some electrons initially in the energy interval dE at E leave this interval by losing a certain amount of energy. Their number is

$$\pi(E,t)\mu_\pi(E)\,dE\,dt,$$

where

$$\mu_\pi(E) = \int_0^E \varphi_{\text{rad}}(E,E')\,dE' + \int^{E/2} \varphi_{\text{col}}(E,E')\,dE'. \tag{3}$$

In this equation the first integral represents the contribution of radiation phenomena and the second integral the contribution of collision processes (note the upper limit $E/2$ in this integral). The formal difficulty arising from the divergence of the integrals in Eq. (3) will be discussed below.

Consider next the change in the number of photons of energy in dE at E that occurs when the radiation traverses the thickness dt. This change is due to the following effects:

(a) Electrons with energy E' greater than E produce a certain number of photons in the energy interval dE at E. This number is:

$$dE\,dt\int_E^\infty \pi(E',t)\varphi_{\pi\gamma}(E',E)\,dE'.$$

Here $\varphi_{\pi\gamma}(E',E)\,dE$ represents the probability per radiation length for an electron of energy E' to produce a photon of energy in dE at E. This can happen only through a radiation process; thus:

$$\varphi_{\pi\gamma}(E',E) = \varphi_{\text{rad}}(E',E). \tag{4}$$

(b) Photons with energy E' greater than E produce a certain number of photons in the energy interval dE at E. This number is:

$$dE\,dt\int_E^\infty \gamma(E',t)\varphi_{\gamma\gamma}(E',E)\,dE'.$$

Here $\varphi_{\gamma\gamma}(E',E)\,dE$ represents the probability per radiation length for a photon of energy E' to produce a photon of energy in dE at E. This can happen only through a Compton collision; thus:

$$\varphi_{\gamma\gamma}(E',E) = \varphi_{\text{com}}(E',E). \tag{5}$$

(c) Some photons, initially in the energy interval dE at E, are absorbed by pair production or undergo Compton scattering. Their number can be written as

$$\gamma(E,t)\mu_\gamma(E)\,dE\,dt$$

where

$$\mu_\gamma(E) = \mu_{\text{pair}}(E) + \mu_{\text{com}}(E). \tag{6}$$

By considering the various processes listed above, one obtains the following equations:

$$\frac{\partial\pi(E,t)}{\partial t} = -\pi(E,t)\mu_\pi(E) + \int_E^\infty \pi(E',t)\varphi_{\pi\pi}(E',E)\,dE' + \int_E^\infty \gamma(E',t)\varphi_{\gamma\pi}(E',E)\,dE' \tag{7a}$$

$$\frac{\partial\gamma(E,t)}{\partial t} = \int_E^\infty \pi(E',t)\varphi_{\pi\gamma}(E',E)\,dE' + \int_E^\infty \gamma(E',t)\varphi_{\gamma\gamma}(E',E)\,dE' - \gamma(E,t)\mu_\gamma(E). \tag{7b}$$

We shall now modify Eq. (7a) so as to introduce a continuous energy loss in place of collisions giving rise to electrons of energy less than η_0. For this purpose, note that Eq. (7a) is not changed if one assumes that μ_π and $\varphi_{\pi\pi}$ have been computed with $\varphi_{\text{col}}(E,E') = 0$ for $E' < \eta_0$ or $E - E' < \eta_0$, and adds, on the right-hand side, the term:

$$M = -\pi(E,t)\int_0^{\eta_0}\varphi_{\text{col}}(E,E')\,dE' + \int_E^{E+\eta_0}\pi(E',t)\varphi_{\text{col}}(E',E)\,dE'.$$

In the second integral one may take $E' - E$ as a new variable and rename it E'. One then obtains, remembering that

$$\varphi_{\text{col}}(E',E) = \varphi_{\text{col}}(E',E'-E),$$

$$M = \int_0^{\eta_0}[-\pi(E,t)\varphi_{\text{col}}(E,E')\,dE' + \pi(E+E',t)\varphi_{\text{col}}(E+E',E')]\,dE',$$

or

$$M = \int_0^{\eta_0}[(\pi\varphi_{\text{col}})_{E+E'} - (\pi\varphi_{\text{col}})_E]\,dE' \tag{8}$$

If, on the other hand, one assumes that electrons undergo a continuous energy loss, whose value is $\epsilon(E)$ per radiation length, one finds that, in the thickness dt, $\pi(E+dE)\epsilon(E+dE)\,dt$ electrons drift into the energy interval (dE) from the upper boundary and $\pi(E)\epsilon(E)\,dt$ drift out of the same interval through the lower boundary. The net change in the number of particles due to this process is therefore:

$$[\pi(E+dE)\epsilon(E+dE) - \pi(E)\epsilon(E)]\,dt = \frac{\partial[\pi(E,t)\epsilon(E)]}{\partial E}\,dE\,dt.$$

Thus inclusion of a continuous energy loss adds to the right-hand side of Eq. (7a) the term:

$$M' = \frac{\partial[\pi(E,t)\epsilon(E)]}{\partial E}. \tag{9}$$

One can easily see that M' coincides with M if η_0 is sufficiently small compared with E so that in the energy interval from E to $E+\eta_0$, the quantity $\pi(E',t)\varphi_{\text{col}}(E',E)$ may be approximated by a linear function of E. In this case, indeed, Eq. (8) yields:

$$\begin{aligned} M &= \int_0^{\eta_0}\frac{\partial}{\partial E}[\pi(E,t)\varphi_{\text{col}}(E,E')]E'\,dE' \\ &= \frac{\partial}{\partial E}\left[\pi(E,t)\int_0^{\eta_0}\varphi_{\text{col}}(E,E')E'\,dE'\right] \\ &= \frac{\partial}{\partial E}[\pi(E,t)\epsilon(E)]. \end{aligned}$$

Thus disregarding collisions that give rise to electrons of energy smaller than η_0 and introducing in their place an equivalent continuous energy loss does not change the shower equations appreciably in the energy region where $E \gg \eta_0$. With the inclusion of a continuous energy loss, Eqs. (7) become:

$$\frac{\partial\pi(E,t)}{\partial t} = -\pi(E,t)\mu_\pi(E) + \int_E^\infty \pi(E',t)\varphi_{\pi\pi}(E',E)\,dE' + \int_E^\infty \gamma(E',t)\varphi_{\gamma\pi}(E',E)\,dE' + \frac{\partial[\pi(E,t)\epsilon(E)]}{\partial E}, \tag{10a}$$

$$\frac{\partial\gamma(E,t)}{\partial t} = \int_E^\infty \pi(E',t)\varphi_{\pi\gamma}(E',E)\,dE' + \int_E^\infty \gamma(E',t)\varphi_{\gamma\gamma}(E',E)\,dE' - \gamma(E,t)\mu_\gamma(E). \tag{10b}$$

As indicated above, *in the computation of μ_π and $\varphi_{\pi\pi}$, one must now take*

$$\varphi_{\text{col}}(E,E') = 0 \quad \text{for} \quad E' < \eta_0 \quad \text{or} \quad E - E' < \eta_0.$$

One will note that there remains a formal difficulty in Eq. (10a). The differential probability for radiation processes diverges as $1/E$ as the energy E of the emitted photon decreases. Thus μ_π, as given by Eq. (3), diverges. The second term on the right-hand side of Eq. (10a) also diverges because $\varphi_{\text{rad}}(E',E' - E)$ tends to infinity as E' approaches E. One could remove this difficulty by a procedure similar to that used in the case of collision losses. That is, one could introduce arbitrarily a low-energy cut-off in the expression for the radiation probability and include the effect of radiation phenomena below cut-off in the continuous energy loss. Usually, however, this is not necessary because one can handle the equation in such a way that the two infinities cancel (see, for instance, § 5.7).

Equations (10) form a system of linear integro-differential equations. Because of their formal similarity to equations occurring in the theory of diffusion phenomena, they are often referred to as the *diffusion equations*. Symbolically, they may be written in the form:

$$\frac{\partial \pi(E,t)}{\partial t} = \mathfrak{A}\pi + \mathfrak{B}\gamma + \frac{\partial(\epsilon\pi)}{\partial E} \tag{11a}$$

$$\frac{\partial \gamma(E,t)}{\partial t} = \mathfrak{C}\pi + \mathfrak{D}\gamma, \tag{11b}$$

where $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ are linear operators, operating on the energy variable of the functions π and γ. Note that these operators operate only on energies equal to or greater than the energy, E, appearing in the functions on the left-hand side of the equations. This expresses the physically obvious fact that the number of electrons and photons of energy E at $t + dt$ depends only on the number of electrons and photons of energy equal to or greater than E at t. The linear character of the equations implies that the sum of any number of solutions of the diffusion equations is another solution of the same equations.

It is hardly necessary to note that the validity of the diffusion equations is not restricted to the case of a shower initiated by a single electron or photon. In fact, the diffusion equations may be solved with any arbitrary initial condition:

$$\pi(E,0) = n_\pi(E); \qquad \gamma(E,0) = n_\gamma(E), \tag{12}$$

where n_π and n_γ are given functions. The solution then describes the shower produced by an incident radiation containing electrons with a differential spectrum $n_\pi(E)$ and photons with a differential spectrum $n_\gamma(E)$. The case of a single incident electron of energy E_0 corresponds to the initial conditions:

$$\pi(E,0) = \delta(E - E_0); \qquad \gamma(E,0) = 0, \tag{13}$$

where δ represents Dirac's improper function:

$$\delta(x) = 0 \quad \text{for } x \neq 0,$$

$$\int_{-\epsilon}^{\epsilon} \delta(x)\,dx = 1 \quad \text{for any } \epsilon \neq 0.$$

Similarly, the case of a single incident photon of energy E_0 corresponds to the initial conditions:

$$\pi(E,0) = 0; \qquad \gamma(E,0) = \delta(E - E_0). \tag{14}$$

One can also extend the diffusion equations to the case that electrons and photons are produced continuously along the path, for instance, through secondary interactions of particles different from electrons or photons. Let $s_\pi(E,t)$ and $s_\gamma(E,t)$ be the "source functions"; i.e., let $s_\pi(E,t)\,dE\,dt$ be the number of electrons of energy between E and $E + dE$ and let $s_\gamma(E,t)\,dE\,dt$ be the number of photons of energy between E and $E + dE$ produced in the infinitesimal layer dt. The diffusion equations become:

$$\frac{\partial \pi(E,t)}{\partial t} = \mathfrak{A}\pi + \mathfrak{B}\gamma + \frac{\partial(\epsilon\pi)}{\partial E} + s_\pi(E,t), \tag{15a}$$

$$\frac{\partial \gamma(E,t)}{\partial t} = \mathfrak{C}\pi + \mathfrak{D}\gamma + s_\gamma(E,t). \tag{15b}$$

5.6. *Stationary distributions. Laplace integrals.* We shall call *stationary distributions* two functions π and γ that satisfy the diffusion equations and are of the following general form:

$$\pi(E,t) = F_\pi(E)f(t); \qquad \gamma(E,t) = F_\gamma(E)f(t). \tag{1}$$

Upon substitution in the diffusion equations (5.5.11) one obtains, remembering that the operators $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ operate on the energy alone:

$$F_\pi(E)\frac{df}{dt} = f(t)\left[\mathfrak{A}F_\pi(E) + \mathfrak{B}F_\gamma(E) + \frac{\partial(\epsilon F_\pi)}{\partial E}\right] \tag{2a}$$

$$F_\gamma(E)\frac{df}{dt} = f(t)\,[\mathfrak{C}F_\pi(E) + \mathfrak{D}F_\gamma(E)]. \tag{2b}$$

The quantities in square brackets are functions of E only. Therefore the ratio f'/f does not depend on t but is equal to some constant λ:

$$\frac{df}{dt} = \lambda f(t).$$

Hence

$$f(t) = \text{const.}\ e^{\lambda t}, \tag{3}$$

and

$$\lambda F_\pi(E) = \mathfrak{A}F_\pi(E) + \mathfrak{B}F_\gamma(E) + \frac{\partial(\epsilon F_\pi)}{\partial E} \tag{4a}$$

$$\lambda F_\gamma(E) = \mathfrak{C}F_\pi(E) + \mathfrak{D}F_\gamma(E). \tag{4b}$$

Equations (1) physically mean that the shape of the energy spectra of electrons and photons, as well as the relative numbers of the two kinds

of particles, remains unchanged as the shower propagates through matter. Equation (3) shows that, when this is the case, the absolute numbers of electrons and photons vary exponentially with thickness. One could have obtained this result directly from the following general argument. In a radiation whose components are absorbed and reproduced as they propagate through matter, the *relative* change of intensity in any given layer depends on the *relative* numbers of particles of different kinds and of different energies incident upon this layer. If these relative numbers do not change with thickness, the relative variations of intensity per unit thickness also remain constant, which implies an exponential dependence of the intensity on t.

We shall see later that the *functional transformations* offer a powerful method of attack for the solution of the diffusion equations. For this reason it is interesting to consider here the general type of equations satisfied by the Laplace integrals (see Appendix 4) with respect to t of π and γ, namely by the functions:

$$\mathfrak{L}_\pi(\lambda,E) = \int_0^\infty \pi(E,t)e^{-\lambda t}\,dt, \tag{5a}$$

$$\mathfrak{L}_\gamma(\lambda,E) = \int_0^\infty \gamma(E,t)e^{-\lambda t}\,dt. \tag{5b}$$

We specifically assume now that the shower under consideration has been produced by a single electron of energy E_0, so that the initial conditions (5.5.13) apply. Let us multiply both sides of Eqs. (5.5.11) by $e^{-\lambda t}$ and integrate from $t = 0$ to $t = \infty$. Since the operators on the right-hand sides operate on the energy, while the integral is made with respect to t, we may reverse the order and apply the operators after the integration over t. On the left-hand sides, we may integrate by parts, remembering that both $\pi(E,t)$ and $\gamma(E,t)$ are zero for $t = \infty$ (i.e., after complete absorption of the shower), whereas for $t = 0$, $\gamma = 0$ and π is a δ-function. We thus obtain:

$$-\delta(E_0 - E) + \lambda\mathfrak{L}_\pi(E,\lambda) = \mathfrak{A}\mathfrak{L}_\pi(E,\lambda) + \mathfrak{B}\mathfrak{L}_\gamma(E,\lambda) + \frac{\partial(\epsilon\mathfrak{L}_\pi)}{\partial E} \tag{6a}$$

$$\lambda\mathfrak{L}_\gamma(E,\lambda) = \mathfrak{C}\mathfrak{L}_\pi(E,\lambda) + \mathfrak{D}\mathfrak{L}_\gamma(E,\lambda). \tag{6b}$$

Note that the equations satisfied by the Laplace transforms are very similar to the equations satisfied by the stationary distributions [Eqs. (4)]. The only difference lies in the term $-\delta(E_0 - E)$ on the left-hand side of Eq. (6a). We may expect that this term does not materially affect the character of the solution at energies very small compared with E_0, so that, for these energies, the functions F and $\mathfrak{L}$ have the same energy dependence. This prediction is borne out, at least in special cases, by results to be described later.

A system of equations similar to Eqs. (6), with a δ-function in the second instead of the first equation, holds in the case of showers initiated by single photons.

From the Laplace integrals of π and γ, etc., one can immediately derive expressions for the corresponding moments (see § 5.1). For instance Eqs. (5.1.2) and (5) show that:

$$\mathfrak{L}_\pi(0,E) = \int_0^\infty \pi(E,t)\,dt = p_0(E),$$

$$\left(\frac{\partial\mathfrak{L}_\pi}{\partial\lambda}\right)_{\lambda=0} = -\int_0^\infty t\pi(E,t)\,dt = -p_1(E),$$

$$\left(\frac{\partial^2\mathfrak{L}_\pi}{\partial\lambda^2}\right)_{\lambda=0} = \int_0^\infty t^2\pi(E,t)\,dt = p_2(E),$$

and, in general,

$$\left(\frac{\partial^n\mathfrak{L}_\pi}{\partial\lambda^n}\right)_{\lambda=0} = (-1)^n\int_0^\infty t^n\pi(E,t)\,dt = (-1)^n p_n(E). \tag{7}$$

5.7. *The diffusion equations and their stationary solutions under Approximation A.* We shall now restrict our consideration to the high-energy region in which the assumptions described in § 5.2 as "Approximation A" are valid. We shall therefore neglect Compton effect and collision losses and describe the probabilities for radiation and pair production by the asymptotic formulae [see Eqs. (2.11.14), (2.11.15), (2.19.14), and (2.19.15)]. This gives:

$$\begin{aligned}
\varphi_{\gamma\pi}(E',E) &= 2\varphi_{\text{pair}}(E',E) = \frac{2}{E'}\psi_{\text{pair}}\left(\frac{E}{E'}\right);\\
\varphi_{\pi\pi}(E',E) &= \varphi_{\text{rad}}(E',E'-E) = \frac{1}{E'}\psi_{\text{rad}}\left(\frac{E'-E}{E'}\right);\\
\varphi_{\pi\gamma}(E',E) &= \varphi_{\text{rad}}(E',E) = \frac{1}{E'}\psi_{\text{rad}}\left(\frac{E}{E'}\right);\\
\varphi_{\gamma\gamma}(E',E) &= 0;\quad \epsilon(E) = 0;\\
\mu_\pi(E') &= \int_0^{E'}\psi_{\text{rad}}\left(\frac{E}{E'}\right)\frac{dE}{E'};\\
\mu_\gamma(E') &= \int_0^{E'}\psi_{\text{pair}}\left(\frac{E}{E'}\right)\frac{dE}{E'} = \mu_0.
\end{aligned} \tag{1}$$

Note that, on a crude approximation, one may use the equations:

$$\varphi_{\text{pair}}(E',E) = \frac{\mu_0}{E'};\qquad \varphi_{\text{rad}}(E',E) = \frac{1}{E'}, \tag{2}$$

which give the correct values for the radiation loss of electrons and for the absorption coefficient of photons.

With Eqs. (1), the diffusion equations become:

$$\frac{\partial \pi}{\partial t} = -\int_0^1 \left[\pi(E,t) - \frac{1}{1-v}\,\pi\left(\frac{E}{1-v},t\right)\right]\psi_{\text{rad}}(v)\,dv + 2\int_0^1 \gamma\left(\frac{E}{v},t\right)\psi_{\text{pair}}(v)\,\frac{dv}{v}; \tag{3a}$$

$$\frac{\partial \gamma}{\partial t} = \int_0^1 \pi\left(\frac{E}{v},t\right)\psi_{\text{rad}}(v)\,\frac{dv}{v} - \mu_0\gamma(E,t). \tag{3b}$$

These equations do not contain explicitly any quantity that depends on the properties of the substance in which the shower propagates. Therefore, as already pointed out, *the shower theory under Approximation A gives exactly the same results for all substances, provided one measures thickness in terms of the radiation length characteristic of each substance.*

To prove Eqs. (3), consider the various terms on the left-hand sides of Eqs. (5.5.11) separately.

$$\mathfrak{A}\pi = -\pi(E,t)\int_0^E \psi_{\text{rad}}\left(\frac{E'}{E}\right)\frac{dE'}{E} + \int_E^\infty \pi(E',t)\psi_{\text{rad}}\left(\frac{E'-E}{E'}\right)\frac{dE'}{E'}.$$

By substituting v for (E'/E) in the first integral and v for $(E'-E)/E'$ in the second, one obtains:

$$\mathfrak{A}\pi = -\int_0^1 \left[\pi(E,t) - \frac{1}{1-v}\,\pi\left(\frac{E}{1-v},t\right)\right]\psi_{\text{rad}}(v)\,dv. \tag{4}$$

In a similar manner one obtains the following expressions for $\mathfrak{B}\gamma$, $\mathfrak{C}\pi$, and $\mathfrak{D}\gamma$:

$$\mathfrak{B}\gamma = 2\int_E^\infty \gamma(E',t)\psi_{\text{pair}}\left(\frac{E}{E'}\right)\frac{dE'}{E'},$$

or, with $v = E/E'$:

$$\mathfrak{B}\gamma = 2\int_0^1 \gamma\left(\frac{E}{v},t\right)\psi_{\text{pair}}(v)\,\frac{dv}{v}. \tag{5}$$

$$\mathfrak{C}\pi = \int_E^\infty \pi(E',t)\psi_{\text{rad}}\left(\frac{E}{E'}\right)\frac{dE'}{E'}.$$

or, with $v = E/E'$,

$$\mathfrak{C}\pi = \int_0^1 \pi\left(\frac{E}{v},t\right)\psi_{\text{rad}}(v)\,\frac{dv}{v}. \tag{6}$$

$$\mathfrak{D}\gamma = -\mu_\gamma(E)\gamma = -\mu_0\gamma. \tag{7}$$

Note that the term $\mathfrak{A}\pi$ is finite even though it results from the difference of two infinite terms [see remark following Eqs. (5.5.10)].

We may write Eqs. (5.6.4) for the stationary distributions as follows:

$$\lambda F_\pi(E) - \int_0^1 \left[F_\pi(E) - \frac{1}{1-v}\,F_\pi\left(\frac{E}{1-v}\right)\right]\psi_{\text{rad}}(v)\,dv + 2\int_0^1 F_\gamma\left(\frac{E}{v}\right)\psi_{\text{pair}}(v)\,\frac{dv}{v}, \tag{8a}$$

$$\lambda F_\gamma(E) = \int_0^1 F_\pi\left(\frac{E}{v}\right)\psi_{\text{rad}}(v)\,\frac{dv}{v} - \mu_0 F_\gamma(E). \tag{8b}$$

Equations (8) are satisfied by power functions of the energy:

$$F_\pi(E) = aE^{-(s+1)}, \qquad F_\gamma(E) = bE^{-(s+1)}, \tag{9}$$

where s is a positive number.

Inserting Eq. (9) into Eq. (8), we obtain:

$$\begin{aligned} \lambda a &= -A(s)a + B(s)b \\ \lambda b &= C(s)a - \mu_0 b; \end{aligned} \tag{10}$$

where

$$\begin{aligned} A(s) &= \int_0^1 [1-(1-v)^s]\psi_{\text{rad}}(v)\,dv \\ B(s) &= 2\int_0^1 v^s\psi_{\text{pair}}(v)\,dv \\ C(s) &= \int_0^1 v^s\psi_{\text{rad}}(v)\,dv \\ \mu_0 &= \int_0^1 \psi_{\text{pair}}(v)\,dv. \end{aligned} \tag{11}$$

Eqs. (10) are linear homogeneous equations in a and b. In order that they may be solved for these variables, λ must satisfy the quadratic equation:

$$[\lambda + A(s)](\lambda + \mu_0) - B(s)C(s) = 0. \tag{12}$$

Hence for every value of the exponent s there are two possible values of λ:

$$\begin{aligned} \lambda_1(s) &= -\frac{[A(s)+\mu_0]}{2} + \frac{1}{2}\left\{[A(s)-\mu_0]^2 + 4B(s)C(s)\right\}^{1/2}, \\ \lambda_2(s) &= -\frac{[A(s)+\mu_0]}{2} - \frac{1}{2}\left\{[A(s)-\mu_0]^2 + 4B(s)C(s)\right\}^{1/2}. \end{aligned} \tag{13}$$

The ratio between the coefficients a and b is:

$$\frac{a_1}{b_1} = \frac{B(s)}{A(s)+\lambda_1(s)} = \frac{\mu_0+\lambda_1(s)}{C(s)},$$

or

$$\frac{a_2}{b_2} = \frac{B(s)}{A(s)+\lambda_2(s)} = \frac{\mu_0+\lambda_2(s)}{C(s)}, \tag{14}$$

according to the two possible choices for λ. In conclusion, we have the two following sets of stationary solutions of the diffusion equations (3):

$$\pi(E,t) = a_1 E^{-(s+1)}\exp\,[\lambda_1(s)t]; \quad \gamma(E,t) = \frac{a_1 C(s)}{\mu_0 + \lambda_1(s)} E^{-(s+1)}\exp\,[\lambda_1(s)t],$$
$$\pi(E,t) = a_2 E^{-(s+1)}\exp\,[\lambda_2(s)t]; \quad \gamma(E,t) = \frac{a_2 C(s)}{\mu_0 + \lambda_2(s)} E^{-(s+1)}\exp\,[\lambda_2(s)t], \tag{15}$$

where a_1 and a_2 are arbitrary constants, and λ_1 and λ_2 are given by Eqs. (13).

One can easily obtain explicit expressions for $A(s)$, $B(s)$, and $C(s)$ by substituting in Eqs. (11) the expressions for ψ_{rad} and ψ_{pair} given by Eqs. (2.11.15) and (2.19.15). The result is:*

$$\begin{aligned}
A(s) &= \left(\frac{4}{3} + 2b\right)\left[\frac{d}{ds}\ln\,(s!) + \gamma - 1 + \frac{1}{s+1}\right] + \frac{1}{2} - \frac{1}{(s+1)(s+2)} \\
&= 1.36\,\frac{d}{ds}\ln\,(s+1)! - \frac{1}{(s+1)(s+2)} - 0.0750, \\
B(s) &= 2\left[\frac{1}{s+1} - \left(\frac{4}{3} + 2b\right)\left(\frac{1}{(s+2)} - \frac{1}{(s+3)}\right)\right] \\
&= 2\left[\frac{1}{(s+1)} - \frac{1.36}{(s+2)(s+3)}\right], \\
C(s) &= \left(\frac{4}{3} + 2b\right)\left(\frac{1}{s} - \frac{1}{s+1}\right) + \frac{1}{s+2} = \frac{1}{s+2} + \frac{1.36}{s(s+1)}, \\
\mu_0 &= \left(\frac{7}{9} - \frac{b}{3}\right) = 0.773.
\end{aligned} \tag{16}$$

The values of $A(s)$, $B(s)$, $C(s)$, $\lambda_1(s)$, $\lambda_2(s)$, $\lambda'_1(s)$ and $\lambda''_1(s)$ corresponding to various values of s are listed in Table 5.24.2. Note that, as s approaches zero, $A(s)$ tends to zero, $B(s)$ tends to $2\mu_0$, and $C(s)$ tends to infinity as $1/s$. Note also that $C(s)$ is always positive and that $\lambda_1(s) > -\mu_0 > \lambda_1(s)$. Therefore a_1/b_1 is always positive and a_2/b_2 always negative. This means that the first of the two solutions (15) has a physical meaning, but the second has not because it always gives either a *positive* number of electrons and a *negative* number of photons, or a *negative* number of electrons and a *positive* number of photons. However, any linear combination of the two solutions that gives positive values for π and γ has physical significance. Such a linear combination represents the general solution of the shower problem under Approximation A for an incident radiation containing electrons and photons distributed in energy according to the same power law. Since λ_2 is always negative and smaller than λ_1, as t increases the terms containing $\exp\,(\lambda_2 t)$ become soon negligible compared with the terms containing $\exp\,(\lambda_1 t)$. Thus for sufficiently large thicknesses, π and γ vary exponentially with depth and their ratio becomes independent of the proportion of electrons and photons in the incident radiation.

* In Eqs. (16), $\frac{d}{ds}\ln\,(s!) = \int_0^1 \frac{1-x^s}{1-x}\,dx - \gamma$ indicates the logarithmic derivative of the gamma-function, $\gamma = 0.5772$ is Euler's constant, and $b = 0.0135$ is defined by Eq. (2.11.16).

According to Table 5.24.2, λ_1 is positive for $s < 1$, zero for $s = 1$, and negative for $s > 1$, approaching $-\mu_0 = -0.7732$ as s tends to infinity. Thus when $s < 1$ the number of shower particles *increases* with depth, when $s = 1$ it *remains constant*, and when $s > 1$ it *decreases*. The case in which the number of shower particles is independent of depth is particularly important. In this case the differential spectra of electrons and photons, under Approximation A, are given by:

$$\begin{aligned}
F_\pi(E) &= a_1 E^{-2}; \\
F_\gamma(E) &= \frac{a_1 C(1)}{\mu_0} E^{-2} = 1.31 a_1 E^{-2}.
\end{aligned} \tag{17}$$

The total energy carried by shower particles of energy larger than any given value turns out to be infinity for $s \leqq 1$. This means that no natural process can give rise to a group of electrons or photons distributed in energy according to a power law $E^{-(s+1)}$ with $s \leqq 1$, which does not break down beyond a certain value of the energy. Correspondingly there does not exist in nature a shower whose intensity either remains indefinitely constant, or continues to increase indefinitely with t.

We wish to emphasize here that *the possibility of finding simple expressions for the stationary distributions under Approximation A is a direct consequence of the assumption that the differential probabilities for all secondary processes are functions only of the ratio of the secondary to the primary energy. The fact that the stationary distributions are represented by power laws of the energy is also a consequence of this assumption.*

5.8. *The Mellin and Laplace integrals of the shower functions under Approximation A.* We turn, in this section, to the more general problem of computing the shower functions under arbitrary initial conditions.

Let us consider the Mellin integrals of π and γ with respect to the energy; i.e., the quantities:

$$\begin{aligned}
\mathfrak{M}_\pi(s,t) &= \int_0^\infty E^s \pi(E,t)\,dE, \\
\mathfrak{M}_\gamma(s,t) &= \int_0^\infty E^s \gamma(E,t)\,dE,
\end{aligned} \tag{1}$$

where s is a complex parameter. $\mathfrak{M}_\pi$ and $\mathfrak{M}_\gamma$ are defined for all those values of s that make the integrals converge. Since π and γ are identically

zero for energies larger than the primary energy, we need to consider the convergence of the integrals only at the lower limit. It follows that the field of convergence of $\mathfrak{M}_\pi$ and $\mathfrak{M}_\gamma$ is the half-plane defined by $R(s) > s_0$, where $R(s)$ indicates the real part of s and s_0 is a real constant. It will be shown later that $s_0 = 0$.

Let us consider also the Laplace integrals of π and γ with respect to t, given by Eqs. (5.6.5). $\mathfrak{L}_\pi$ and $\mathfrak{L}_\gamma$ are defined for all those values of the complex parameter λ that make the integrals converge. The field of convergence is defined by $R(\lambda) > \lambda_0$, where λ_0 is a real constant. It will be shown later that $\lambda_0 = -\mu_0$.

Lastly, we shall consider the Mellin integrals with respect to energy of the Laplace integrals with respect to thickness; i.e., the quantities:

$$\begin{aligned} \mathfrak{N}_\pi(s,\lambda) &= \int_0^\infty E^s\,dE \int_0^\infty e^{-\lambda t}\pi(E,t)\,dt, \\ \mathfrak{N}_\gamma(s,\lambda) &= \int_0^\infty E^s\,dE \int_0^\infty e^{-\lambda t}\gamma(E,t)\,dt, \end{aligned} \tag{2}$$

which are functions of the two complex parameters s and λ.

The reason for introducing the transforms $\mathfrak{M}$, $\mathfrak{L}$, $\mathfrak{N}$ is that they are more easily determined than the original functions π and γ. As noted before, some properties of showers can be derived directly from knowledge of the integrals above, while the functions π and γ themselves can be obtained from the transforms, by means of the inversion formulae given in Appendix 4.

The boundary conditions of the problem are determined by the functions, $\pi(E,0)$ and $\gamma(E,0)$, that describe the radiation incident at $t = 0$. We shall consider, in particular, the boundary conditions expressed by Eqs. (5.5.13) and (5.5.14), representing respectively a single primary electron of energy E_0 and a single primary photon of energy E_0.

The Mellin integrals, as pointed out by Landau and Rumer (LLD38) satisfy a simple system of differential equations that one can obtain by multiplying both sides of the diffusion equations (5.7.3) by E^s and integrating with respect to energy from 0 to ∞. These equations are:

$$\frac{\partial \mathfrak{M}_\pi(s,t)}{\partial t} = -A(s)\mathfrak{M}_\pi(s,t) + B(s)\mathfrak{M}_\gamma(s,t), \tag{3a}$$

$$\frac{\partial \mathfrak{M}_\gamma(s,t)}{\partial t} = C(s)\mathfrak{M}_\pi(s,t) - \mu_0\mathfrak{M}_\gamma(s,t), \tag{3b}$$

where $A(s)$, $B(s)$, $C(s)$ and μ_0 are given by Eqs. (5.7.11).

To prove Eqs. (3), consider that:

$$\int_0^\infty E^s\,dE\;\mathfrak{A}\pi = -\int_0^\infty E^s\,dE \int_0^1 \left[\pi(E,t) - \frac{1}{1-v}\,\pi\left(\frac{E}{1-v},t\right)\right]\psi_{\rm rad}(v)\,dv;$$

or, with $E' = \dfrac{E}{1-v}$:

$$\begin{aligned} \int_0^\infty E^s\,dE\;\mathfrak{A}\pi &= -\int_0^\infty E^s\pi(E,t)\,dE \int_0^1 \psi_{\rm rad}(v)\,dv \\ &\quad + \int_0^\infty (E')^s\pi(E',t)\,dE' \int_0^1 (1-v)^s\psi_{\rm rad}(v)\,dv \\ &= -\int_0^\infty E^s\pi(E,t)\,dE \int_0^1 [1-(1-v)^s]\psi_{\rm rad}(v)\,dv \\ &= -\mathfrak{M}_\pi(s,t)A(s). \end{aligned}$$

Similarly

$$\int_0^\infty E^s\,dE\;\mathfrak{B}\gamma = B(s)\mathfrak{M}_\gamma(s,t);$$

$$\int_0^\infty E^s\,dE\;\mathfrak{C}\pi = C(s)\mathfrak{M}_\pi(s,t);$$

$$\int_0^\infty E^s\,dE\;\mu_0\gamma = \mu_0\mathfrak{M}_\gamma(s,t).$$

If one now multiplies both sides of Eqs. (3) by $e^{-\lambda t}$ and integrates with respect to t from 0 to ∞, one obtains the following system of algebraic equations for $\mathfrak{N}_\pi$ and $\mathfrak{N}_\gamma$:

$$\lambda\mathfrak{N}_\pi(s,\lambda) - \mathfrak{M}_\pi(s,0) = -A(s)\mathfrak{N}_\pi(s,\lambda) + B(s)\mathfrak{N}_\gamma(s,\lambda), \tag{4a}$$

$$\lambda\mathfrak{N}_\gamma(s,\lambda) - \mathfrak{M}_\gamma(s,0) = C(s)\mathfrak{N}_\pi(s,\lambda) - \mu_0\mathfrak{N}_\gamma(s,\lambda), \tag{4b}$$

where $\mathfrak{M}_\pi(s,0)$ and $\mathfrak{M}_\gamma(s,0)$ are the Mellin integrals of $\pi(E,0)$ and $\gamma(E,0)$. In particular,

(a) for an incident electron of energy E_0:

$$\mathfrak{M}_\pi(s,0) = E_0^s, \qquad \mathfrak{M}_\gamma(s,0) = 0; \tag{5}$$

(b) for an incident photon of energy E_0:

$$\mathfrak{M}_\pi(s,0) = 0, \qquad \mathfrak{M}_\gamma(s,0) = E_0^s. \tag{6}$$

Solution of Eqs. (4) with the boundary conditions (5) or (6) yields the functions $\mathfrak{N}_\pi^{(\pi)}$ and $\mathfrak{N}_\gamma^{(\pi)}$ corresponding to π and γ in electron-initiated showers, and the functions $\mathfrak{N}_\pi^{(\gamma)}$ and $\mathfrak{N}_\gamma^{(\gamma)}$ corresponding to π and γ in photon-initiated showers. These functions are of the general form:

$$\mathfrak{N}(E_0,s,\lambda) = N(s,\lambda)E_0^s. \tag{7}$$

The expressions for the four corresponding functions N are listed in Table 1, where $\lambda_1(s)$ and $\lambda_2(s)$ are the functions of s defined by Eqs. (5.7.15).

Either by solving directly Eqs. (3) with the boundary conditions (5) or (6), or by applying the inverse Laplace transformation to the expressions

Table 5.8.1. Expressions for the functions $N(s,\lambda)$ in Eq. (5.8.7)

Secondary Particles \ Primary Particle	Electron	Photon
Electrons	$N_\pi^{(\pi)} = \frac{\mu_0 + \lambda}{[\lambda - \lambda_1(s)][\lambda - \lambda_2(s)]}$	$N_\pi^{(\gamma)} = \frac{B(s)}{[\lambda - \lambda_1(s)][\lambda - \lambda_2(s)]}$
Photons	$N_\gamma^{(\pi)} = \frac{C(s)}{[\lambda - \lambda_1(s)][\lambda - \lambda_2(s)]}$	$N_\gamma^{(\gamma)} = \frac{A(s) + \lambda}{[\lambda - \lambda_1(s)][\lambda - \lambda_2(s)]}$

for $\mathfrak{N}$ (see Appendix 4), one easily obtains the expressions for the Mellin integrals of π and γ corresponding to electron-initiated showers [$\mathfrak{M}_\pi^{(\pi)}$ and $\mathfrak{M}_\gamma^{(\pi)}$] or to photon-initiated showers [$\mathfrak{M}_\pi^{(\gamma)}$ and $\mathfrak{M}_\gamma^{(\gamma)}$]. These expressions are of the general form:

$$\mathfrak{M}(E_0,s,t) = E_0^s[M_1(s)e^{\lambda_1(s)t} - M_2(s)e^{\lambda_2(s)t}]. \tag{8}$$

The functions M_1 and M_2, relative to the four different cases, are given in Table 2. For any given value of t, the functions on the right-hand side

Table 5.8.2. Expressions for the functions $M_1(s)$ and $M_2(s)$ in Eq. (5.8.8)

Secondary Particle \ Primary Particle	Electron	Photon
Electrons	$M_{1\pi}^{(\pi)} = \frac{\mu_0 + \lambda_1(s)}{\lambda_1(s) - \lambda_2(s)}$	$M_{1\pi}^{(\gamma)} = \frac{-[\mu_0 + \lambda_1(s)][\mu_0 + \lambda_2(s)]}{C(s)[\lambda_1(s) - \lambda_2(s)]}$
	$M_{2\pi}^{(\pi)} = \frac{\mu_0 + \lambda_2(s)}{\lambda_1(s) - \lambda_2(s)}$	$M_{2\pi}^{(\gamma)} = \frac{-[\mu_0 + \lambda_1(s)][\mu_0 + \lambda_2(s)]}{C(s)[\lambda_1(s) - \lambda_2(s)]}$
Photons	$M_{1\gamma}^{(\pi)} = \frac{C(s)}{\lambda_1(s) - \lambda_2(s)}$	$M_{1\gamma}^{(\gamma)} = \frac{-[\mu_0 + \lambda_2(s)]}{\lambda_1(s) - \lambda_2(s)}$
	$M_{2\gamma}^{(\pi)} = \frac{C(s)}{\lambda_1(s) - \lambda_2(s)}$	$M_{2\gamma}^{(\gamma)} = \frac{-[\mu_0 + \lambda_1(s)]}{\lambda_1(s) - \lambda_2(s)}$

of Eq. (8) are regular for $s > 0$ and tend to infinity at the limit for $s = 0$, where $\lambda_1(s)$ becomes positively infinite. This shows that the field of convergence of the functions $\mathfrak{M}$ is the positive half-plane. The expressions (8) represent the functions $\mathfrak{M}$ only in this half-plane, and give their analytical continuations in the negative half-plane.

One obtains the Laplace integrals, $\mathfrak{L}$, by applying the inverse Mellin transformation to the functions $\mathfrak{N}$ (see Appendix 4), as follows:

$$\mathfrak{L}(E_0,E,\lambda) = \frac{1}{2\pi i}\frac{1}{E}\int_{s-i\infty}^{s+i\infty}\left(\frac{E_0}{E}\right)^s N(s,\lambda)\,ds. \tag{9}$$

The integration path runs parallel to the imaginary axis, to the right of all poles. From the expressions for the functions N (see Table 1), one sees that for $\lambda > -\mu_0$ there is only one pole on the positive half-plane of s; this pole lies on the real axis and is defined by the equation:

$$\lambda_1(s) = \lambda. \tag{10}$$

It can be shown that there is an infinite number of poles on the negative half-plane. For $E < E_0$, the integrand tends to zero at the limit for $s = -\infty$. Hence, deforming the contour to the left, the integral can be expressed as the sum of the residues at all the poles. Each pole contributes a term proportional to $(E_0/E)^s$. For the pole on the positive real axis, s is real and positive, whereas for those on the negative half-plane, $R(s) < 0$. The latter poles contribute terms that are negligible with respect to the first one if $E \ll E_0$. Hence, *for energies E small compared with the initial energy E_0*, the Laplace integrals have expressions of the form:

$$\mathfrak{L}(E_0,E,\lambda) = -\frac{1}{E}\left(\frac{E_0}{E}\right)^s L(\lambda). \tag{11}$$

The functions $L(\lambda)$ relative to the functions π and γ in an electron- or a photon-initiated shower are given in Table 3.

Table 5.8.3. Expressions for the functions $L(\lambda)$ in Eqs. (5.8.11)

Secondary Particles \ Primary Particle	Electron	Photon
Electrons	$L_\pi^{(\pi)} = \frac{\mu_0 + \lambda_1(s)}{[\lambda_1(s) - \lambda_2(s)]\lambda_1'(s)}$	$L_\pi^{(\gamma)} = \frac{B(s)}{[\lambda_1(s) - \lambda_2(s)]\lambda_1'(s)}$
Photons	$L_\gamma^{(\pi)} = \frac{C(s)}{[\lambda_1(s) - \lambda_2(s)]\lambda_1'(s)}$	$L_\gamma^{(\gamma)} = \frac{A(s) + \lambda_1(s)}{[\lambda_1(s) - \lambda_2(s)]\lambda_1'(s)}$

In Eq. (11), as well as in Table 3, s is the function of λ defined by Eq. (10), hence λ coincides with $\lambda_1(s)$. $\lambda_1'(s)$ indicates the derivative of λ_1 with respect to s and is given as a function of s in Table 5.24.2. For $\lambda = -\mu_0$, $s = -\infty$ and the expressions for $\mathfrak{L}$ go to infinity. It follows that the half-plane of convergence of the Laplace integrals is to the right of the point $\lambda = -\mu_0$.

One obtains the expression for the Mellin integral of Π by means of Eq. (A.4.7):

$$\mathfrak{M}_\Pi(E_0,s-1,t) = \frac{1}{s}\,\mathfrak{M}_\pi(E_0,s,t). \tag{12}$$

The expression for the Laplace integral of Π is:

$$\mathfrak{L}_\Pi(E_0,E,\lambda) = \int_E^\infty \mathfrak{L}_\pi(E_0,E',\lambda)\,dE' = \frac{E}{s}\,\mathfrak{L}_\pi(E_0,E,\lambda). \tag{13}$$

As pointed out before, *the expressions for the Laplace integrals given above are valid only for energies of the shower particles small compared with the energy of the primary electron or photon. Note that, in this energy range, the functions* $\mathfrak{L}_\pi(E_0,E,\lambda)$ *and* $\mathfrak{L}_\gamma(E_0,E,\lambda)$ *are proportional to* $E^{-(s+1)}$. *Thus they have the same energy dependence as the stationary distributions whose variation with depth obeys the exponential law* $\exp(\lambda t)$ *(see* § 5.7).

From the equations developed in this section one can easily derive the expression for the straggling in the radiation loss quoted in § 2.13, Eq. (2.13.2). For this purpose, notice that the function $w(E_0,E,t)$ defined in § 2.13 is identical to the function $\pi^{(\pi)}(E_0,E,t)$ computed under the assumption that $\varphi_{\gamma\pi} = 0$ (i.e., that there is no reproduction of electrons by materialization of photons). With this assumption, $B(s) = 0$ and Eq. (3a) yields:

$$\frac{\partial \mathfrak{M}_\pi(s,t)}{\partial t} = -A(s)\mathfrak{M}_\pi(s,t), \tag{14}$$

or, with the boundary condition (5):

$$\mathfrak{M}_\pi(s,t) = E_0^s e^{-A(s)t}. \tag{15}$$

The inversion formula of the Mellin transform, Eq. (A.4.9), gives the following expression for $w(E_0,E,t) = \pi^{(\pi)}(E_0,E,t)$:

$$w(E_0,E,t) = \frac{1}{2\pi i E_0}\int_c \left(\frac{E_0}{E}\right)^{s+1} e^{-A(s)t}\,ds. \tag{16}$$

If one introduces the approximate expression (2.13.1) into the first of Eqs. (5.7.11), one obtains:

$$A(s) = \frac{1}{\ln 2}\ln(1+s). \tag{17}$$

Eq. (16) then becomes:

$$w(E_0,E,t) = \frac{1}{2\pi i E_0}\int_c \frac{(E_0/E)^{s+1}}{(1+s)^{t/\ln 2}}\,ds \tag{18}$$

In this equation the integration path runs parallel to the imaginary axis, to the right of the origin. If one completes the contour by an infinite semicircle on the left plane and evaluates the singularity at $s = -1$, one obtains:

$$w(E_0,E,t) = \frac{1}{E_0}\frac{[\ln(E_0/E)]^{(t/\ln 2)-1}}{\Gamma(t/\ln 2)}. \tag{19}$$

This equation coincides with Eq. (2.13.2).

5.9. ***The track length, the center of gravity, and the longitudinal spread of a shower under Approximation A.*** As pointed out in § 5.6, the track lengths, or zero moments (p_0, g_0, P_0) of the functions π, γ and Π are identical to the corresponding Laplace integrals with $\lambda = 0$. Eq. (5.8.11) together with Table 5.8.3 and Table 5.24.2 yields the following results:

$$p_0^{(\pi)}(E_0,E) = p_0^{(\gamma)}(E_0,E) = 0.437\,\frac{E_0}{E^2}, \tag{1}$$

$$g_0^{(\pi)}(E_0,E) = g_0^{(\gamma)}(E_0,E) = 0.572\,\frac{E_0}{E^2}, \tag{2}$$

$$P_0^{(\pi)}(E_0,E) = P_0^{(\gamma)}(E_0,E) = 0.437\,\frac{E_0}{E}. \tag{3}$$

The position of the center of gravity, $\bar{t}$, and the longitudinal spread, τ, corresponding to the functions π, γ, and Π, can also be obtained from the expressions for the corresponding Laplace integrals, by means of Eqs. (5.1.4), (5.1.6), and (5.6.7). The quantities $\bar{t}$ and τ are given by expressions of the form:

$$\bar{t} = -\frac{1}{\lambda_1'(1)}y + h = 1.01y + h, \tag{4}$$

$$\tau^2 = \frac{\lambda_1''(1)}{[\lambda_1'(1)]^3}y + k = 1.61y + k, \tag{5}$$

where $y = \ln(E_0/E)$ and the constants h and k relative to the various functions have the values listed in Table 1. *The expressions above, of course, break down when the energies of the shower particles approach the primary energy.*

Table 5.9.1. Values of the constants h and k in Eqs. (4) and (5)

Function \ Primary Particle	Electron		Photon	
	h	k	h	k
π	1.0	−0.1	1.8	1.2
γ	1.2	1.1	2.0	2.3
Π	0.03	−0.6	0.8	0.6

Note the proportionality relation between P_0 *and* E_0/E *expressed by Eq.* (3), which confirms the result obtained through the qualitative analysis presented in § 5.3. *Note also the logarithmic dependence of* $\bar{t}$ *and* τ^2 *on* E_0/E.

5.10. ***Expressions for the differential and integral spectra under Approximation A.*** From Eqs. (5.8.8) and (5.8.12) one can obtain formal expressions for the differential spectra of electrons and photons and for the integral spectra of electrons by using the inversion formulae of the Mellin transformation. For instance:

$$\pi(E,t)\,dE = \frac{1}{2\pi i}\frac{dE}{E}\int_{\delta-i\infty}^{\delta+i\infty} E^{-s}\,\mathfrak{M}_\pi(s,t)\,ds, \tag{1}$$

where the integration path is any line parallel to the imaginary axis in the positive half-plane.

The complex integrals can be evaluated by the saddle point method. Each integral contains a term proportional to $\exp(\lambda_1 t)$ and a term proportional to $\exp(\lambda_2 t)$. Since $\lambda_2 < \lambda_1$, for sufficiently large thicknesses the second term may be neglected. For small thicknesses, however, both terms must be taken into consideration. We refer the reader to the review article of Rossi and Greisen (RB41.1) and to Chapter VI in the book *Cosmic*

*Rays** by Janossy for a detailed discussion of the mathematical calculations. The result of these calculations is a set of parametric equations giving the quantities π, γ, etc., and the thickness t, as functions of a variable, s. For sufficiently large thicknesses, these equations are:

(a) Primary electron of energy E_0:

$$\pi^{(\pi)}(E_0,E,t)\,dE = \left\{\frac{H_\pi{}^{(\pi)}(s)}{\sqrt{2\pi}[\lambda_1''(s)t]^{1/2}}\right\}\left(\frac{E_0}{E}\right)\frac{dE}{E}\,e^{\lambda_1(s)t}, \tag{2}$$

$$t = -\frac{1}{\lambda_1'(s)}\ln\left(\frac{E_0}{E}\right).$$

$$\gamma^{(\pi)}(E_0,E,t)\,dE = \left\{\frac{H_\gamma{}^{(\pi)}(s)}{\sqrt{2\pi}[\lambda_1''(s)t + (1/2s^2)]^{1/2}}\right\}\frac{1}{\sqrt{s}}\left(\frac{E_0}{E}\right)^s\frac{dE}{E}\,e^{\lambda_1(s)t}, \tag{3}$$

$$t = -\frac{1}{\lambda_1'(s)}\left[\ln\left(\frac{E_0}{E}\right) - \frac{1}{2s}\right].$$

$$\Pi^{(\pi)}(E_0,E,t) = \left\{\frac{H_\pi{}^{(\pi)}(s)}{\sqrt{2\pi}[\lambda_1''(s)t + (1/s^2)]^{1/2}}\right\}\frac{1}{s}\left(\frac{E_0}{E}\right)^s e^{\lambda_1(s)t}, \tag{4}$$

$$t = -\frac{1}{\lambda_1'(s)}\left[\ln\left(\frac{E_0}{E}\right) - \frac{1}{s}\right].$$

(b) Primary photon of energy E_0:

$$\pi^{(\gamma)}(E_0,E,t)\,dE = \left\{\frac{H_\pi{}^{(\gamma)}(s)}{\sqrt{2\pi}[\lambda_1''(s)t - (1/2s^2)]^{1/2}}\right\}\sqrt{s}\left(\frac{E_0}{E}\right)^s\frac{dE}{E}\,e^{\lambda_1(s)t}, \tag{5}$$

$$t = -\frac{1}{\lambda_1'(s)}\left[\ln\left(\frac{E_0}{E}\right) + \frac{1}{2s}\right].$$

$$\gamma^{(\gamma)}(E_0,E,t)\,dE = \left\{\frac{H_\gamma{}^{(\gamma)}(s)}{\sqrt{2\pi}[\lambda_1''(s)t]^{1/2}}\right\}\left(\frac{E_0}{E}\right)^s\frac{dE}{E}\,e^{\lambda_1(s)t}, \tag{6}$$

$$t = -\frac{1}{\lambda_1'(s)}\ln\frac{E_0}{E}.$$

$$\Pi^{(\gamma)}(E_0,E,t) = \left\{\frac{H_\pi{}^{(\gamma)}(s)}{\sqrt{2\pi}[\lambda_1''(s)t + (1/2s^2)]^{1/2}}\right\}\frac{1}{\sqrt{s}}\left(\frac{E_0}{E}\right)^s e^{\lambda_1(s)t}. \tag{7}$$

$$t = -\frac{1}{\lambda_1'(s)}\left[\ln\frac{E_0}{E} - \frac{1}{2s}\right].$$

The functions H in Eqs. (2) to (7) are related to the functions M listed in Table 5.8.2:

$$H_\pi{}^{(\pi)}(s) = M_{1\pi}{}^{(\pi)}(s); \qquad H_\gamma{}^{(\pi)}(s) = \sqrt{s}\,M_{1\gamma}{}^{(\pi)}(s);$$
$$H_\pi{}^{(\gamma)}(s) = \frac{1}{\sqrt{s}}M_{1\pi}{}^{(\gamma)}; \qquad H_\gamma{}^{(\gamma)}(s) = M_{1\gamma}{}^{(\gamma)}(s). \tag{8}$$

* Clarendon Press, Oxford (1948).

Unlike the quantities M, however, all of the quantities H are finite and different from zero at $s = 0$. Their numerical values are given as functions of s in Table 5.24.3.

Note that the expressions for $\pi\,dE$, $\gamma\,dE$ and Π depend only on the thickness, t, and on the ratio of the initial energy, E_0, to the energy, E, of the observed shower particles. For the dependence on t, the leading term is $\exp(\lambda_1 t)$; for the dependence on E the leading term is $E^{-(s+1)}$ in the case of the differential spectra, E^{-s} in the case of the integral spectra. In other words, if we restrict ourselves to a small range of energies and thicknesses, the differential spectra depend on energy approximately according to a power law with exponent $-(s+1)$ and their intensity varies with depth approximately according to an exponential law with coefficient λ_1. The quantities λ_1 and s have the same relation to one another as the quantities λ and s that enter in the expressions for the stationary solutions. Thus, in the neighborhood of any given energy and any given thickness, a shower behaves approximately as the shower described by a stationary solution. This means, in particular, that the "local" variation with depth for particles of a given energy, E, is completely determined by the shape of the energy spectrum near E. The physical reason for this result is that most of the shower particles with energy near E observed at $t + dt$ originate from shower particles that had energies not much greater than E at t.

In order to determine the position of the maxima of the functions π, γ, and Π we consider the quantities in curled brackets of Eqs. (2) to (7) as slowly varying functions of t and equate to zero the derivative with respect to t of the functions that multiply these quantities. Such functions have the general form:

$$\exp[\lambda_1(s)t + sy - n\ln s], \tag{9}$$

where

$$y = \ln\left(\frac{E_0}{E}\right).$$

One will note that the corresponding expressions for t are equivalent to equations of the form

$$\lambda_1'(s)t + y - \frac{n}{s} = 0. \tag{10}$$

The "optimum thickness" T obeys the equation:

$$\lambda_1(s) + \left[\lambda_1'(s)T + y - \frac{n}{s}\right]\frac{\partial s}{\partial t} = 0.$$

This equation together with Eq. (10) gives:

$$\lambda_1(s) = 0 \qquad \text{or} \qquad s = 1$$

and yields the following general expression for the optimum thickness.

$$T = \frac{-(y-n)}{\lambda_1'(1)} = 1.01\left[\ln\left(\frac{E_0}{E}\right) - n\right]. \tag{11}$$

One obtains the corresponding maximum values of the functions π, γ, and Π by placing $s = 1$, $t = T$ in Eqs. (2) to (7). The maximum values have expressions of the general form:

$$\left.\begin{matrix}\pi_{\max}(E_0,E)\,dE\\ \gamma_{\max}(E_0,E)\,dE\end{matrix}\right\} = \frac{l}{\left[\ln\left(\frac{E_0}{E}\right) - m\right]^{1/2}}\frac{E_0}{E^2}\,dE; \tag{12}$$

$$\Pi_{\max}(E_0,E) = \frac{l}{\left[\ln\left(\frac{E_0}{E}\right) - m\right]^{1/2}}\frac{E_0}{E}. \tag{13}$$

The constants l, m, and n have the values listed in Table 1. Note that, in agreement with the results of the qualitative analysis in § 5.3, *the optimum thickness increases logarithmically with the ratio E_0/E, and the total number of electrons with energy greater than E at the maximum* ($\Pi_{\max}$) *is approximately proportional to this ratio. The energy dependence of the electron and photon spectra at the maximum is approximately equal to the energy dependence of the corresponding track lengths.* This is due to the fact that the *average* properties of shower particles reflect mainly the properties of these particles near the maximum.

Table 5.10.1. Values of constants l, m, and n in Eqs. (11), (12), (13)

Function \ Primary Particle	Electron			Photon		
	l	m	n	l	m	n
π	0.137	0	0	0.137	−0.18	−0.5
γ	0.180	0.18	0.5	0.180	0	0
Π	0.137	0.37	1	0.137	0.18	0.5

If one compares the expressions for the optimum thickness with the corresponding expressions for the position of the center of gravity $\bar{t}$, Eq. (5.9.4), one sees that $\bar{t}$ exceeds T by a quantity independent of energy and of the order of one radiation length. This indicates that the increase of number of particles with thickness before the maximum is somewhat more rapid than the decrease beyond the maximum.

As pointed out before, the approximations made in calculating the differential and integral spectra limit the validity of the expressions obtained to values of t not smaller than about one radiation length and to energies not too close to the initial energy. On the other hand, for small thicknesses and for energies close to the primary energy, the method of

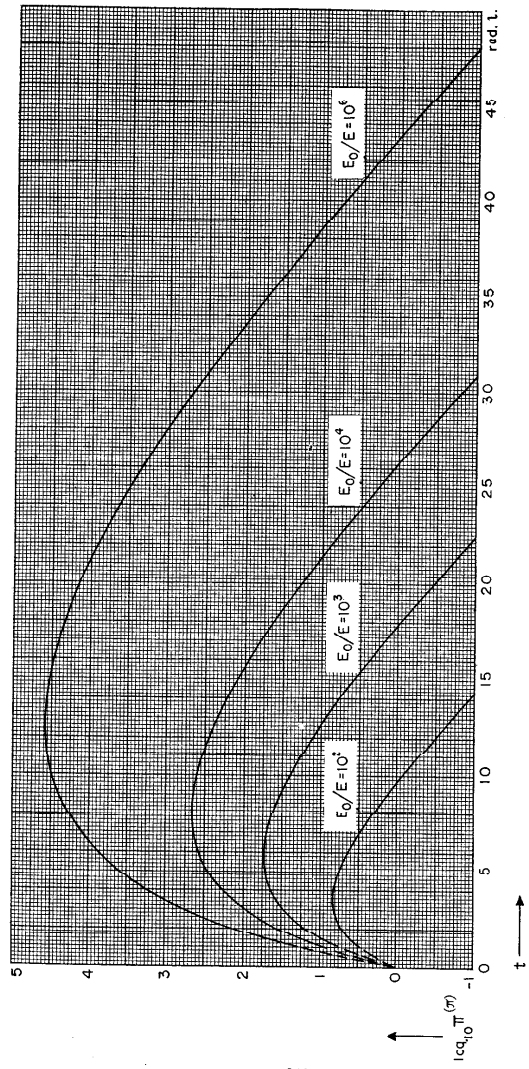

Fig. 5.10.1. The number of electrons of energy greater than E in a shower initiated by an electron of energy E_0, $\Pi^{(e)}(E_0,E,t)$, as a function of t. Computed for various values of E_0/E according to Approximation A. From Rossi and Greisen (RB41.1).

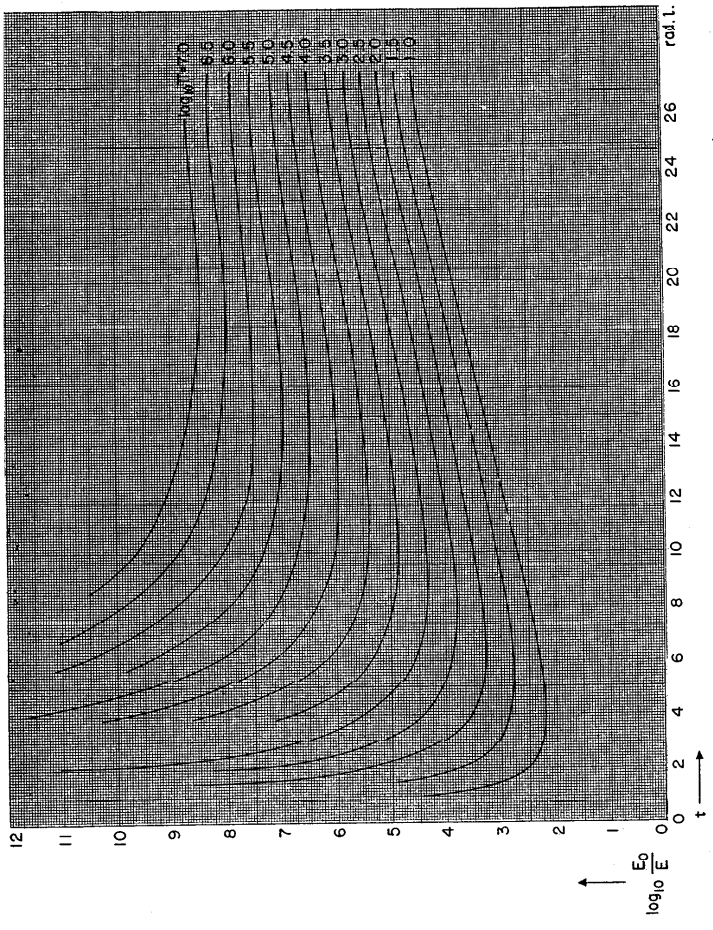

Fig. 5.10.2. Energy E_0 of the primary electron required to produce a shower with Π electrons of energy greater than E at different depths t. Computed according to Approximation A.

successive collisions gives the desired result in the form of a rapidly convergent series. Therefore the two methods supplement one another. For small thicknesses one can also, according to Eyges,* compute π and γ by developing the integrand in Eq. (1) and in the similar equation for γ in a Taylor series of t.

The graph in Fig. 1 gives $\Pi^{(\pi)}(E_0,E,t)$ as a function of t for various values of E_0/E. Figure 2 presents the same results in a different form, namely as the energy E_0 required to produce a shower with $\Pi^{(\pi)}$ electrons of energy greater than E at different depths t.

5.11. ***Simplified expression for the integral electron spectrum under Approximation A.*** In their original paper, Landau and Rumer (LLD38) computed approximately the integral spectrum of electrons starting directly from the equation for the corresponding Mellin integral. Heisenberg,† following the method of Landau and Rumer, arrived at the following approximate analytic expression for $\Pi(E_0,E,t)$:

$$\Pi(E_0,E,t) = \left(\frac{y-\beta}{t-\alpha}\right)^{1/2} \frac{1}{t^{1/2}} \exp\left\{-t + 2[(t-\alpha)(y-\beta)]^{1/2}\right\}, \qquad (1)$$

where $y = \ln(E_0/E)$, $\alpha = 1.4$, $\beta = 0.56$. Within the approximations of the method the same expression also gives the integral electron spectrum for a photon-initiated shower. The agreement between the values of the function Π computed from Eq. (1) and from the equations of the previous section is fairly good.

5.12. ***The diffusion equations under Approximation B.*** Until now we have developed the shower theory under Approximation A, namely with the following assumptions: (a) collision processes of electrons are negligible, and (b) radiation phenomena and pair production can be described by the asymptotic formulae for complete screening. In what follows, we shall adopt the point of view of Approximation B (§ 5.2), namely we shall maintain (b) but shall replace (a) with the assumption that all electrons lose a constant amount of energy, ϵ_0, per radiation length.

In order to derive the diffusion equations corresponding to Approximation B, note that the terms $\mathfrak{A}\pi$, $\mathfrak{B}\gamma$, $\mathfrak{C}\pi$ and $\mathfrak{D}\gamma$ in Eqs. (5.5.11) have the same expressions for Approximation B as for Approximation A [see Eqs. (5.7.4), (5.7.5), (5.7.6), and (5.7.7)], whereas the term $\partial(\epsilon\pi)/\partial E$, which is zero under Approximation A, becomes equal to $\epsilon_0(\partial\pi/\partial E)$ under Approximation B. Thus the diffusion equations under Approximation B are as follows:

$$\frac{\partial\pi}{\partial t} = -\int_0^1 \left[\pi(E,t) - \frac{1}{1-v}\,\pi\left(\frac{E}{1-v},t\right)\right]\psi_{\text{rad}}(v)\,dv + 2\int_0^1 \gamma\left(\frac{E}{v},t\right)\psi_{\text{pair}}(v)\,\frac{dv}{v} + \epsilon_0\,\frac{\partial\pi(E,t)}{\partial E}; \qquad (1a)$$

* L. Eyges, private communication.

† W. Heisenberg, *Cosmic Radiation*, Dover Publications, New York (1946), p. 11.

$$\frac{\partial \gamma}{\partial t} = \int_0^1 \pi\left(\frac{E}{v},t\right)\psi_{\text{rad}}(v)\,\frac{dv}{v} - \mu_0\gamma(E,t). \tag{1b}$$

Note that if one were to measure energies in terms of the critical energy, ϵ_0, Eqs. (1) would not contain any quantity that depends on the medium in which the shower propagates. Therefore, as already pointed out, *the shower theory under Approximation B gives identical results for all substances, provided one measures thicknesses in radiation lengths and energies in terms of the critical energy.*

The addition of the term $\epsilon_0(\partial\pi/\partial E)$ makes the diffusion equations much more difficult to handle mathematically. One sees this clearly if one tries to solve the diffusion equations by the transform method followed in the preceding sections. Consider again the functions $\mathfrak{N}_\pi(s,\lambda)$ and $\mathfrak{N}_\gamma(s,\lambda)$ representing the Mellin integrals with respect to E of the Laplace integrals with respect to t of the functions π and γ (see Eqs. 5.8.2). One may find a system of equations for $\mathfrak{N}_\pi$ and $\mathfrak{N}_\gamma$ by starting from Eqs. (1) and following a procedure entirely similar to that employed in deriving Eqs. (5.8.4) from Eqs. (5.7.3). Considering that:

$$\int_0^\infty \frac{\partial \pi}{\partial E}E^s\,dE = -s\,\mathfrak{M}_\pi(s-1,t)$$

[see Eq. (A.4.7)], one obtains:

$$\lambda\,\mathfrak{N}_\pi(s,\lambda) - \mathfrak{M}_\pi(s,0) = -A(s)\,\mathfrak{N}_\pi(s,\lambda) + B(s)\,\mathfrak{N}_\gamma(s,\lambda) - s\epsilon_0\,\mathfrak{N}_\pi(s-1,\lambda), \tag{2a}$$

$$\lambda\,\mathfrak{N}_\gamma(s,\lambda) - \mathfrak{M}_\gamma(s,0) = C(s)\,\mathfrak{N}_\pi(s,\lambda) - \mu_0\,\mathfrak{N}_\gamma(s,\lambda), \tag{2b}$$

where all the symbols have the same meanings as in Eq. (5.8.4). In particular, $\mathfrak{M}_\pi(s,0)$ and $\mathfrak{M}_\gamma(s,0)$ represent the Mellin integrals of the incident spectra, so that $\mathfrak{M}_\pi(s,0) = E_0^s$, $\mathfrak{M}_\gamma(s,0) = 0$ for an incident electron, $\mathfrak{M}_\pi(s,0) = 0$, $\mathfrak{M}_\gamma(s,0) = E_0^s$ for an incident photon [see Eqs. (5.8.5), (5.8.6)]. Equations (2) differ from Eqs. (5.8.4) only because of the extra term $-s\epsilon_0\,\mathfrak{N}_\pi(s-1,\lambda)$ in Eq. (2a). This, however, is a very important difference. Equations (5.8.4) form a system of algebraic equations that can be solved by elementary means and give explicit expressions for $\mathfrak{N}_\pi$ and $\mathfrak{N}_\gamma$ as functions of s and λ. Equation (2a), instead, is a difference equation because one of its terms contains $s-1$ whereas all other terms contain s. The solution of a system of the type of Eqs. (2) requires the application of difficult mathematical procedures, which do not lead to closed mathematical expressions for $\mathfrak{N}_\pi$ and $\mathfrak{N}_\gamma$.

In the following sections we shall outline briefly the methods that have been used to find a partial solution of the shower problem under Approximation B and describe some of the results obtained. For the details of the mathematical computations we refer the reader to the original papers and to the review article by Rossi and Greisen (RD41.1).

5.13. ***The Method of Snyder and Serber.*** In this method the first step consists in investigating the *stationary distributions* under Approximation B; i.e., in finding solutions of Eqs. (5.12.1) of the general type:

$$\pi(E,t) = F_\pi(E)e^{-\lambda t}; \qquad \gamma(E,t) = F_\gamma(E)e^{-\lambda t}.$$

For energies large compared with ϵ_0, the collision term can be neglected and Eqs. (5.12.1) become identical to Eqs. (5.7.3), which describe showers under Approximation A. When Eqs. (5.7.3) are valid, F_π and F_γ are power functions of E, see Eqs. (5.7.9). This suggests the following expressions for F_π and F_γ under Approximation B;

$$\begin{aligned} F_\pi(E) &\equiv aE^{-(s+1)}z_\pi(s,E/\epsilon_0), \\ F_\gamma(E) &= bE^{-(s+1)}z_\gamma(s,E/\epsilon_0), \end{aligned} \tag{1}$$

where s is a positive number and the functions z_π and z_γ tend to 1 for energies large compared with ϵ_0. It follows that λ must satisfy Eq. (5.7.12) and has, for each value of s, two possible values, λ_1 and λ_2, as given by Eqs. (5.7.13). The corresponding ratios between the coefficients a and b are given by Eqs. (5.7.14).

Substitution of the expressions for the stationary solutions given above into the diffusion equations yields a system of integral equations whose solution is of the following form (SHS38; SbR38; RB41.1):

$$z_\pi\left(s,\frac{E}{\epsilon_0}\right) = \frac{1}{2\pi i}\int_{-\delta-i\infty}^{-\delta+i\infty}\frac{\Gamma(-r)\Gamma(s+r+1)}{\Gamma(s+1)}K(s,r)\left(\frac{E}{\epsilon_0}\right)^{-r}dr; \tag{2a}$$

$$z_\gamma\left(s,\frac{E}{\epsilon_0}\right) = \frac{1}{2\pi i}\int_{-\delta-i\infty}^{-\delta+i\infty}\frac{\Gamma(-r)\Gamma(s+r+1)}{\Gamma(s+1)}\frac{C(s+r)}{C(s)}K(s,r)\left(\frac{E}{\epsilon_0}\right)^{-r}dr. \tag{2b}$$

In Eq. (2a) the integration path in the complex plane of the variable, r, runs parallel to the imaginary axis, between $r = 0$ and $r = -(s+1)$. In Eq. (2b) the integration path is between $r = 0$ and $r = -s$. Γ signifies the gamma function and $K(s,r)$ is a function of the complex variable r and of the parameter s that satisfies the difference equation:

$$\left[\lambda(s) + A(s+r) - \frac{B(s+r)C(s+r)}{\mu_0 + \lambda(s)}\right]K(s,r) = rK(s,r-1), \tag{3}$$

and the boundary condition:

$$K(s,0) = 1. \tag{4}$$

For each value of s there are, of course, two values of λ and therefore two solutions, $K_1(s,r)$ and $K_2(s,r)$, of Eq. (3). Correspondingly, there are two sets of functions $z_{\pi 1}$, $z_{\gamma 1}$, and $z_{\pi 2}$, $z_{\gamma 2}$.

The determination of F_π and F_γ thus requires evaluation of integrals in the complex plane. This is a difficult task in the general case. How-

ever, when $E \ll \epsilon_0$, the method of the residues can be applied and it yields the following expressions for F_π, F_γ and for the corresponding integral electron spectrum, F_Π:

$$\begin{aligned} F_\pi(E) &= \epsilon_0^{-(s+1)}\left[q_1(s) + q_2(s)\ln\frac{\epsilon_0}{E}\right]; \\ F_\gamma(E) &= \epsilon_0^{-s} q_3(s)\frac{1}{E}; \\ F_\Pi(E) &= \epsilon_0^{-s} q_4(s). \end{aligned} \tag{5}$$

The functions q_2, q_3, q_4 are given by:

$$\begin{aligned} q_2(s) &= \frac{a}{s}\frac{2.1}{[\mu_0 + \lambda(s)]}K(s,-s) \\ q_3(s) &= \frac{a}{s}\frac{1.36}{[\mu_0 + \lambda(s)]}K(s,-s) \\ q_4(s) &= \frac{a}{s}K(s,-s). \end{aligned} \tag{6}$$

For sufficiently small values of E, the quantity $q_1(s)$ in the equation for F_π obviously can be disregarded in comparison with the logarithmic term; therefore its expression will not be given explicitly here.

Equations (5) show that *when the energy tends to zero, the differential spectrum of electrons diverges as* $-\ln E$, *the differential spectrum of photons diverges as* $1/E$, *and the integral spectrum of electrons tends to a finite value.*

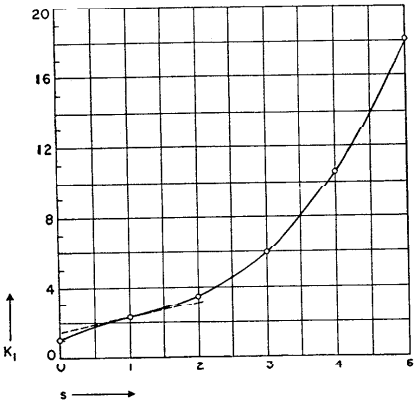

Fig. 5.13.1. The function $K_1(s,-s)$ defined by Eqs. (3) and (4).

The latter result is an obvious consequence of the assumption that electrons undergo a continuous energy loss by collision.

From the difference equation (3) and the boundary condition (4) it is possible to determine the values of $K(s,r)$ and of $\partial K(s,r)/\partial s$ for integral values of r. By graphical interpolation one may then find the values of $K(s,r)$ for non-integral values of r.* Equations (6) contain only values of K corresponding to $r = -s$. With this condition, $K(s,r)$ becomes a function of one variable, s. A plot of $K_1(s,-s)$ is given in Fig. 1.

In order to obtain expressions for the energy spectra of electrons and photons *initiated by a single electron or photon*, we return to the formal expressions of the differential spectra of electrons and photons *under Approximation A* [see Eqs. (5.10.1)] and we rewrite them explicitly for the case of an electron-initiated shower:

$$\pi^{(\pi)}(E_0,E,t) = \frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty}\left[\frac{\mu_0+\lambda_1(s)}{\lambda_1(s)-\lambda_2(s)}e^{\lambda_1(s)t} - \frac{\mu_0+\lambda_2(s)}{\lambda_1(s)-\lambda_2(s)}e^{\lambda_2(s)t}\right]\frac{E_0^s}{E^{s+1}}\,ds, \tag{7a}$$

$$\gamma^{(\pi)}(E_0,E,t) = \frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty}\frac{C(s)}{\lambda_1(s)-\lambda_2(s)}\left(e^{\lambda_1(s)t} - e^{\lambda_2(s)t}\right)\frac{E_0^s}{E^{s+1}}\,ds. \tag{7b}$$

One may regard the functions $\pi^{(\pi)}$ and $\gamma^{(\pi)}$ given by the above equations as linear combinations of the stationary solutions (5.7.15) of the diffusion equations (5.7.3). Here, however, the parameter s and the coefficients a_1, a_2, b_1 and b_2 are complex quantities. For example:

$$a_1 = \frac{ds}{2\pi i}\frac{\mu_0+\lambda_1(s)}{\lambda_1(s)-\lambda_2(s)}E_0^s; \qquad a_2 = -\frac{ds}{2\pi i}\frac{\mu_0+\lambda_2(s)}{\lambda_1(s)-\lambda_2(s)}E_0^s.$$

Snyder (SHS38) showed that one can obtain an approximate solution of the shower problem *under Approximation B* by substituting in Eqs. (7) the stationary solutions of the diffusion equations with collision loss (5.12.1) for the stationary solutions of the diffusion equations without collision loss (5.7.3). This substitution yields:

$$\pi^{(\pi)}(E_0,E,t) = \frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty}\left[\frac{\mu_0+\lambda_1(s)}{\lambda_1(s)-\lambda_2(s)}z_{\pi 1}\left(s,\frac{E}{\epsilon_0}\right)e^{\lambda_1(s)t} - \frac{\mu_0+\lambda_2(s)}{\lambda_1(s)-\lambda_2(s)}z_{\pi 2}\left(s,\frac{E}{\epsilon_0}\right)e^{\lambda_2(s)t}\right]\frac{E_0^s}{E^{s+1}}\,ds. \tag{8a}$$

$$\gamma^{(\pi)}(E_0,E,t) = \frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty}\frac{C(s)}{\lambda_1(s)-\lambda_2(s)}\left[z_{\gamma 1}\left(s,\frac{E}{\epsilon_0}\right)e^{\lambda_1(s)t} - z_{\gamma 2}\left(s,\frac{E}{\epsilon_0}\right)e^{\lambda_2(s)t}\right]\frac{E_0^s}{E^{s+1}}\,ds. \tag{8b}$$

* Snyder (SHS49.2) has given a general solution of the difference equation (3) that proves the analytical character of the function $K(s,r)$.

In Eqs. (8a) and (8b) $z_{\pi 1}$, $z_{\pi 2}$, $z_{\gamma 1}$, and $z_{\gamma 2}$ are the functions z_π and z_γ corresponding to λ_1 and λ_2 respectively.

The functions π and γ defined by Eqs. (8) are linear combinations of the elementary solutions of the diffusion equations (5.12.1). Therefore π and γ are themselves solutions of these equations. We want to investigate their behavior at $t = 0$. Putting $t = 0$ in Eqs. (8) we obtain:

$$\pi^{(\pm)}(E_0,E,0) = \frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty}\left[\frac{\mu_0 \mid \lambda_1(s)}{\lambda_1(s) - \lambda_2(s)} z_{\pi 1}\left(s,\frac{E}{\epsilon_0}\right) - \frac{\mu_0 + \lambda_2(s)}{\lambda_1(s) - \lambda_2(s)} z_{\pi 2}\left(s,\frac{E}{\epsilon_0}\right)\right]\frac{E_0{}^s}{E^{s+1}}\,ds; \quad (9a)$$

$$\gamma^{(\pm)}(E_0,E,0) = \frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty}\frac{C(s)}{\lambda_1(s) - \lambda_2(s)}\left[z_{\gamma 1}\left(s,\frac{E}{\epsilon_0}\right) - z_{\gamma 2}\left(s,\frac{E}{\epsilon_0}\right)\right]\frac{E_0{}^s}{E^{s+1}}\,ds. \quad (9b)$$

The quantities $z_{\pi 1}$, $z_{\pi 2}$, $z_{\gamma 1}$, and $z_{\gamma 2}$ differ from 1 by terms of the order of ϵ_0/E. If, in Eqs. (9), we set $z_{\pi 1} = z_{\pi 2} = z_{\gamma 1} = z_{\gamma 2} = 1$, $\pi(E_0,E,0)$ and $\gamma(E_0,E,0)$ reduce to $\delta(E_0 - E)$ and zero, respectively, i.e., we return to the earlier treatment where collision loss was neglected. For energies E large compared with ϵ_0, therefore, $\pi(E_0,E,0)$ coincides approximately with $\delta(E_0 - E)$ and $\gamma(E_0,E,0)$ is approximately zero. Moreover, for $E > E_0$, $\pi(E_0,E,0)$ and $\gamma(E_0,E,0)$, as given by Eqs. (9), are identically zero. (One can prove this by deforming the contour of integration to the right; the integrands go to zero when $R(s)$ goes to $+\infty$ and have no poles on the positive half-plane.) On the other hand, both $\pi(E_0,E,0)$ and $\gamma(E_0,E,0)$ may differ considerably from zero for energies of the order of ϵ_0 or smaller.

We conclude that the expressions for π and γ given by Eqs. (8) satisfy *approximately* the boundary conditions describing a single incident electron of energy E_0. More precisely, Eqs. (8) represent the shower produced by a primary electron of energy E_0 accompanied by a virtual distribution of electrons and photons that does not extend beyond E_0 and has appreciable intensity only in the neighborhood of the critical energy ϵ_0. When $E_0 \gg \epsilon_0$ and $t \gg 1$, such a shower cannot differ appreciably from the shower produced by a single electron of energy E_0, because the contribution of primaries of small energy to the development of the shower at large thicknesses is negligible.

The expressions for the differential spectra of electrons and photons in the case of a single incident photon can be found by exactly the same procedure.

It now remains to evaluate the complex integrals entering in the formal expressions of π, γ, and Π. When $E \ll \epsilon_0$ one can use the expressions (2) for z_π and z_γ. For large values of t, one can neglect the terms

containing $\exp(-\lambda_2 t)$ in Eqs. (8) and in the similar equations for the other shower functions.

Computations have been made only for the *integral electron spectrum* at $E = 0$, namely for the quantity $\Pi(E_0,0,t)$ giving the *total* number of electrons as a function of thickness. The result may be expressed by the following equation:

$$\Pi(E_0,0,t) = \Pi_A(E_0,\epsilon_0,t)K_1(s,-s), \quad (10)$$

where $\Pi_A(E_0,\epsilon_0,t)$ is the *integral spectrum of electrons at* $E = \epsilon_0$, *computed under Approximation A*, and is given by the parametric equations (5.10.4) for an electron-initiated shower, or (5.10.7) for a photon-initiated shower. The parameter s is the same that enters in these equations and the function $K_1(s,-s)$ is the solution of Eq. (3) corresponding to $\lambda = \lambda_1$ (see Fig. 1).

Figure 2 shows $\Pi^{(\pm)}(E_0,0,t)$ under Approximation B as a function of t computed from Eq. (10) for various values of E_0/ϵ_0. Figure 3 gives the energy E_0 required to produce a shower with a total number Π of electrons at the thickness t, as a function of t for various values of Π.

Since $K_1(s,-s)$ is a slowly varying function, Eq. (10) shows that the maximum of the function $\Pi(E_0,0,t)$ occurs approximately at the same thickness, T, at which the function $\Pi_A(E_0,\epsilon_0,t)$ has its maximum. At the optimum thickness, $s = 1$; therefore the maximum value of the function $\Pi(E_0,0,t)$ equals the maximum value of the function $\Pi_A(E_0,\epsilon_0,t)$ multiplied by $K_1(1,-1) = 1/0.437$.

We list below the expressions for the optimum thickness, the maximum value, the Laplace integral, the track length, the position of the center of gravity, and the longitudinal spread of the function $\Pi(E_0,0,t)$ computed under Approximation B with the method outlined above:

(a) Primary electron of energy E_0:

$$T_\Pi{}^{(\pm)}(E_0,0) = [T_\Pi{}^{(\pm)}(E_0,\epsilon_0)]_A = 1.01\left[\ln\left(\frac{E_0}{\epsilon_0}\right) - 1\right]. \quad (11)$$

$$\Pi_{\max}^{(\pm)}(E_0,0) = [\Pi_{\max}^{(\pm)}(E_0,\epsilon_0)]_A K_1(1,-1) = \frac{0.31}{[\ln(E_0/\epsilon_0) - 0.37]^{1/2}}\frac{E_0}{\epsilon_0} \quad (12)$$

$$\mathfrak{L}_\Pi{}^{(\pm)}(E_0,0,\lambda) = [\mathfrak{L}_\Pi{}^{(\pm)}(E_0,\epsilon_0,\lambda)]_A K_1(s,-s). \quad (13)$$

$$P_0{}^{(\pm)}(E_0,0) - 0.437K_1(1, \quad 1)\frac{E_0}{\epsilon_0} - \frac{E_0}{\epsilon_0}. \quad (14)$$

$$\bar{l}_\Pi{}^{(\pm)}(E_0,0) = 1.01\ln\frac{E_0}{\epsilon_0} + 0.1. \quad (15)$$

$$[\tau_\Pi{}^{(\pm)}(E_0,0)]^2 = 1.61\ln\frac{E_0}{\epsilon_0} - 0.2. \quad (16)$$

(b) Primary photon of energy E_0:

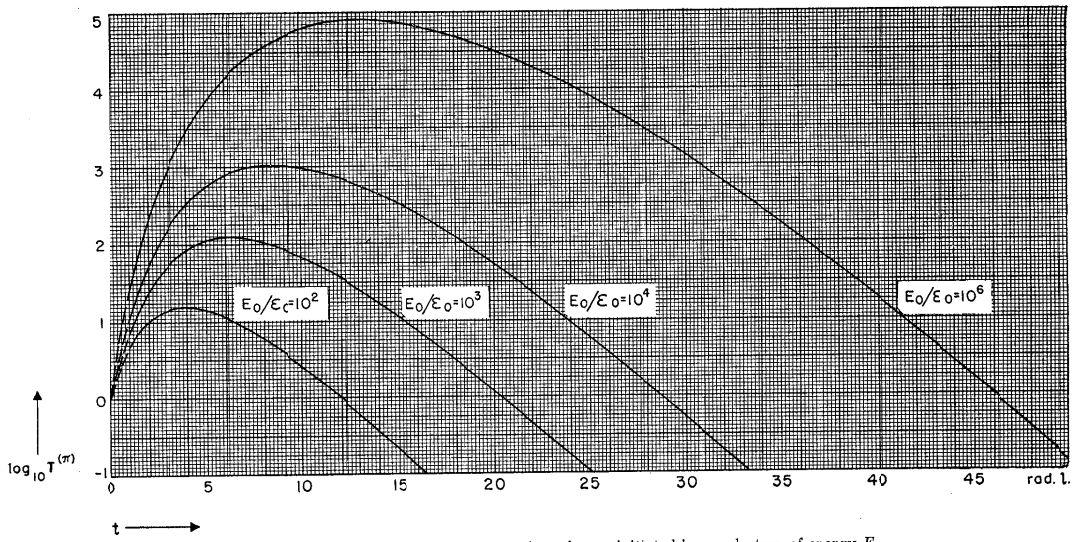

Fig. 5.13.2. The total number of electrons in a shower initiated by an electron of energy E_0. $\Pi^{(\pi)}(E_0,0,t)$, as a function of t. Computed for various values of E_0/ϵ_0 under Approximation B, according to the method of Snyder and Serber. From Rossi and Greisen (RB41.1).

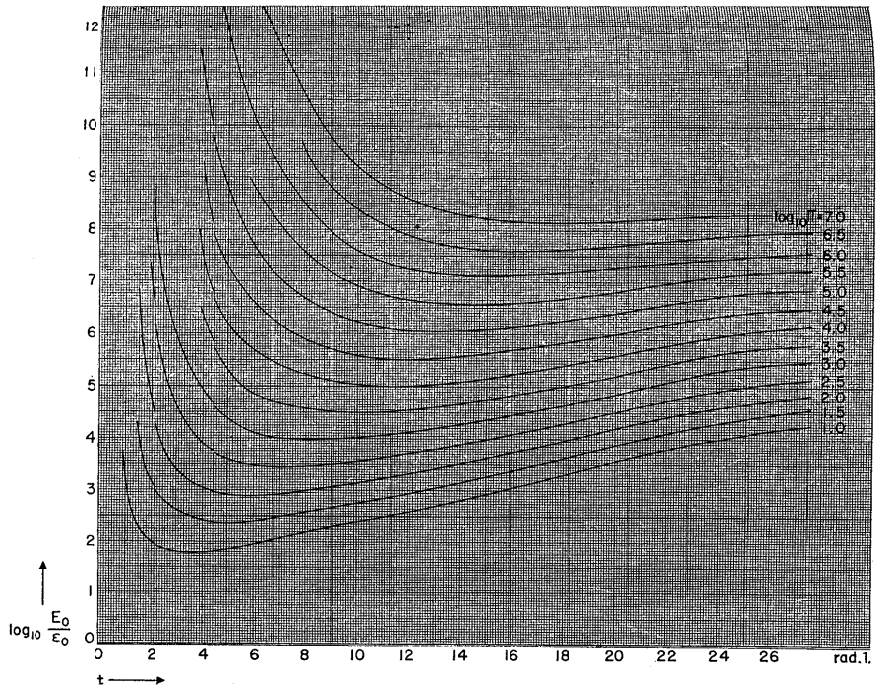

Fig. 5.13.3. Energy E_0 of the primary electron required to produce a shower with a total number Π of electrons at different depths t. Computed under Approximation B, according to the method of Snyder and Serber.

$$T_{\Pi}^{(\gamma)}(E_0,0) = [T_{\Pi}^{(\gamma)}(E_0,\epsilon_0)]_A = 1.01\left[\ln\left(\frac{E_0}{\epsilon_0}\right) - \frac{1}{2}\right]. \tag{17}$$

$$\Pi_{\max}^{(\gamma)}(E_0,0) = [\Pi_{\max}^{(\gamma)}(E_0,\epsilon_0)]_A K(1,-1) = \frac{0.31}{[\ln(E_0/\epsilon_0) - 0.18]^{1/2}}\frac{E_0}{\epsilon_0} \tag{18}$$

$$\mathfrak{L}_{\Pi}^{(\gamma)}(E_0,0,\lambda) = [\mathfrak{L}_{\Pi}^{(\gamma)}(E_0,\epsilon_0,\lambda)]_A K_1(s,-s). \tag{19}$$

$$P_0^{(\gamma)}(E_0,0) = 0.437K_1(1,-1)\frac{E_0}{\epsilon_0} = \frac{E_0}{\epsilon_0}. \tag{20}$$

$$\bar{t}_{\Pi}^{(\gamma)}(E_0,0) = 1.01\ln\left(\frac{E_0}{\epsilon_0}\right) + 1.2. \tag{21}$$

$$[\tau_{\Pi}^{(\gamma)}(E_0,0)]^2 = 1.61\ln\left(\frac{E_0}{\epsilon_0}\right) + 0.9. \tag{22}$$

In the equations above, the subscript A indicates a quantity computed under Approximation A, and s is defined by the equation $\lambda_1(s) = \lambda$. The equations for $P_0(E_0,0)$ express the fact that the track length of all shower electrons equals the initial energy divided by the energy loss per radiation length. This is an obvious consequence of the assumption that electrons undergo a constant energy loss.

5.14. ***The solutions of Bhabha and Chakrabarty and the second Snyder's solution.*** Bhabha and Chakrabarty (BHJ42), (BHJ43), (BHJ48), and Snyder (SHS49.2) have attempted to solve the diffusion equations under Approximation B, using the rigorous boundary conditions representing one incident electron or one incident photon. The formal solutions given by Bhabha and Chakrabarty and by Snyder are mathematically equivalent to one another. It has been shown by Scott (SWT50.2) that one can obtain these solutions by first solving Eqs. (5.12.2) and then applying the inverse Laplace transformation in t and the inverse Mellin transformation in E to the functions $\mathfrak{N}_\pi$ and $\mathfrak{N}_\gamma$ thus determined. This yields a set of equations giving π, γ, and Π as multiple integrals of complex parameters. The method of Bhabha and Chakrabarty and that of Snyder differ in the mathematical procedure used in the evaluation of these integrals.

Bhabha and Chakrabarty obtained their solution in the form of a series whose first two terms they evaluated explicitly. The series converges more rapidly the larger the value of E. Thus the method of Bhabha and Chakrabarty appears to be best suited for the computation of the electron and photon spectra at energies of the order of the critical energy or greater. At $E = 0$ the first two terms of the expansion give values of Π that are about 30 per cent too low. One can see this easily by measuring the areas under the curves that represent $\Pi(E_0,0,t)$ as a function of t. These areas should be equal to E_0/ϵ_0, whereas, according to the computations of Bhabha and Chakrabarty, they range in value from $0.75E_0/\epsilon_0$ [for $\ln(E_0/\epsilon_0) = 2$] to $0.67E_0/\epsilon_0$ [for $\ln(E_0/\epsilon_0) = 8$; see (SHS49.2)].

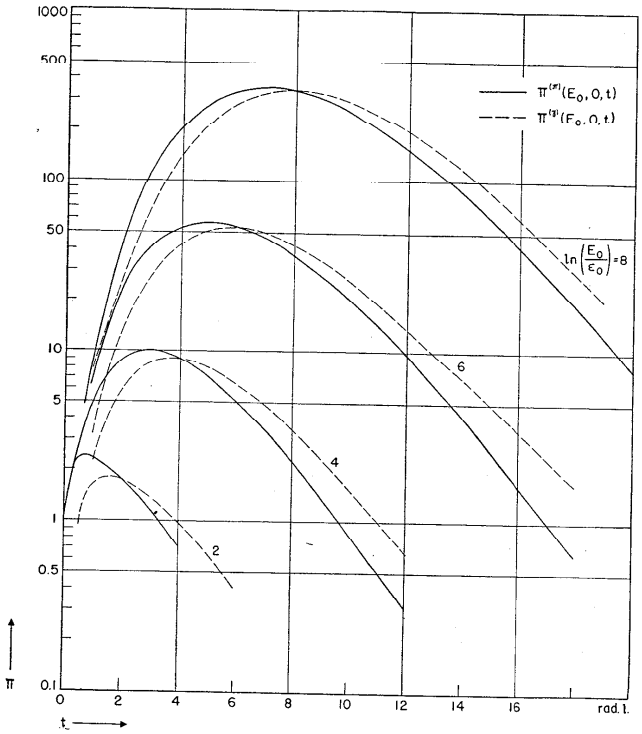

Fig. 5.14.1. The solid lines are graphs of the function $\Pi^{(\pi)}(E_0,0,t)$, as computed by Snyder under Approximation B (SHS49.2). The dashed lines are graphs of the function $\Pi^{(\gamma)}(E_0,0,t)$, according to the same author. The numbers attached to each pair of curves indicate the corresponding value of $\ln(E_0/\epsilon_0)$.

Snyder, too, arrived at a solution in the form of an expansion that, however, converges best for small values of E. The method of Snyder has been used to compute only the integral electron spectrum at $E = 0$. The results are in good agreement with those obtained by the less rigorous method described in the preceding section. The areas under the curves representing $\Pi(E_0,0,t)$ as a function of t do not differ from E_0/ϵ_0 by more than a few per cent.

Figure 1 shows the functions $\Pi(E_0,0,t)$ computed by Snyder (SHS49.2) for electron-initiated and photon-initiated showers. In Fig. 2 the results of Snyder (SHS49.2), of Bhabha and Chakrabarty (BHJ48) and of Snyder and Serber, Eq. (5.13.10), are compared with one another for the case $\ln(E_0/\epsilon_0) = 6$. One sees that the method of Snyder and that of Snyder and Serber give practically the same result, whereas the method of Bhabha and Chakrabarty gives values of $\Pi(E_0,0,t)$ that are too small, as already

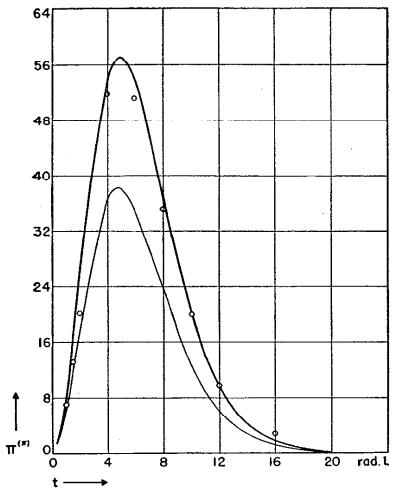

Fig. 5.14.2. The heavy line represents the function $\Pi^{(\pi)}(E_0,0,t)$ for $\ln(E_0/\epsilon_0) = 6$, computed according to the method of Snyder (SHS49.2). The thin line represents the same function computed according to the method of Bhabha and Chakrabarty (BHJ48). The circles are values of $\Pi^{(\pi)}$, computed according to the method of Snyder and Serber, (§ 5.13).

pointed out. The formulae developed by Bhabha and Chakrabarty yield information on the energy distribution of electrons at various thicknesses. We refer the reader to the original papers for details on this subject.

5.15. ***Remarks on the physical significance of shower theory under Approximation B.*** The arguments developed in § 5.2 show that in substances of low atomic number the shower theory under Approximation B is capable of yielding valid results on the number and energy distribution of shower particles with energy not much smaller than the critical energy. It also gives fairly accurate values for the *energy dissipation*, $\rho(E_0,t)$, whose expression, under Approximation B, is:

$$\rho(E_0,t) = \epsilon_0 \Pi(E_0,0,t). \tag{1}$$

In the preceding sections we have sometimes referred to the quantity $\Pi(E_0,0,t)$ computed under Approximation B as to the *total number of electrons at the depth t.* This expression, even though formally correct, has little relation to physical reality because Approximation B cannot give the correct number of electrons near zero energy. Nevertheless, knowledge of $\Pi(E_0,0,t)$ is physically significant because, as shown by Eq. (1), this function multiplied by ϵ_0 correctly represents the energy per radiation length dissipated by all shower particles in excitation and ionization of atoms at the depth t.

In substances of high atomic number, the assumption of a constant absorption coefficient for γ-rays becomes grossly incorrect, even at energies above the critical energy. For this reason, if for no others, Approximation B cannot give the correct dependence of the energy dissipation on depth in heavy elements. In fact, as the thickness increases, the logarithmic slope of the function $\rho(t)$ (i.e., the quantity $-d \ln \rho/dt$) computed under Approximation B tends to the value $\mu_0 = 0.773$, which represents the asymptotic value for the absorption coefficient of γ-rays. In heavy elements the absorption coefficient of γ-rays in the neighborhood of the critical energy is much smaller than μ_0. Therefore the shower must have a much longer "tail" than the theory predicts. This "tail" consists mainly of low-energy photons and of their secondary electrons and accounts for an appreciable fraction of the total energy dissipation. A method for the computation of the number of low-energy photons in the "tail" of a shower has been developed by Greisen (GKI49). The influence of the large penetration of low-energy photons on the position and the size of the shower maximum will be discussed in § 5.22.

Another difficulty of the shower theory in heavy elements lies in the large scattering of electrons in the neighborhood of the critical energy. This effect tends to shift the position of the shower maximum toward smaller thicknesses, but it does not compensate for the opposite effect of the small absorption of low-energy photons (see § 5.22).

One should note that, even in substances of low atomic number, the use of the asymptotic expressions for the probabilities of radiation and pair production introduces an appreciable error in the computation of $\rho(E_0,t)$. This has been shown by the work of Bernstein (BIB50), who developed an iteration method for the solution of the exact diffusion equations of the shower theory. This author used the solution obtained by Snyder under Approximation B (see preceding section) as the zero-order approximation and then computed a first-order correction by introducing

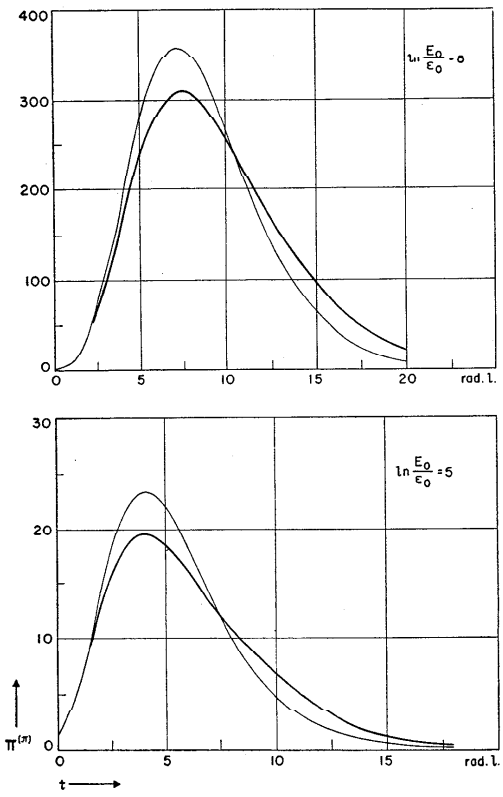

Fig. 5.15.1. The function $\Pi^{(\pi)}(E_0,0,t)$ in air, computed according to Snyder (SIG49.2) (thin lines), and according to Bernstein (BIB50) (heavy lines) for $\ln (E_0/\epsilon_0) = 8$, and $\ln (E_0/\epsilon_0) = 5$. The computations of Bernstein, unlike those of Snyder, take into account the energy dependence of the cross-sections for radiation and pair production.

more accurate expressions for the probabilities of radiation and pair production. The method yields accurate results only in the case of light elements and high initial energies. Some of these results are shown in Fig. 1, giving $\Pi^{(\pi)}(E_0,0,t)$ as a function of t for $\ln (E_0/\epsilon_0) = 8$ and $\ln (E_0/\epsilon_0) = 5$.

One sees that the use of the accurate expressions for φ_{rad} and φ_{pair} in place of the asymptotic expressions for these functions makes the computed values of $\Pi(E_0,0,t)$ somewhat smaller at small thicknesses and somewhat greater at large thicknesses. This result is qualitatively understandable because the asymptotic expressions represent an overestimate for the probabilities of radiation and pair production.

5.16. ***The method of the moments. Integral equations for the track length.*** The direct computation of moments offers a promising approach to the shower problem. As pointed out in § 5.1, the moments enter directly into many important problems and, in principle at least, knowledge of all the moments of a shower function is sufficient to determine the shower function itself.

As a first step, we shall derive the equations for the zero moments or track lengths, from the general diffusion equations (5.5.10). We shall assume that the shower is initiated by an electron of energy E_0, so that the boundary conditions (5.5.13) apply. Integration of Eqs. (5.5.10) with respect to t yields:

$$-\delta(E-E_0) = -p_0^{(\pi)}(E_0,E)\mu_\pi(E) + \int_E^{E_0} p_0^{(\pi)}(E_0,E')\varphi_{\pi\pi}(E',E)\,dE' + \int_E^{E_0} g_0^{(\pi)}(E_0,E')\varphi_{\gamma\pi}(E',E)\,dE' + \frac{\partial}{\partial E}[p_0^{(\pi)}(E_0,E)\epsilon(E)]; \tag{1a}$$

$$0 = \int_E^{E_0} p_0^{(\pi)}(E_0,E')\varphi_{\pi\gamma}(E',E)\,dE' + \int_E^{E_0} g_0^{(\pi)}(E_0,E')\varphi_{\gamma\gamma}(E',E)\,dE' - g_0^{(\pi)}(E_0,E)\mu_\gamma(E). \tag{1b}$$

In the case of a shower initiated by a photon instead of an electron, the integral equations for the differential track lengths [$p_0^{(\gamma)}$ and $g_0^{(\gamma)}$] are identical to Eqs. (1) except that, on the left-hand side, there is a zero in the first equation and a δ-function in the second one. However, when the equations for an electron-initiated shower are solved, the track lengths for a photon-initiated shower can be determined by simple integrations, as we shall now proceed to show. Consider a photon of energy E_0 incident upon matter. In its first interaction it changes either into a negaton-positon pair (pair production) or into an electron-photon pair (Compton effect). The probability that the photon, in this first interaction, produces an electron of energy in dE' at E' is $\varphi_{\gamma\pi}(E_0,E')\,dE'/\mu_\gamma(E_0)$; the probability that it produces a photon of energy in dE' at E' is $\varphi_{\gamma\gamma}(E_0,E')\,dE'/\mu_\gamma(E_0)$. For the computation of the differential electron track length, the original photon is equivalent to the secondary particles that it produces in its first interaction. Therefore:

$$p_0^{(\gamma)}(E_0,E) = \frac{1}{\mu_\gamma(E_0)}\left[\int_E^{E_0}\varphi_{\gamma\pi}(E_0,E')p_0^{(\pi)}(E',E)\,dE' + \int_E^{E_0}\varphi_{\gamma\gamma}(E_0,E')p_0^{(\gamma)}(E',E)\,dE'\right]. \quad (2)$$

Similarly, one obtains the following expression for the integral electron track length $P_0^{(\gamma)}$:

$$P_0^{(\gamma)}(E_0,E) = \frac{1}{\mu_\gamma(E_0)}\left[\int_E^{E_0}\varphi_{\gamma\pi}(E_0,E')P_0^{(\pi)}(E',E)\,dE' + \int_E^{E_0}\varphi_{\gamma\gamma}(E_0,E')P_0^{(\gamma)}(E',E)\,dE'\right]. \quad (3)$$

To derive the corresponding equation for the photon track length, one must consider that the original photon travels, on the average, a distance $1/\mu_\gamma(E_0)$ with energy E_0 before undergoing the first interaction. Therefore, the *integral* photon track length of the original photon is $1/\mu_\gamma(E_0)$ plus the average combined track lengths of the secondary particles produced in the first interaction. These considerations lead to the following equation:

$$G_0^{(\gamma)}(E_0,E) = \frac{1}{\mu_\gamma(E_0)}\left[1 + \int_E^{E_0}\varphi_{\gamma\pi}(E_0,E')G_0^{(\pi)}(E',E)\,dE' + \int_E^{E_0}\varphi_{\gamma\gamma}(E_0,E')G_0^{(\gamma)}(E',E)\,dE'\right]. \quad (4)$$

Differentiation with respect to E yields:

$$g_0^{(\gamma)}(E_0,E) = \frac{1}{\mu_\gamma(E_0)}\left[\delta(E-E_0) + \int_E^{E_0}\varphi_{\gamma\pi}(E_0,E')g_0^{(\pi)}(E',E)\,dE' + \int_E^{E_0}\varphi_{\gamma\gamma}(E_0,E')g_0^{(\gamma)}(E',E)\,dE'\right]. \quad (5)$$

Photons produce secondary photons only by Compton effect. At high energies this phenomenon is rare compared with pair production and, when it occurs, most of the energy of the primary photon usually goes into the recoil electron. Therefore, on a good approximation, one can neglect the integrals containing $\varphi_{\gamma\gamma}$ in Eqs. (2) and (5). These equations then give $p_0^{(\gamma)}$ and $g_0^{(\gamma)}$ in terms of $p_0^{(\pi)}$ and $g_0^{(\pi)}$.

One can write equations corresponding to Eqs. (2), (3), (4), and (5) for electron-initiated showers. The integral electron track length, for instance, satisfies formally an equation similar to Eq. (4):

$$\mu_\pi(E_0)P_0^{(\pi)}(E_0,E) = 1 + \int_E^{E_0}\varphi_{\pi\pi}(E_0,E')P_0^{(\pi)}(E',E)\,dE' + \int_E^{E_0}\varphi_{\pi\gamma}(E_0,E')P_0^{(\gamma)}(E',E)\,dE'. \quad (6)$$

Here $\mu_\pi(E)$ represents the probability per radiation length for an electron of energy E_0 to undergo either a collision or a radiation process (Eq. 5.5.3). This quantity is infinitely large because of the divergent character of $\varphi_{\rm col}(E_0,E')$ and $\varphi_{\rm rad}(E_0,E')$ at $E' = 0$. For the same reason, $\varphi_{\pi\pi}(E_0,E')$ becomes infinite at $E' = E_0$. One may eliminate these infinities with a procedure similar to that followed in § 5.5; i.e., by substituting a continuous energy loss, $\epsilon(E_0)$, for that part of the discontinuous loss that corresponds to energy transfers smaller than a given value, η_0. In this case it is convenient to include in $\epsilon(E_0)$ not only "small collision losses," but also "small radiation losses." We define, therefore, $\epsilon(E_0)$ as follows:

$$\epsilon(E_0) = \int_0^{\eta_0} E'[\varphi_{\rm rad}(E_0,E') + \varphi_{\rm col}(E_0,E')]\,dE'. \quad (7)$$

After some mathematical transformations, similar to those leading from Eq. (5.5.7a) to Eq. (5.5.10a), Eq. (6) becomes:

$$\epsilon(E_0)\frac{\partial P_0^{(\pi)}(E_0,E)}{\partial E_0} + \mu_\pi(E_0)P_0^{(\pi)}(E_0,E) = 1 + \int_E^{E_0}\varphi_{\pi\pi}(E_0,E')P_0^{(\pi)}(E',E)\,dE' + \int_E^{E_0}\varphi_{\pi\gamma}(E_0,E')P_0^{(\gamma)}(E',E)\,dE'. \quad (8)$$

The quantities $\mu_\pi(E_0)$ and $\varphi_{\pi\pi}(E_0,E')$ entering in this equation, unlike those entering in Eq. (6), are computed with $\varphi_{\rm rad}(E_0,E') = 0$ and $\varphi_{\rm col}(E_0,E') = 0$ for $E' < \eta_0$. Therefore they are everywhere finite. For the validity of Eq. (8) it is necessary that $P_0^{(\pi)}(E_0,E)$ may be approximated by a linear function of E_0 in any interval η_0 of this variable. This condition is less restrictive than the corresponding condition for the validity of Eq. (5.5.10a) because the dependence of $P_0^{(\pi)}(E_0,E)$ on E_0 is represented by a smoother function than the dependence of $\Pi^{(\pi)}(E_0,E)$ on E. In fact, when E is sufficiently small, one may take $\eta_0 = E$. Remembering the expressions for μ_π, $\varphi_{\pi\pi}$ and $\varphi_{\pi\gamma}$ (see § 5.5) one may then rewrite Eq. (8) in the following form:

$$\epsilon(E_0)\frac{\partial P_0^{(\pi)}(E_0,E)}{\partial E_0} = 1 - \int_E^{E_0}[P_0^{(\pi)}(E_0,E) - P_0^{(\pi)}(E',E) - P_0^{(\pi)}(E_0-E',E)]\varphi_{\rm col}(E_0,E')\,dE' - \int_E^{E_0}[P_0^{(\pi)}(E_0,E) - P_0^{(\gamma)}(E',E) - P_0^{(\pi)}(E_0-E',E)]\varphi_{\rm rad}(E_0,E')\,dE'. \quad (9)$$

Equations (3) and (8) offer an alternate starting point for the mathematical treatment of the track length problem. The solution of these equations gives the integral electron track lengths for electron-initiated and for photon-initiated showers as functions of the initial energy E_0.

The above arguments show that the general track length problem is reduced to the solution of Eqs. (1). It is convenient to eliminate the photon track length, $g_0^{(\pi)}$, from these two equations and obtain an equation for the electron track length, $p_0^{(\pi)}$. This can be done very easily if one neglects the production of secondary photons by Compton effect. In this case $\varphi_{\gamma\gamma} = 0$ and Eq. (1b) can be solved for $g_0^{(\pi)}$. Substitution in Eq. (1a) yields then the desired equation.

This equation contains the term:

$$\int_E^{E_0} g_0^{(\pi)}(E_0,E')\varphi_{\gamma\pi}(E',E)\,dE' = \int_E^{E_0}\varphi_{\gamma\pi}(E',E)\,\frac{dE'}{\mu_\gamma(E')}\int_{E'}^{E_0} p_0^{(\pi)}(E_0,E'')\varphi_{\pi\gamma}(E'',E')\,dE''.$$

Since $p_0^{(\pi)}(E_0,E'')$ is identically zero for $E'' > E_0$, one can substitute ∞ for E_0 in the upper limits of the integrals. By changing the order of integration one then obtains:

$$\int_E^{\infty} g_0^{(\pi)}(E_0,E')\varphi_{\gamma\pi}(E',E)\,dE' = \int_E^{\infty} p_0^{(\pi)}(E_0,E'')\,dE''\int_E^{E''}\frac{\varphi_{\pi\gamma}(E'',E')\varphi_{\gamma\pi}(E',E)}{\mu_\gamma(E')}\,dE',$$

or, by interchanging the symbols E' and E'':

$$\int_E^{\infty} g_0^{(\pi)}(E_0,E')\varphi_{\gamma\pi}(E',E)\,dE' = \int_E^{\infty} p_0^{(\pi)}(E_0,E')\varphi_{\pi\gamma\pi}(E',E)\,dE', \tag{10}$$

where:
$$\varphi_{\pi\gamma\pi}(E',E) = \int_E^{E'}\frac{\varphi_{\pi\gamma}(E',E'')\varphi_{\gamma\pi}(E'',E)}{\mu_\gamma(E'')}\,dE'' \tag{11}$$

The quantity $\varphi_{\pi\gamma\pi}(E',E)\,dE$ represents the probability per radiation length that an electron of energy E' produces a photon which, in turn, when interacting again with matter, gives rise to an electron of energy in dE at E. Define also the quantity:

$$\varphi(E',E)\,dE = \varphi_{\pi\pi}(E',E)\,dE + \varphi_{\pi\gamma\pi}(E',E)\,dE, \tag{12}$$

which represents the total probability per radiation length that an electron of energy E' produces an electron of energy in dE at E either directly or through the intermediary of a photon. With these notations, one obtains from Eq. (1a):

$$\delta(E-E_0) = p_0^{(\pi)}(E_0,E)\mu_\pi(E) - \int_E^{\infty} p_0^{(\pi)}(E_0,E')\varphi(E',E)\,dE' - \frac{\partial}{\partial E}[p_0^{(\pi)}(E_0,E)\epsilon(E)]. \tag{13}$$

After Eq. (13) is solved, Eq. (1b) with $\varphi_{\gamma\gamma} = 0$ yields the photon track length, $g_0^{(\pi)}$.

Equation (13) is the fundamental equation of the track length problem. The first term represents the catastrophic absorption of electrons; the second term represents the reproduction (direct or indirect); the third term represents the continuous energy loss. Of course, in the computation

of $\mu_\pi(E)$ and of $\varphi(E',E)$ one must disregard those processes that are included in $\epsilon(E)$. Thus if $\epsilon(E)$ includes collision losses corresponding to energy transfers smaller than η_0, one must take $\varphi_{\text{col}}(E',E) = 0$ for $E < \eta_0$ (or $E > E' - \eta_0$) in the computation of $\mu_\pi(E)$ and $\varphi(E',E)$. If, in addition, $\epsilon(E)$ also includes radiation losses giving rise to photons of energy less than η_0, one must also take $\varphi_{\text{rad}}(E',E) = 0$ for $E < \eta_0$.

Note that neglecting the production of secondary photons by Compton effect does simplify the mathematical procedure considerably, but is not a necessary requirement of the method outlined. Friedman (FFL49) has shown that one can solve Eq. (1b) for $g_0^{(\pi)}$ without putting $\varphi_{\gamma\gamma} = 0$ by a method of successive approximations. The first improvement upon the approximation $\varphi_{\gamma\gamma} = 0$ consists of considering those photons that undergo one Compton collision between the place of production and the place of absorption. Next one can consider those photons that undergo two Compton collisions, etc.

5.17. *Mathematical elaboration of the track length equations.* * Write Eq. (5.16.13) in the symbolic form:

$$\left[\mathfrak{F} - \frac{\partial}{\partial E}\epsilon(E)\right] p_0^{(\pi)} = \delta(E_0 - E), \tag{1}$$

where the operator $\mathfrak{F}$ has the following meaning:

$$\mathfrak{F}p_0^{(\pi)} = p_0^{(\pi)}(E_0,E)\mu_\pi(E) - \int_E^{\infty} p_0^{(\pi)}(E_0,E')\varphi(E',E)\,dE'. \tag{2}$$

Define another operator $\mathfrak{F}^\dagger$ such that:

$$\int_{E_1}^{\infty} u(E,E_1)\mathfrak{F}p_0^{(\pi)}(E_0,E)\,dE = \int_{E_1}^{\infty} p_0^{(\pi)}(E_0,E)\mathfrak{F}^\dagger u(E,E_1)\,dE \tag{3}$$

where $u(E,E_1)$ is an arbitrary function that satisfies the condition:

$$u(E,E_1) = 0 \qquad \text{for} \quad E < E_1. \tag{4}$$

$\mathfrak{F}^\dagger$ operates on the variable E. Substitution of (2) into (3) and change in the order of integration yields the following expression for $\mathfrak{F}^\dagger$:

$$\mathfrak{F}^\dagger u(E,E_1) = \mu_\pi(E)u(E,E_1) - \int_{E_1}^{E}\varphi(E,E')u(E',E_1)\,dE'. \tag{5}$$

From Eq. (1) one obtains:

$$\int_0^{\infty} u(E,E_1)\,dE\left[\mathfrak{F} - \frac{\partial}{\partial E}\epsilon(E)\right] p_0^{(\pi)}(E_0,E) = u(E_0,E_1). \tag{6}$$

Since $u(E,E_1) = 0$ for $E < E_1$ and $p_0^{(\pi)}(E_0,E) = 0$ for $E > E_0$,

$$\int_0^{\infty} u(E,E_1)\frac{\partial}{\partial E}[\epsilon(E)p_0^{(\pi)}(E_0,E)]\,dE = -\int_0^{\infty} p_0^{(\pi)}(E_0,E)\epsilon(E)\frac{\partial u(E,E_1)}{\partial E}\,dE. \tag{7}$$

* See ref. (FFL49).

This equation, together with Eqs. (3) and (6), gives:

$$\int_0^\infty p_0^{(-)}(E_0,E)\left[\mathfrak{T}^\dagger + \epsilon(E)\frac{\partial}{\partial E}\right]u(E,E_1)\,dE - u(E_0,E_1). \tag{8}$$

If the function u is given, Eq. (8) is an integral equation for $p_0^{(\pi)}$. One may use the freedom in the choice of u to obtain an equation easier to handle mathematically than the original equation (1).

Suppose now that the function $u(E,E_1)$ is chosen so as to satisfy the equation:

$$\left[\mathfrak{T}^\dagger + \epsilon(E)\frac{\partial}{\partial E}\right]u(E,E_1) - \delta(E - E_1). \tag{9}$$

Then Eq. (8) yields:

$$u(E_0,E_1) = p_0^{(\pi)}(E_0,E_1). \tag{10}$$

In other words, the solution of the original Eq. (1) is equivalent to the solution of Eq. (9).

One may use Eq. (9) to obtain an equation for the integral track length, $P_0^{(\pi)}(E_0,E)$. For this purpose, integrate both sides of Eq. (9) with respect to E_1 from a given value E_2 of this variable to infinity. Consider that the operators on the left-hand side operate on the variable E and that, when Eq. (9) is satisfied, $u(E,E_1) = p_0^{(\pi)}(E,E_1)$. Replace E_2 by E, and E by E_0. The result is:

$$\mathfrak{T}^\dagger P_0^{(\pi)}(E_0,E) + \epsilon(E_0)\frac{\partial P_0^{(\pi)}(E_0,E)}{\partial E_0} = 1. \tag{11}$$

In mathematical language, Eq. (9) is called the *adjoint equation* to Eq. (1). One can understand its physical significance by the following considerations. The track length is a function of two variables: the energy E_0 of the primary electron and the energy E of the secondary electrons. For a given primary energy one can compute the track length corresponding to the secondary energy E if one knows the track lengths of particles with energies greater than E, because particles of a given energy are produced only by particles of greater energy. One thus arrives at Eq. (5.16.13), where the primary energy, E_0, is a parameter and the secondary energy, E, is the independent variable.

Alternately, for any secondary energy, E, one can compute the track length corresponding to a primary energy E_0 if one knows the track lengths for all primary particles of energy less than E_0, because an electron of energy E_0 can produce secondary particles of energy E either by a direct process or through the intermediary of particles with energies between E_0 and E. In the preceding section we have followed this second point of view in deriving the system of Eqs. (5.16.3) and (5.16.8), where the secondary energy E appears as a parameter and the primary energy E_0 is the independent variable. Equation (11), obtained by means of the adjoint theory, is entirely equivalent to Eqs. (5.16.3) and (5.16.8), as one can easily recognize by eliminating $P_0^{(\gamma)}$ from these two equations after placing $\varphi_{\gamma\gamma} = 0$.

5.18. ***Some general conclusions concerning the track lengths.*** For values of the secondary energy, E, sufficiently large to justify the use of Approximation A, yet small compared with the initial energy E_0, the track lengths are given by the following equations [see Eqs. (5.9.1), (5.9.2), (5.9.3)].

$$p_0^{(\pi)}(E_0,E) = p_0^{(\gamma)}(E_0,E) = 0.437\,\frac{E_0}{E^2}, \tag{1}$$

$$g_0^{(\pi)}(E_0,E) = g_0^{(\gamma)}(E_0,E) = 0.572\,\frac{E_0}{E^2}, \tag{2}$$

$$P_0^{(\pi)}(E_0,E) = P_0^{(\gamma)}(E_0,E) = 0.437\,\frac{E_0}{E}. \tag{3}$$

These equations lose their validity when E approaches E_0 and when E becomes comparable with or smaller than the critical energy. We shall investigate the behavior of the track lengths at the limits $E \approx 0$ and $E \approx E_0$, *assuming the validity of Approximation B.*

At $E = 0$, the integral electron track length is given by Eqs. (5.13.14) and (5.13.20):

$$P_0^{(\pi)}(E_0,0) = P_0^{(\gamma)}(E_0,0) = \frac{E_0}{\epsilon_0}. \tag{4}$$

Equations (4) are a direct consequence of the principle of the conservation of energy.

According to our assumptions, $\varphi_{\gamma\gamma} = 0$, $\mu_\gamma(E) = \mu_0 = \text{const.}$, and $\varphi_{\pi\gamma}(E',E) = \varphi_{\text{rad}}(E',E)$ is approximately equal to $1/E$ [see Eqs. (5.7.2)]. Therefore it follows from Eqs. (5.16.1b) and (4) that, at the limit for very small values of E, $g_0^{(\pi)}$ has the approximate expression:

$$g_0^{(\pi)}(E_0,E) \approx \frac{P_0^{(\pi)}(E_0,E)}{\mu_0}\frac{1}{E} \approx \frac{E_0}{\epsilon_0\mu_0}\frac{1}{E}. \tag{5}$$

An identical expression gives $g_0^{(\gamma)}(E_0,E)$. Thus the differential photon track length diverges as $1/E$ at the limit for $E = 0$.

In order to determine the behavior of P_0 near $E = 0$, consider Eq. (5.16.1a). For small values of E and large values of E_0, the dominant terms on the right hand side of this equation are the two last ones, because electrons of low energy are mainly produced by materialization of photons and are mainly absorbed by collision loss. Considering Eq. (5) and taking $\varphi_{\gamma\gamma}(E',E) = 2\varphi_{\text{pair}}(E',E) = 2\mu_0/E'$ [see Eqs. (5.7.2)] one thus obtains, for $E_0 \gg \epsilon_0 \gg E$:

$$\epsilon_0\frac{\partial p_0^{(\pi)}(E_0,E)}{\partial E} \approx -2\,\frac{E_0}{\epsilon_0}\frac{1}{E},$$

which gives: $$p_0^{(\pi)}(E_0,E) \approx \frac{2E_0}{\epsilon_0^2} \ln\left(\frac{\epsilon_0}{E}\right) + \text{const.} \tag{6}$$

An identical expression gives $p_0^{(\gamma)}(E_0,E)$. Thus the differential electron track length diverges logarithmically at the limit for $E = 0$.

Next, we turn our attention to the behavior of the track lengths at energies near the initial energy E_0. In an electron-initiated shower, the differential electron track length, $p_0^{(\pi)}(E_0,E)$, must be finite at $E = E_0$ because, on account of the continuous energy loss, an electron of energy E_0 does not travel any finite distance without losing a finite amount of energy. Equation (5.16.1b) then shows that

$$g_0^{(\pi)}(E_0,E_0) = 0 \tag{7}$$

Consider now Eq. (5.16.1a) and assume that the continuous energy loss, $\epsilon(E)$, includes a collision loss as well as the effects of radiation processes in which the secondary photons acquire energies smaller than a given value, η_0. Then $\mu_\pi(E)$ and $\varphi_{\pi\pi}(E',E)$ are finite everywhere. As E approaches E_0 the two integral terms in Eq. (5.16.1a) tend to zero and the equation reduces to:

$$-\delta(E - E_0) = -p_0^{(\pi)}(E_0,E)\mu_\pi(E) + \frac{\partial}{\partial E}[p_0^{(\pi)}(E_0,E)\epsilon(E)].$$

This equation shows that $p_0^{(\pi)}(E_0,E)$ undergoes a discontinuous change of $-1/\epsilon(E_0)$ at $E = E_0$. Since $p_0^{(\pi)}(E_0,E) = 0$ for $E > E_0$, the value of $p_0^{(\pi)}(E_0,E)$ at the left of E_0 is:

$$p_0^{(\pi)}(E_0,E_0^-) = \frac{1}{\epsilon(E_0)}. \tag{8}$$

Equation (8) is valid in general. In order to apply it to the case of Approximation B, one has to take a constant value, ϵ_0, for the collision loss and let the limiting energy for the "continuous" radiation loss go to zero. Since the energy lost by radiation to photons of energy less than a given value is proportional to this limiting value, the radiation loss goes to zero and Eq. (8) becomes:

$$p_0^{(\pi)}(E_0,E_0^-) = \frac{1}{\epsilon_0} \tag{9}$$

The reader can easily prove the following equations:

from Eq. (8) and the definition of P_0:

$$P_0^{(\pi)}(E_0,E_0) = P_0^{(\gamma)}(E_0,E_0) = 0; \tag{10}$$

from Eqs. (8) and (5.16.2):

$$p_0^{(\gamma)}(E_0,E_0) = 0; \tag{11}$$

from Eqs. (7) and (5.16.5) (for E close to E_0):

$$g_0^{(\gamma)}(E_0,E) = \frac{1}{\mu_\gamma(E_0)}\,\delta(E_0 - E). \tag{12}$$

To illustrate the foregoing conclusions Figs. 1 and 2 represent, schematically, the functions $p_0^{(\pi)}(E_0,E)$ and $P_0^{(\pi)}(E_0,E)$ as given by Eqs. (1) and (3), and as computed according to Approximation B.

Qualitatively the behavior of the track lengths does not change if one abandons the restrictive assumptions of Approximation B, but retains the assumption of a continuous energy loss. In this case Eq. (8), for instance,

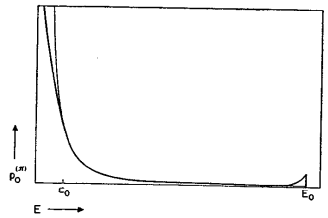

Fig. 5.18.1. Schematic representation of the function $p_0^{(\pi)}(E_0,E)$, according to Eq. (1) (thin line) and according to Approximation B (heavy line).

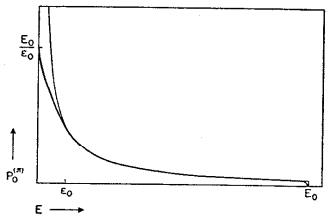

Fig. 5.18.2. Schematic representation of the function $P_0^{(\pi)}(E_0,E)$, according to Eq. (3) (thin line) and according to Approximation B (heavy line).

still holds and it is still true that $P_0(E_0,E_0) = 0$. Equation (4) is replaced by the equation:

$$\int_0^{E_0} p_0^{(\pi)}(E_0,E)\epsilon(E)\,dE = E_0 \tag{13}$$

expressing the principle of conservation of energy. Since $\epsilon(E)$ is a slowly varying function of E, Eq. (13) indicates that the integral track length at $E = 0$ is given by an expression of the type of Eq. (4) where ϵ_0 now represents some kind of an average value of $\epsilon(E)$.

One may rightly question the significance of results concerning the behavior of the track lengths at energies close to zero or to E_0, as we have obtained these results by disregarding the production of secondary rays

of energy less than η_0 and introducing a continuous energy loss in their stead. Obviously we may not expect the consequences of this approximation to be correct for energies less than η_0, or between $E_0 - \eta_0$ and E_0. This is clearly shown by the fact that $\epsilon(E)$ is a decreasing function of η_0, and therefore the value of $P_0^{(\pi)}(E_0,E)$ at $E = 0$, and that of $p_0^{(\pi)}(E_0,E)$ at $E = E_0$, depend on the arbitrary choice of this energy. This dependence, however, is not a very critical one, as $\epsilon(E)$ varies only logarithmically with η_0.

5.19. ***Approximate analytical solution of the track length equation under Approximation B.*** Rewrite Eq. (5.17.8) with $\epsilon = \epsilon_0$ as follows:

$$\int_0^\infty p_0^{(\pi)}(E_0,E)\left[\mathfrak{F}^\dagger + \epsilon_0 \frac{\partial}{\partial E}\right] u(E,E_1)\,dE = u(E_0,E_1). \tag{1}$$

In the computation of the operator $\mathfrak{F}^\dagger$ neglect Compton effect as well as production of secondary electrons by collision and use the asymptotic expressions for the probabilities of radiation and pair production. Equation (1) with a function $u(E,E_1)$, chosen as discussed in § 5.17, is then an integral equation giving $p_0^{(\pi)}$ under Approximation *B*.

Consider now the equation:

$$\mathfrak{F}^\dagger u(E,E_1) = 1. \tag{2}$$

This is the equation of the integral track length under Approximation *A* [see Eq. (5.17.11)]. An approximate solution of Eq. (2) is therefore [see Eq. (5.18.3)]:

$$\begin{aligned} &u(E,E_1) = 0 \qquad \text{for} \quad E < E_1; \\ &u(E,E_1) = \frac{1}{q}\frac{E}{E_1} \qquad \text{for} \quad E > E_1 \quad \left(\frac{1}{q} = 0.437\right). \end{aligned} \tag{3}$$

This solution is probably quite accurate for values of E not too close to E_1.

In Eq. (1), $u(E,E_1)$ is a function of E that becomes zero for $E < E_1$ but is otherwise completely arbitrary. We shall use for u the function given by Eqs. (3) and assume that this function satisfies Eq. (2) exactly. Since $u(E,E_1)$ changes discontinuously from 0 to $(1/q)$ at $E = E_1$, the following equation holds:

$$\frac{\partial u(E,E_1)}{\partial E} = \frac{1}{q}\frac{1}{E_1} + \frac{1}{q}\delta(E - E_1),$$

Eq. (1) now becomes

$$\int_{E_1}^\infty p_0^{(\pi)}(E_0,E)\left[1 + \frac{\epsilon_0}{qE_1}\right] dE + \frac{\epsilon_0}{q} p_0^{(\pi)}(E_0,E_1) = \frac{1}{q}\frac{E_0}{E_1},$$

or, if one introduces the integral track length, $P_0^{(\pi)}$, and substitutes E for E_1,

$$P_0^{(\pi)}(E_0,E)\left[1 + \frac{\epsilon_0}{qE}\right] - \frac{\epsilon_0}{q}\frac{\partial P_0^{(\pi)}(E_0,E)}{\partial E} = \frac{E_0}{qE}. \tag{4}$$

The solution of this equation is

$$P_0^{(\pi)}(E_0,E) = \frac{E_0}{\epsilon_0} x e^x \int_x^{x_0} \frac{e^{-s}}{s^2}\,ds, \tag{5}$$

where: $$x = q\frac{E}{\epsilon_0}; \qquad x_0 = q\frac{E_0}{\epsilon_0}.$$

Equation (5) was first given by Tamm and Belenky (TIE39). The method of derivation presented here is due to Friedman (FFL49). Equation (5) may be rewritten in the following form:

$$P_0^{(\pi)}(E_0,E) = \frac{E_0}{\epsilon_0} e^x\left[\mathcal{E}_1(x) - \frac{x}{x_0}\mathcal{E}_1(x_0)\right], \tag{6}$$

where the function: $$\mathcal{E}_1(x) = x\int_x^\infty \frac{e^{-s}}{s^2}\,ds = \int_1^\infty \frac{e^{-xs}}{s^2}\,ds \tag{7}$$

is a special case of the so-called Gold integrals. Some of its properties and some of its numerical values are given in Appendix 5.

For $E = 0$, Eq. (6) gives:

$$P_0^{(\pi)}(E_0,0) = \frac{E_0}{\epsilon_0}\mathcal{E}_1(0) = \frac{E_0}{\epsilon_0},$$

in agreement with Eq. (5.18.4).

For $E = E_0$, Eq. (5) gives:

$$P_0^{(\pi)}(E_0,E_0) = 0$$

and: $$p_0^{(\pi)}(E_0,E_0) = -\left[\frac{\partial P_0^{(\pi)}(E_0,E)}{\partial E}\right]_{E=E_0} = \frac{1}{\epsilon_0},$$

in agreement with Eqs. (5.18.10) and (5.18.9).

From Eq. (1) it is possible to develop a method of successive approximations that determines the function $P_0^{(\pi)}(E_0,E)$ exactly, starting from the Tamm and Belenky solution. Friedman has computed, in this way, the first correction to the Tamm and Belenky formula, without changing the basic assumptions of Approximation *B*. Figure 1 shows the results of this computation for the case $x_0 = 1$ ($E_0 = 0.437\epsilon_0$). One sees that in the case under consideration, the correction to the Tamm and Belenky formula never exceeds 4.5 per cent. One may predict, and it has been checked by direct computations of Friedman, that the correction becomes even smaller at higher primary energies.

How much error is introduced in the Tamm and Belenky solution by the simplifications included in Approximation *B* can be determined only by comparison with the results of calculations in which these simplifications are not made (see next section). Here we wish only to recall that the formula developed under the limiting assumption of complete screening

grossly underestimates the probability for pair production at energies close to the critical energy in heavy elements; whereas, in the same energy range, the asymptotic expression for the radiation probability still represents an acceptable approximation.

One can easily prove that neglecting the energy dependence of pair production does not introduce any serious error in the equations for the

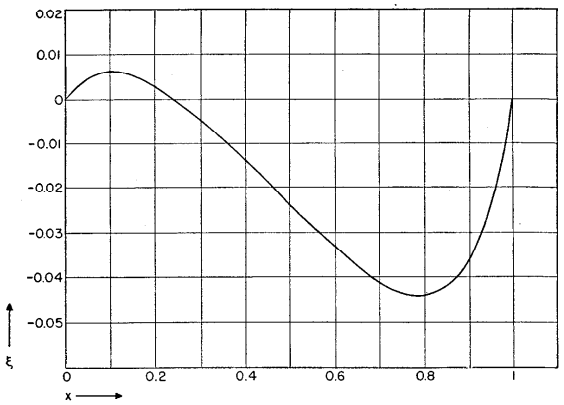

Fig. 5.19.1. The per cent correction, ξ, to the Tamm and Belenky formula, for the case $x_0 = 1$, plotted as a function of $x = qE/\epsilon_0$. To obtain the corrected value of $P_0^{(\pi)}(E_0,E)$, the expression of this quantity given by Eq. (5) should be multiplied by $(1 + \xi)$. Friedman, unpublished result.

electron track length. In fact Figs. (2.19.1) and (2.19.2) show that even in the energy range where

$$\mu_{\text{pair}}(E) = \int_0^E \varphi_{\text{pair}}(E,E')\, dE'$$

varies rapidly with E, the shape of the function $\varphi_{\text{pair}}(E,E')$ does not change very drastically. Therefore one can always take with good approximation [see Eq. (2.19.14)]:

$$\varphi_{\text{pair}}(E,E')\, dE' = \frac{\mu_{\text{pair}}(E)}{\mu_0}\, \psi_{\text{pair}}\left(\frac{E'}{E}\right) d\left(\frac{E'}{E}\right). \tag{8}$$

With this expression for φ_{pair} and with the asymptotic expression for the radiation probability one obtains again the value of $\varphi_{\pi\gamma\pi}$ given by Eq. (5.16.11). Thus the fact that μ_{pair} depends on energy does not alter the

equation for the electron track length. The physical reason for this is that when Compton effect is disregarded each photon is eventually converted into an electron pair, and that, as far as the electron track length is concerned, the place where conversion occurs is of no importance.

Of course the variation of μ_{pair} with energy does affect the *photon* track length, $g_0^{(\pi)}$. This quantity is given, in terms of the electron track length, $p_0^{(\pi)}$, by Eq. (5.16.1b), where we may take $\varphi_{\gamma\gamma} = 0$ and $\mu_\gamma(E) = \mu_{\text{pair}}(E)$:

$$\int_E^{E_0} p_0^{(\pi)}(E_0,E')\varphi_{\pi\gamma}(E',E)\, dE' = g_0^{(\pi)}(E_0,E)\mu_{\text{pair}}(E).$$

On the other hand, if we call $p_B^{(\pi)}$ and $g_B^{(\pi)}$ the quantities $p_0^{(\pi)}$ and $g_0^{(\pi)}$ computed under Approximation B,

$$\int_E^{E_0} p_B^{(\pi)}(E_0,E')\varphi_{\pi\gamma}(E',E)\, dE' = g_B^{(\pi)}(E_0,E)\mu_0.$$

Since, as explained above, $p_B^{(\pi)} = p_0^{(\pi)}$, it follows that

$$g_0^{(\pi)}(E_0,E) = g_B^{(\pi)}(E_0,E)\, \frac{\mu_0}{\mu_{\text{pair}}(E)}. \tag{9}$$

For another analytical solution of the track length equation, the reader may consult a paper by Ferretti (FB42).

5.20. ***Numerical computation of the electron track length.*** Rossi and Klapman (RB42.2) have integrated Eqs. (5.16.3) and (5.16.9) by a straightforward numerical method and have thus obtained the integral electron track lengths, $P_0^{(\pi)}(E_0,E)$ and $P_0^{(\gamma)}(E_0,E)$, for a fixed value of E, as functions of the initial energy E_0. They made computations for air, with $E = 10^7$ ev, using Eqs. (2.3.10), (2.5.1), (2.11.3), (2.11.4), (2.11.5), (2.11.6), (2.11.7), (2.18.7), (2.19.4), (2.19.5), (2.19.6), (2.19.7), and (2.19.8) to describe the elementary processes. That is, they neglected pair production and radiation processes in the field of electrons and disregarded the density effect in the collision losses of electrons. Figure 1 shows the functions $P_0^{(\pi)}(E_0,E)$ and $P_0^{(\gamma)}(E_0,E)$, in g cm^{-2}, as given by the calculations of Rossi and Klapman. It also shows, for comparison, the function $P_0^{(\pi)}(E_0,E)$ computed from the formula of Tamm and Belenky, Eq. (5.19.5), with the values $X_0 = 43$ g cm^{-2}, and $\epsilon_0 = 98$ Mev, which are consistent with the cross-sections used by Rossi and Klapman.

The agreement between the values of the function $P_0^{(\pi)}(E_0,E)$ computed by the two methods is quite satisfactory. Note, in particular, that for $E = 10^7$ ev, and for sufficiently large energies, E_0, both the method of Tamm and Belenky and that of Rossi and Klapman give the result:

$$P_0^{(\pi)}(E_0,E) = hE_0,$$

where $h = 2.96 \cdot 10^{-7}$ g cm^{-2} (ev)$^{-1}$, according to the equation of Tamm

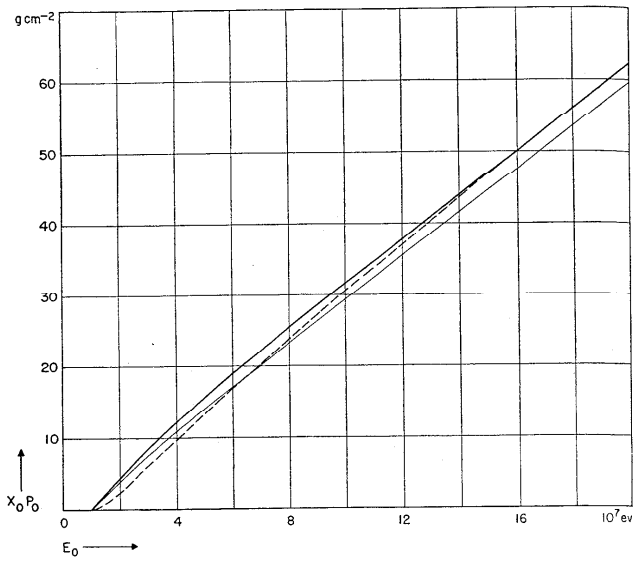

Fig. 5.20.1. The track lengths in g cm^{-2}, according to the numerical computations of Rossi and Klapman (based on approximately correct cross-sections at all energies for all important processes) and according to the formula of Tamm and Belenky (derived under Approximation *B*). Heavy line: $X_0P_0^{(\pi)}(E_0,E)$ from Rossi and Klapman. Dashed line: $X_0P_0^{(\gamma)}(E_0,E)$ from Rossi and Klapman. Thin line: $X_0P_0^{(\pi)}(E_0,E)$ from Tamm and Belenky [Eq.(5.19.6)]. The quantities above are computed for $E = 10^7$ ev and plotted as functions of E_0.

and Belenky, and $h = 3.06 \cdot 10^{-7}$ g cm^{-2} (ev)$^{-1}$, according to Rossi and Klapman. (Notice also that, under Approximation *B*,

$$P_0^{(\pi)}(E_0,0) = E_0X_0/\epsilon_0 = 4.4 \cdot 10^{-7}E_0 \text{ g cm}^{-2}.)$$

Richards and Nordheim (RJA48) have developed a different numerical procedure for the solution of the track length problem. This procedure is valid for large values of the initial energy, E_0, and it gives the electron and photon track lengths as functions of the energy, E, of the secondary particles.

Equation (5.16.1a) contains two terms representing loss and gain of particles by radiation. The sum of these terms is:

$$D(E) = -p_0^{(\pi)}(E_0,E)\int_0^E \varphi_{\text{rad}}(E,E')\,dE' + \int_E^\infty p_0^{(\pi)}(E_0,E')\varphi_{\text{rad}}(E',E'-E)\,dE'.$$

The method of Richards and Nordheim is based largely upon the assumption that one can replace $D(E)$ with a term describing a continuous energy loss:

$$D_1(E) = \frac{\partial}{\partial E}\left[\epsilon_{\text{rad}}(E)p_0^{(\pi)}(E_0,E)\right], \tag{1}$$

where:
$$\epsilon_{\text{rad}}(E) = \int_0^E E'\varphi_{\text{rad}}(E_0,E')\,dE'. \tag{2}$$

To justify this assumption, consider that, at high energies, $p_0^{(\pi)}(E_0,E)$ is inversely proportional to E^2. Therefore if one writes:

$$p_0^{(\pi)}(E_0,E) = \frac{f(E)}{E^2}$$

$f(E)$ is a slowly varying function of E. By developing this function in a Taylor's series of $E' - E$ and keeping only the first two terms, one obtains:

$$D(E) = -\int_0^E \varphi_{\text{rad}}(E,E')f(E)\frac{dE'}{E^2} + \int_E^\infty \varphi_{\text{rad}}(E',E'-E)\left[f(E) + (E'-E)\frac{df}{dE}\right]\frac{dE'}{(E')^2}.$$

At high energies, one may use for φ_{rad} the asymptotic expression for complete screening, which depends only on the fractional radiation loss [see Eq. (2.11.14)]. The substitutions: $E'/E = v$ in the first integral and $(E' - E)/E' = v$ in the second give:

$$D(E) = -\int_0^1 \psi_{\text{rad}}(v)f(E)\frac{dv}{E^2} + \int_0^1 \psi_{\text{rad}}(v)\left[f(E) + E\frac{v}{1-v}\frac{df}{dE}\right]\frac{(1-v)}{E^2}\,dv$$

$$= -\frac{f(E)}{E^2}\int_0^1 v\psi_{\text{rad}}(v)\,dv + \frac{1}{E}\frac{df}{dE}\int_0^1 v\psi_{\text{rad}}(v)\,dv$$

$$= \frac{d}{dE}\left[\frac{f(E)}{E}\int_0^1 v\psi_{\text{rad}}(v)\,dv\right]$$

$$= \frac{d}{dE}\left[\frac{f(E)}{E^2}\epsilon_{\text{rad}}(E)\right].$$

Thus Eq. (1) is justified at high energies. At low energies, on the other hand, radiation phenomena are unimportant compared with the collision loss and therefore do not need to be taken into account accurately.

The substitution of a continuous energy loss for the discontinuous radiation processes simplifies the track-length equation sufficiently to afford the possibility of a solution by a numerical iteration method.

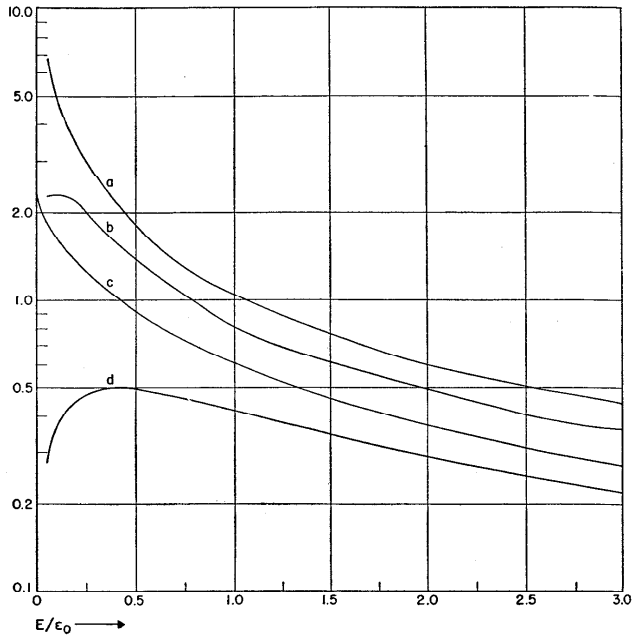

Fig. 5.20.2. The results of Richards and Nordheim for air ($E_0 3\epsilon_0$) (RJA48). The four curves represent the following functions:

a: $(\epsilon_0/0.437E_0)\, G_0^{(\pi)}(E_0,E)$.
b: $(E/0.437E_0)\, g_0^{(\pi)}(E_0,E)$.
c: $(\epsilon_0/0.437E_0)\, P_0^{(\pi)}(E_0,E)$.
d: $(E/0.437E_0)\, p_0^{(\pi)}(E_0,E)$.

The computations are based on approximately correct cross-sections for all important processes.

Richards and Nordheim used this method to compute the track lengths in air and in lead. They used correct expressions for the probabilities of Compton effect, radiation, and pair production, considering also radiation and pair production in the field of electrons. They took into account the production of secondary electrons by collision processes, using the following approximate expression for the probability of this effect:

$$\varphi_{\mathrm{col}}(E,E')\,dE' = 2CX_0 m_e c^2 \frac{dE'}{(E')^2}\left[1 - \frac{2E'}{E}\right]$$

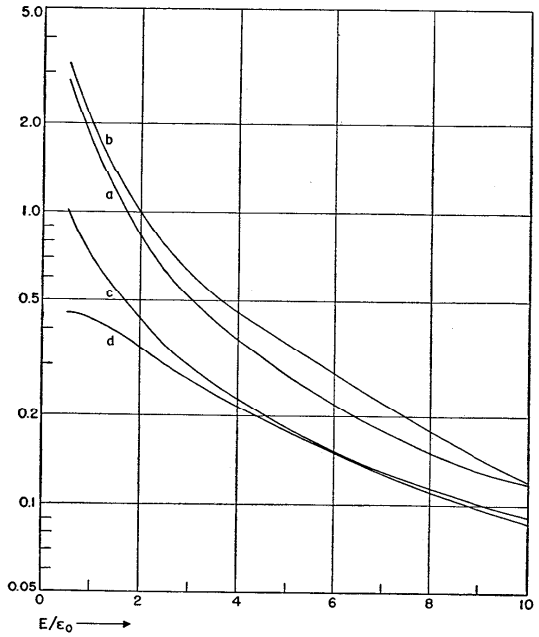

Fig. 5.20.3. The results of Richards and Nordheim for lead ($E_0 >> 10\epsilon_0$) (RJA48). The four curves represent the following functions:

a: $(\epsilon_0/0.437E_0)\, G_0^{(\pi)}(E_0,E)$.
b: $(E/0.437E_0)\, g_0^{(\pi)}(E_0,E)$.
c: $(\epsilon_0/0.437E_0)\, P_0^{(\pi)}(E_0,E)$.
d: $(E/0.437E_0)\, p_0^{(\pi)}(E_0,E)$.

The computations are based on approximately correct cross-sections for all important processes.

For the radiation length and critical energy they used $X_0 = 38$ g cm^{-2} and $\epsilon_0 = 86$ Mev in air, $X_0 = 5.9$ g cm^{-2} and $\epsilon_0 = 6.7$ Mev in lead.

Figures (2) and (3) show the results. In Fig. 4 the differential electron track length found by Richards and Nordheim for air is compared with that given by the Tamm and Belenky formula. The agreement is quite good.

Note that the computations of Richards and Nordheim refer to values of E small compared with the primary energy (indeed the method is not applicable when E is near E_0). For $E \ll E_0$, there is no appreciable difference between the track lengths of electron-initiated and of photon-initiated showers.

Before closing this section we again note that the numerical computations of Rossi and Klapman, as well as those of Richards and Nordheim, take into account Compton effect and production of secondary electrons by collision processes; whereas the analytical treatment of Tamm and Belenky disregards these phenomena. Thus the general agreement between the results obtained by the numerical and analytical methods (Figs. 1 and 4) indicates that Compton effect and collision processes have a relatively small influence upon the track length.

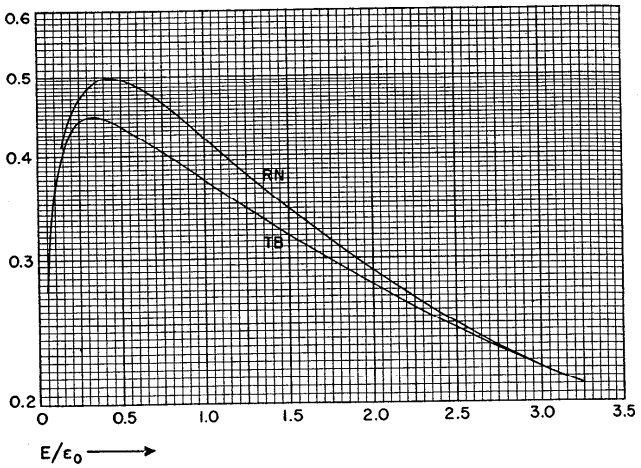

Fig. 5.20.4. Comparison between the differential electron track lengths in air computed by Richards and Nordheim (RN), and by Tamm and Belenky (TB). For both curves the ordinate represents the quantity $(E/0.437E_0)\ p_0^{(\pi)}(E_0,E)$.

5.21. ***Computation of higher moments.*** The nth moments of the shower functions are related to the zero—and to the $(n-1)$—moments by recursion formulae of the following type [see (FFL49)]:

$$p_n^{(\pi)}(E_0,E) = n\left[\int_E^{E_0} p_{n-1}^{(\pi)}(E_0,E')p_0^{(\pi)}(E',E)\,dE' + \int_E^{E_0} g_{n-1}^{(\pi)}(E_0,E')p_0^{(\gamma)}(E',E)\,dE'\right] \quad (1)$$

$$P_n^{(\pi)}(E_0,E) = n\left[\int_E^{E_0} p_{n-1}^{(\pi)}(E_0,E')P_0^{(\pi)}(E',E)\,dE' + \int_E^{E_0} g_{n-1}^{(\pi)}(E_0,E')P_0^{(\gamma)}(E',E)\,dE'\right] \quad (2)$$

$$g_n^{(\pi)}(E_0,E) = n\left[\int_E^{E_0} p_{n-1}^{(\pi)}(E_0,E')g_0^{(\pi)}(E',E)\,dE' + \int_E^{E_0} g_{n-1}^{(\pi)}(E_0,E')g_0^{(\gamma)}(E',E)\,dE'\right] \quad (3)$$

$$p_n^{(\gamma)}(E_0,E) = n\left[\int_E^{E_0} p_{n-1}^{(\gamma)}(E_0,E')p_0^{(\pi)}(E',E)\,dE' + \int_E^{E_0} g_{n-1}^{(\gamma)}(E_0,E')p_0^{(\gamma)}(E',E)\,dE'\right]. \quad (4)$$

These formulae reduce the evaluation of the higher moments to a computation of integrals, once the zero moments, or track lengths, have been determined. One should notice, however, that small errors in the expressions for the zero moments become gradually amplified as one proceeds to compute the higher moments.

We shall prove Eq. (1) as an example. Assume that an electron of energy E_0 is incident at $t = 0$ and consider the development of the shower arising from this electron at two different depths, t' and $t(t' < t)$. All of the electrons of energy E observed at the depth t arise from cascade multiplication of electrons and photons of energy greater than E that were present at the depth t'. Therefore the following identity holds:

$$\pi^{(\pi)}(E_0,E,t) = \int_E^{E_0} \pi^{(\pi)}(E_0,E',t')\pi^{(\pi)}(E',E,t-t')\,dE' + \int_E^{E_0} \gamma^{(\pi)}(E_0,E',t')\pi^{(\gamma)}(E',E,t-t')\,dE'. \quad (5)$$

Consider further the identities:

$$\begin{aligned} p_n^{(\pi)}(E_0,E) &= \int_0^\infty t^n \pi^{(\pi)}(E_0,E,t)\,dt \\ &= \int_0^\infty \pi^{(\pi)}(E_0,E,t)\,dt\; n\int_0^t (t')^{n-1}\,dt' \\ &= n\int_0^\infty (t')^{n-1}\,dt' \int_{t'}^\infty \pi^{(\pi)}(E_0,E,t)\,dt. \end{aligned} \quad (6)$$

By combining Eq. (5) with Eq. (6) one obtains:

$$p_n^{(\pi)}(E_0,E) = n\int_0^\infty (t')^{n-1}\,dt'\int_{t'}^\infty dt\left[\int_E^{E_0}\pi^{(\pi)}(E_0,E',t')\pi^{(\pi)}(E',E,t-t')\,dE' + \int_E^{E_0}\gamma^{(\pi)}(E_0,E',t')\pi^{(\gamma)}(E',E,t-t')\,dE'\right]. \quad (7)$$

For the evaluation of the integrals in Eq. (7), note that the integrations with respect to t and E can be interchanged. Thus

$$\int_0^\infty (t')^{n-1}\,dt'\int_{t'}^\infty dt\int_E^{E_0}\pi^{(\pi)}(E_0,E',t')\pi^{(\pi)}(E',E,t-t')\,dE' = \int_E^{E_0}dE'\int_0^\infty (t')^{n-1}\pi^{(\pi)}(E_0,E',t')\,dt'\int_{t'}^\infty \pi^{(\pi)}(E',E,t-t')\,dt. \quad (8)$$

The substitution:

$$t - t' = t''$$

yields:

$$\int_{t'}^\infty \pi^{(\pi)}(E',E,t-t')\,dt = p_0^{(\pi)}(E',E),$$

and therefore:

$$\int_0^\infty (t')^{n-1}\,dt'\int_{t'}^\infty dt\int_E^{E_0}\pi^{(\pi)}(E_0,E',t')\pi^{(\pi)}(E',E,t-t')\,dE' = \int_E^{E_0} p_{n-1}^{(\pi)}(E_0,E')p_0^{(\pi)}(E',E)\,dE'. \quad (9)$$

In a similar manner one obtains:

$$\int_0^\infty (t')^{n-1}\,dt'\int_{t'}^\infty dt\int_E^{E_0}\gamma^{(\pi)}(E_0,E',t')\pi^{(\gamma)}(E',E,t-t')\,dE' = \int_E^{E_0} g_{n-1}^{(\pi)}(E_0,E')p_0^{(\gamma)}(E',E)\,dE', \quad (10)$$

and one thus concludes that Eq. (7) is equivalent to Eq. (1).

If one recalls, in particular, that the zero moment of the function $\rho(E_0,t)$ representing the energy dissipation is the energy, E_0, of the primary particle, one sees that the recursion equations for the higher moments, R_n, of the function ρ are of the form:

$$R_n^{(\pi)}(E_0) = n\left[\int_0^{E_0} p_{n-1}^{(\pi)}(E_0,E')E'\,dE' + \int_0^{E_0} g_{n-1}^{(\pi)}(E_0,E')E'\,dE'\right]. \quad (11)$$

5.22. *Positions and magnitudes of the maxima of the shower functions.* Knowledge of the zero, first, and second moments of a shower function makes it possible to compute approximate values for the position and the magnitude of the corresponding maximum. On a first approximation one may assume, with Belenky (BS44.1), that the shower functions in the neighborhood of the maximum may be represented by Gaussian functions, e.g.:

$$\Pi^{(\pi)}(E_0,E,t) = \Pi_{\max}^{(\pi)}(E_0,E)e^{-(t-T)^2/2\tau^2} \quad (1)$$

where T is the abscissa of the maximum and τ is the longitudinal spread of the shower function (see § 5.1).

Under this approximation the shower function is symmetric with respect to the maximum, and therefore the abscissa of the maximum coincides with that of the center of gravity:

$$T = \bar{t}. \quad (2)$$

The integral of the shower function with respect to t, i.e., the zero moment, has the expression:

$$P_0^{(\pi)}(E_0,E) = \int_0^\infty \Pi^{(\pi)}(E_0,E,t)\,dt = \sqrt{2\pi}\,\tau\Pi_{\max}^{(\pi)}(E_0,E). \quad (3)$$

Equations (2) and (3), together with Eqs. (5.1.4) and (5.1.6), give T and $\Pi_{\max}$ in terms of P_0, P_1, and P_2. If one uses for $\bar{t}$ and τ the expressions obtained from Approximation B (see § 5.13) one arrives at the following result for the case of electron-initiated showers:

$$T_\Pi^{(\pi)}(E_0,0) = 1.01\ln\left(\frac{E_0}{\epsilon_0}\right) + 0.4; \quad (4)$$

$$\Pi_{\max}^{(\pi)}(E_0,0) = \frac{0.31}{\sqrt{\ln\left(\frac{E_0}{\epsilon_0}\right) - 0.1}}\frac{E_0}{\epsilon_0}. \quad (5)$$

These expressions may be compared with the more accurate expressions of $T_\Pi^{(\pi)}$ and $\Pi_{\max}^{(\pi)}$ given in § 5.13.

Belenky followed the method outlined above to compute the position and the magnitude of the maximum of the function $\Pi^{(\pi)}(E_0,0,t)$ for lead. He used the formula of Tamm and Belenky, Eq. (5.19.5), as the expression for the electron track length, $P_0^{(\pi)}$; computed the photon track length, $g_0^{(\pi)}$, by means of Eq. (5.16.1b) with $\varphi_{\gamma\gamma} = 0$, $\mu_\gamma(E') = \mu_0$, and $\varphi_{\text{rad}}dE' = \psi_{\text{rad}}dv$; corrected the result by means of Eq. (5.19.9) to take into account the variation of μ_{pair} with energy and then used the recurrence equations (§ 5.21) to determine the higher moments. With certain mathematical approximations, he obtained the results expressed by the following equations:

$$T_\pi^{(\pi)}(E_0,0) = Q_1\left(\frac{E_0}{\epsilon_0}\right)\ln\left(\frac{E_0}{\epsilon_0}\right), \quad (6)$$

$$\Pi_{\max}^{(\pi)}(E_0,0) = \frac{Q_2\,(E_0/\epsilon_0)}{\sqrt{\ln\,(E_0/\epsilon_0)}}\frac{E_0}{\epsilon_0}, \quad (7)$$

where Q_1 and Q_2 are slowly varying functions of (E_0/ϵ_0), some values of which are listed in Table 1.

Table 5.22.1. Values of the functions Q_1 and Q_2 in Eqs. (6) and (7)
[from Belenky (BS44.1)]

E_0	$5 \cdot 10^8$ ev	10^9 ev	10^{10} ev
E_0/ϵ_0*	71.5	143	1430
Q_1	1.4	1.33	1.23
Q_2	0.172	0.18	0.20

* Computed for $\epsilon_0 = 7$ Mev.

The same method, without the correction for the variation of μ_{pair} with E, yields the following equations:

$$T_\Pi^{(\pi)}(E_0,0) = 0.96 \ln\left(\frac{E_0}{\epsilon_0}\right); \tag{8}$$

$$\Pi_{\max}^{(\pi)}(E_0,0) = \frac{0.3}{\sqrt{\ln (E_0/\epsilon_0)}} \frac{E_0}{\epsilon_0}. \tag{9}$$

The differences between Eqs. (5.13.11), (5.13.12), Eqs. (4), (5), and Eqs. (8), (9) (all of which are based on Approximation *B*) correspond to the different mathematical approximations made in the computations.

In Figs. 1 and 2, the quantities $T_\Pi^{(\pi)}$ and of $\Pi_{\max}^{(\pi)}$ computed from Eqs. (6) and (7) are plotted against E_0/ϵ_0.

Belenky (BS44.2) has also investigated the influence of scattering on the position and magnitude of the maximum of the function $\Pi^{(\pi)}(E_0,0,t)$ in lead. For an initial energy $E_0 = 5 \cdot 10^8$ ev, he found that scattering decreases the value of T by 6 per cent and increases the value of $\Pi_{\max}$ by 4 per cent. Thus the large scattering of low-energy electrons in lead seems to affect the magnitude and the position of the shower maximum only slightly. According to Belenky, this is due to the fact that T and $\Pi_{\max}$ are mainly determined by the behavior of the high-energy particles in the shower. Presumably, however, scattering has an important effect on the "tail" of the shower curve.

It may be appropriate to recall here that the quantity that retains physical significance in spite of the occurrence of scattering is not the function $\Pi(E_0,0,t)$ itself, but rather the function $\rho(E_0,t) = \epsilon_0\Pi(E_0,0,t)$, which represents the energy dissipation of all shower particles in a layer of unit thickness at the depth t. Note that when scattering is important (as it is in the case of lead) many of the low-energy particles travel at wide angles to the axis of the shower and even in backward directions (see Fig. 5.1.1). Strictly speaking, the quantity $\rho(E_0,t)$, computed as discussed above,

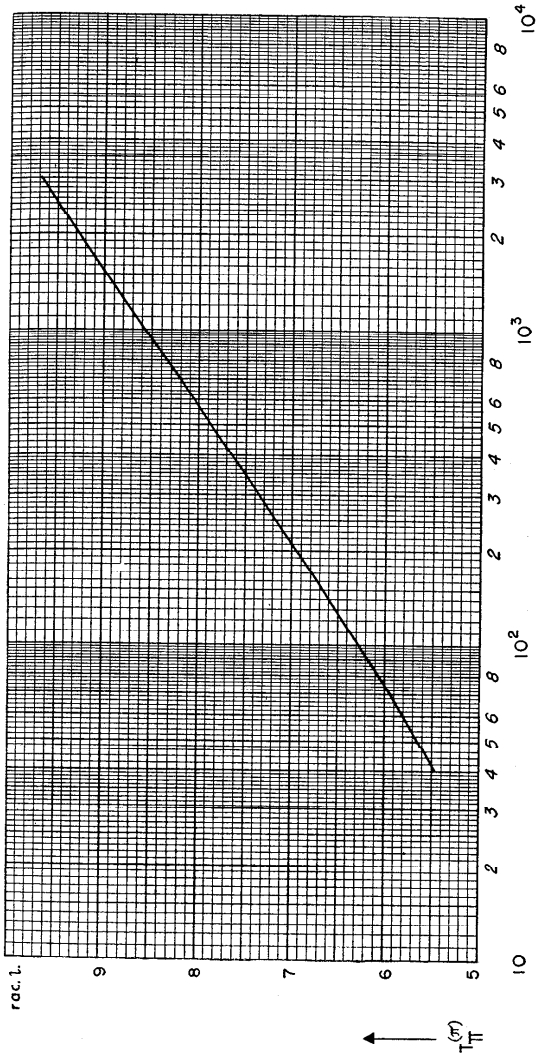

Fig. 5.22.1. $T_\Pi^{(\pi)}(E_0/\epsilon_0)$ according to Eq. (6).

gives the energy dissipation in a layer of unit thickness perpendicular to the shower axis for the case of a shower that develops in an infinite medium. In practice the theoretical results apply to regions sufficiently far from the boundary of the absorber that the contribution of scattering is not affected appreciably by escape of particles at the boundary. The effect of scattering, of course, increases strongly with increasing Z. For example,

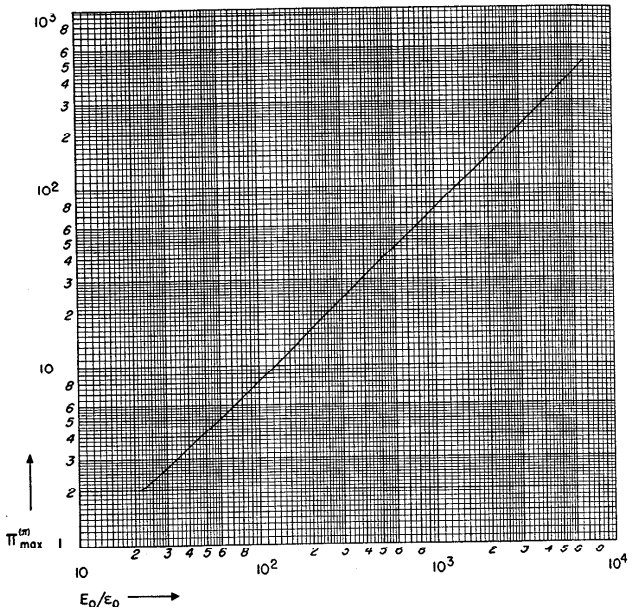

Fig. 5.22.2. $\Pi^{(\pi)}_{max}(E_0/\epsilon_0)$, according to Eq. (7).

within a lead slab the number of electrons traversing a given layer in the forward direction is probably not much greater than the number of those scattered back by the material beyond this layer. Therefore the number of shower electrons emerging from a slab of finite thickness may be close to one-half the computed value.

5.23. ***The fluctuation problem.*** The theoretical results that we have described so far concern the average behavior of showers. The study of the fluctuations from the average behavior represents a much more difficult mathematical problem. As already mentioned in § 5.1, this problem has not yet received a satisfactory solution. In this volume, therefore, we shall limit ourselves to a few remarks on the subject of fluctuations.

According to our definition, $\Pi(E_0,0,t)$ represents the *average* number of electrons of all energies at the depth t in a shower initiated by an electron or photon of energy E_0. If these electrons were genetically independent, one could use Poisson's law (Appendix 3) to compute the probability, $W(N,t)$, that the actual number of electrons in a given shower is N. One would thus obtain:

$$W(N,t) = \frac{e^{-\Pi}}{N!}\Pi^N. \tag{1}$$

The corresponding mean square deviation from the average is (Appendix 3):

$$\langle(N-\Pi)^2\rangle_{av} = \sum_0^\infty (N-\Pi)^2 W(N) = \langle N^2\rangle_{av} - \Pi^2 = \Pi. \tag{2}$$

The particles of a shower, however, are not genetically independent. From qualitative arguments one can easily see that this fact must *increase* the statistical fluctuations in the number of particles observed at a given depth. Consider, for example, a shower initiated by a photon of energy E_0. On the average this photon undergoes materialization at a distance $\langle t\rangle_{av} = 1/\mu_0 = 1.3$ radiation lengths from its point of incidence. However the probability that the materialization process occurs, say, at a distance greater than 3 times $\langle t\rangle_{av}$ is $\exp(-3) \approx 1/20$. Thus in one case in 20 the number of electrons at $3\langle t\rangle_{av} = 3.9$ radiation lengths is zero irrespective of the initial photon energy. On the other hand, the average shower produced in air by a photon of 10^{10} ev, for example, contains about 18 electrons at $t = 3.9$ (see § 5.13). According to the Poisson formula, Eq. (1), the probability that no electrons are present at this position is of the order of $\exp(-18) \sim 10^{-8}$.

Clearly the reason why, in this case, Poisson's formula grossly underestimates the fluctuations is the fact that the probability for any one electron to be present at the thickness t depends on the behavior of the primary photon in the first t units of thickness. In general, one may say that any deviation from the average behavior occurring in the early development of the shower reflects itself upon the successive generation, amplified by the multiplication process.

Furry (FWH37) determined the function $W(N,t)$ for a drastically simplified model of the cascade process. The Furry model ignores the distinction between electrons and photons, neglects ionization loss, and assumes that each shower particle has a probability dt of splitting into two particles on traversing a thickness dt of matter. According to these assumptions, the probabilities that one, two, etc., particles emerge from

a layer of thickness t when one particle is incident upon this layer satisfy the following differential equations:

$$\begin{aligned}
\frac{dW(1,t)}{dt} &= -W(1,t), \\
\frac{dW(2,t)}{dt} &= -2W(2,t) + W(1,t), \\
\frac{dW(3,t)}{dt} &= -3W(3,t) + 2W(2,t), \\
&\cdots\cdots\cdots\cdots\cdots \\
\frac{dW(n,t)}{dt} &= -nW(n,t) + (n-1)W(n-1,t).
\end{aligned} \tag{3}$$

The third equation, for example, says that as the absorber thickness changes from t to $t + dt$, the probability for 3 and only 3 particles to emerge from the absorber changes because of the two following effects: (a) if there are 3 particles at t, any one of them may split into 2 particles on traversing dt, in which case there will be 4 particles at $t + dt$; (b) if there are only 2 particles at t, either of them may split into 2 particles in dt, in which case there will be 3 particles at $t + dt$. The probability that two or more particles interact in dt is a differential of the second order and is therefore negligible.

The boundary conditions are:

$$\begin{aligned}
W(1,0) &= 1, \\
W(n,0) &= 0 \qquad \text{for} \quad n \neq 1.
\end{aligned} \tag{4}$$

By successive integration and application of the boundary conditions one obtains:

$$\begin{aligned}
W(1,t) &= e^{-t}, \\
W(2,t) &= e^{-t}(1 - e^{-t}), \\
\text{and in general:} \qquad W(N,t) &= e^{-t}(1 - e^{-t})^{N-1}.
\end{aligned} \tag{5}$$

One sees from Eq. (5) that the *most probable* number of particles emerging from an absorber of any thickness is one.

For the *average* number of particles,

$$\Pi(t) = W(1,t) + 2W(2,t) + \ldots + nW(n,t) + \ldots \tag{6}$$

Eqs. (3) yield the following equation:

$$\frac{d\Pi(t)}{dt} = \Pi(t), \tag{7}$$

from which it follows:

$$\Pi = e^t. \tag{8}$$

Thus the average number of particles increases exponentially with thickness, which is a direct consequence of neglecting ionization loss.

With the expression (8) for the average number of particles, Eq. (5) becomes:

$$W(N,t) = \frac{1}{\Pi(t)}\left[1 - \frac{1}{\Pi(t)}\right]^{N-1}. \tag{9}$$

In this form, the expression of W gives the probability for N shower particles to be present at a place where the average number of shower particles is Π.

The two curves in Fig. 1 show the Furry distribution and the Poisson distribution for $\Pi = 5$. One sees that the Furry model gives much greater fluctuations than the Poisson formula. Indeed one can show that Eq. (9) yields the following value for the mean square fluctuation:

$$\langle (N - \Pi)^2 \rangle_{av} = \Pi(\Pi - 1). \tag{10}$$

The most serious source of error in Furry's calculations arises from the neglect of ionization loss. The effect of ionization loss becomes increasingly noticeable with increasing thickness. Indeed, the ionization loss causes the average number of particles to go through a maximum and

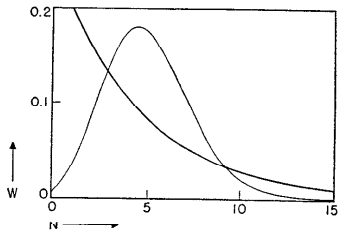

Fig. 5.23.1. The Furry distribution (heavy line), and the Poisson distribution (thin line) for $\Pi = 5$.

then to decrease again, instead of growing continuously as indicated by Eq. (8). Therefore we may expect Eq. (9) to be approximately valid for small thicknesses but not for large thicknesses.

The inclusion of ionization loss introduces very serious mathematical difficulties even if one uses the most drastic simplifications to describe the ionization loss and the multiplication process.

Nordsieck, Lamb, and Uhlenbeck (NA40) and Scott and Uhlenbeck (SWT42), for example, maintained the basic hypotheses of the Furry model; however they added the condition that particles with energy less

than a certain value, ϵ_1, are immediately stopped by ionization loss and therefore do not contribute to the further propagation of the shower.* As in the original Furry model, they neglected ionization loss of particles with energy greater than ϵ_1.

They were not able to compute the distribution function $W(N,t)$ but did compute some values for the mean square fluctuation. The results are summarized in Table 1. The quantity $\langle(N - \Pi)^2\rangle_{av}/\Pi$ listed in this

Table 5.23.1. The average number of particles, Π, and the relative mean square fluctuation $\langle(N - \Pi)^2\rangle_{av}/\Pi$ [as computed by Scott and Uhlenbeck (SWT42) by means of the modified Furry model. The computations are made for $\ln(E_0/\epsilon_1) = 4.75$, where E_0 is the primary energy and ϵ_1 the cut-off energy].

t (radiation lengths)	Π	$\langle(N - \Pi)^2\rangle_{av}/\Pi$
2.20	5.57	2.33
3.95	11.93	3.25
5.67	17.9	2.60
7.50	21.15	1.57
8.48	21.25	1.16
9.47	20.4	1.14
10.54	18.5	1.34
12.8	13.3	2.00
15.2	7.96	3.01
17.8	3.97	3.26
19.1	2.65	3.19

table represents the relative mean square fluctuations. This quantity is one according to the Poisson formula and $\Pi - 1$ according to Furry's formula. One sees from Table 1 that its actual value lies mostly between these two extremes. Scott and Uhlenbeck also investigated the fluctuation problem under Approximation A, again taking the ionization loss into account by the cut-off method. They carried out the computations only for one primary energy and one thickness and obtained the following results:

$$\ln(E_0/\epsilon_1) = 5.67; \quad t = 5.67; \quad \Pi = 16.12;$$
$$\langle(N - \Pi)^2\rangle_{av}/\Pi = 3.$$

* The cut-off energy, ϵ_1, is related to the critical energy, ϵ_0, but is not identical to it. One may define the critical energy with the relation, Eq. (5.18.4):

$$\epsilon_0 \int_0^\infty \Pi(E_0,\epsilon_1,t)\, dt = E_0.$$

One can then show that, for the model considered here, $\epsilon_1 \approx 2\epsilon_0$, whereas, for the shower problem under Approximation A, the cut-off energy that yields the correct energy dissipation is $\epsilon_1 = 0.437\epsilon_0$.

A further contribution to the fluctuation problem may be found in a book by Arley.* Arley assumed that the distribution function has a form intermediate between the Poisson and the Furry distributions (Polya distribution) and, with other drastic simplifications, obtained numerical values for this function that are tabulated in his volume. Janossy (JL50) has recently presented some interesting considerations of a general character on the same problem.

R. R. Wilson made a different approach to the fluctuation problem.† This approach has been described as the "Monte Carlo method" from the name of the well-known gambling place on the French Riviera.‡

To apply the Monte Carlo method to the shower problem, one begins by subdividing the thickness of the material where the shower propagates into a large number of intervals, each small compared with one radiation length. One then follows the primary particle and all of its secondary products through successive intervals and determines the fate of any given particle in any given interval by spinning a "wheel of chance." This wheel of chance is a cylinder carrying a set of graphs. The cylinder is

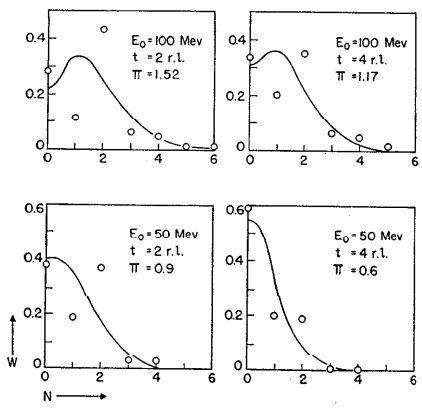

Fig. 5.23.2. The probability W of finding N electrons after two or four radiation lengths of lead, in a shower initiated by a photon of 100 or 50 Mev energy. The circles represent results of computations by the Monte Carlo method. The curves represent the corresponding Poisson distributions. (Private communication by R. R. Wilson.)

* N. Arley, *Stochastic Processes*, G. E. C. Gads Forlag, Copenhagen (1947).

† The writer is greatly indebted to Dr. Wilson for kindly making available the unpublished results of his calculations.

‡ The "Monte Carlo" method was originally suggested by Ulam and von Neumann (US47). It was first used by Goldberger (GML48) to investigate nuclear disintegrations induced by high-energy particles, see § 7.9.

spun rapidly and then stopped at random. When the cylinder has come to rest, the position of a stationary index on the map drawn upon the cylinder decides the fate of the particle in question. For example, if this particle is a photon the index will show whether the photon undergoes a process of materialization, a Compton scattering, or no interaction whatsoever. If, for example, the photon undergoes materialization, the index will also determine the energies of the two electrons thereby produced. A similar procedure determines the radiation processes of electrons, and their ionization loss is then taken into account as a constant loss per unit thickness.

Figure 2 shows some of the preliminary results obtained with the method outlined above. The four graphs in this figure give the probabilities of observing various numbers of electrons at 2 radiation lengths and at 4 radiation lengths of lead in showers initiated by photons with 100 and 50 Mev of energy, respectively. The circles represent the results of the Monte Carlo method, applied to one hundred 50-Mev photon showers and to eighty-five 100-Mev photon showers. The curves show the Poisson distribution, adjusted to the average number of electrons, Π, obtained from the Monte Carlo method. One sees that in each graph the Monte Carlo points lie alternately above and below the Poisson curve. In particular, the point at $N = 1$ is very low and the point at $N = 2$ is very high. This result is a direct consequence of the fact that electrons always arise in pairs.

The Monte Carlo method appears to be valuable not only for the solution of the fluctuation problem, but also for the investigation of the average behavior of showers, when the initiating electrons or photons have energies not much greater than the critical energy. This is also the case where the analytical method is least reliable.

5.24. *Summary and tables.* For the convenience of the reader we list here some of the most useful results of the shower theory.

(a) *Radiation lengths and critical energies* for various substances are given in Table 1.

(b) *Stationary solutions.* For $E \gg \epsilon_0$, the stationary spectra, i.e., the quantities $F_\pi(E)$, $F_\gamma(E)$, and $F_\Pi(E)$ defined in 5.6, are represented by power functions of E_0/E. Under Approximation A, they are given by Eqs. (5.7.9), with the values of a and b shown by Eqs. (5.7.14). Numerical values of the functions appearing in Eqs. (5.7.14) are listed in Table 2.

For $E \ll \epsilon_0$, $F_\pi(E)$, $F_\gamma(E)$, and $F_\Pi(E)$ are given, under Approximation B, by Eqs. (5.13.5) and (5.13.6). Approximate expressions for the stationary spectra, valid when E is greater than about twice the critical energy, will be found in the review article by Rossi and Greisen (RB41.1).

(c) *Solutions for a single incident electron or photon of energy E_0.* For $E \gg \epsilon_0$ the functions π, γ, and Π are given, under Approximation A, by Eqs. (5.10.2) to (5.10.7). Mathematical approximations limit the validity

Table 5.24.1. Values for the radiation length, X_0, and the critical energy, ϵ_0, for various substances

Substance	Z	A	X_0 (g cm^{-2}), Eq. (5.2.1)	ϵ_0 (Mev) Without density effect	ϵ_0 (Mev) With density effect
Carbon	6	12	44.6	102	76
Nitrogen	7	14	39.4	88.7	
Oxygen	8	16	35.3	77.7	
Aluminum	13	27	24.5	48.8	
Argon	18	39.9	19.8	35.2	
Iron	26	55.84	14.1	24.3	21
Copper	29	63.57	13.1	21.8	
Lead	82	207.2	6.5	7.8	7.6
Air	7.37	14.78	37.7	84.2	
Water	7.23	14.3	37.1	83.8	65

of these equations to cases where $E \ll E_0$ and $t > 1$. Numerical values of the functions entering in Eqs. (5.10.2) to (5.10.7) are listed in Table 3. Eq. (5.11.1) is a simpler but less accurate expression for Π. For small values of t, and E near E_0, one may compute π, γ, and Π by the method of successive collisions (§ 5.4), by a development in series of t (§ 5.11), or by the Monte Carlo method (§ 5.23).

For $E_0 \gg \epsilon_0$, the total number of electrons [i.e., the function $\Pi(E_0,0,t)$ is given, under Approximation B, by Eq. (5.13.10) or by the computations of Snyder (§ 5.14). Again under Approximation B, the function $\Pi(E_0,E,t)$ may be computed with the method of Bhabha and Chakrabarty; the accuracy is good when E is not much smaller than ϵ_0 (§ 5.14).

The work of Bernstein (§ 5.15) gives $\Pi(E_0,0,t)$ computed with the correct values of the cross-sections for pair production and radiation.

(d) *Position and magnitude of the maximum for the various shower functions.* For $E_0 \gg E \gg \epsilon_0$, $\pi_{max}(E_0,E)$ and $\gamma_{max}(E_0,E)$ are approximately proportional to E_0/E^2, $\Pi_{max}(E_0,E)$ is approximately proportional to E_0/E. Equations (5.10.12) and (5.10.13) are the expressions for the above quantities under Approximation A. The corresponding optimum thicknesses, T_π, T_γ, and T_Π, are approximately proportional to $\ln(E_0/E)$ and are given by Eq. (5.10.11).

For $E_0 \gg \epsilon_0$, the maximum of the function representing the total number of electrons, $\Pi_{max}(E_0,0)$, is approximately proportional to E_0/ϵ_0 and is given, under Approximation B, by Eqs. (5.13.12) and (5.13.18). The corresponding optimum thickness, T_Π, is approximately proportional to $\ln(E_0/\epsilon_0)$ and is given by Eqs. (5.13.11) and (5.13.17).

Table 5.24.2. Numerical values of some functions that enter in the theory of cascade showers under Approximation *A*. [From Rossi and Greisen (RB41.1).]

s	$A(s)$	$B(s)$	$C(s)$	$\lambda_2(s)$	$\lambda_1(s)$	$\lambda_1'(s)$	$\lambda_1''(s)$
0.0	0.0000	1.546	∞	$-\infty$	$+\infty$	$-\infty$	$+\infty$
0.1	0.1520	1.400	12.842	−4.715	+3.789	−25.005	—
0.2	0.2863	1.280	6.123	−3.330	2.270	−9.488	+75
0.3	0.4007	1.180	3.923	−2.749	1.569	5.415	1 26
0.4	0.5152	1.095	2.846	−2.415	1.127	−3.654	12.5
0.5	0.6146	1.022	2.214	−2.201	0.813	−2.693	7.6
0.6	0.706	0.959	1.802	−2.055	0.576	−2.093	4.95
0.7	0.791	0.905	1.513	−1.953	0.389	−1.685	3.50
0.8	0.870	0.855	1.3014	−1.878	0.235	−1.389	2.55
0.9	0.943	0.812	1.1400	−1.824	0.108	−1.1660	1.97
1.0	1.0135	0.7733	1.0135	−1.787	0.000	−0.9908	1.563
1.1	1.078	0.7383	0.9112	−1.760	−0.092	−0.8501	1.275
1.2	1.142	0.7065	0.8276	−1.744	−0.171	−0.7333	1.060
1.3	1.200	0.6778	0.7580	−1.734	−0.239	−0.6362	0.893
1.4	1.257	0.6514	0.6988	−1.732	−0.298	−0.5531	0.764
1.5	1.311	0.6272	0.6484	−1.734	−0.350	−0.4825	0.655
1.6	1.363	0.6049	0.6047	−1.741	−0.395	−0.4214	0.565
1.7	1.412	0.5842	0.5666	−1.751	−0.435	−0.3691	0.487
1.8	1.460	0.5650	0.5329	−1.762	−0.470	−0.3238	0.423
1.9	1.506	0.5473	0.5032	−1.780	−0.500	−0.2841	0.370
2.0	1.550	0.5306	0.4767	−1.797	−0.526	−0.2498	0.320
2.1	1.592	0.5148	0.4528	−1.816	−0.550	−0.2202	0.277
2.2	1.634	0.5004	0.4313	−1.837	−0.570	−0.1943	0.241
2.3	1.674	0.4866	0.4117	−1.859	−0.589	−0.1719	0.210
2.4	1.713	0.4736	0.3940	−1.882	−0.605	−0.1523	0.182
2.5	1.750	0.4614	0.3776	−1.904	−0.619	−0.1354	0.159
2.6	1.787	0.4499	0.3627	−1.928	−0.632	−0.1205	0.138
2.7	1.821	0.4389	0.3489	−1.951	−0.643	−0.1077	0.120
2.8	1.857	0.4285	0.3362	−1.977	−0.654	−0.0964	0.107
2.9	1.892	0.4186	0.3243	−2.003	−0.663	−0.0863	0.093
3.0	1.923	0.4093	0.3134	−2.026	−0.671	−0.0777	0.080
4.0	2.211	0.3352	0.2347	−2.264	−0.720	−0.0307	—
5.0	2.448	0.2847	0.1882	−2.480	−0.742	−0.0146	—
6.0	2.648	0.2479	0.1574	−2.669	−0.752	−0.0080	—
7.0	2.822	0.2198	0.1354	−2.837	−0.759	−0.0048	—
8.0	2.977	0.1975	0.1189	−2.988	−0.763	−0.0031	—
9.0	3.115	0.1794	0.1060	−3.123	−0.765	−0.0021	—
10.0	3.239	0.1644	0.0957	−3.246	−0.766	−0.0015	—

Equations (5.22.7) and (5.22.6) give $\Pi_{\max}(E_0,0)$ and the corresponding optimum thickness *in lead*, computed with the correct values for the absorption coefficient of γ-rays near critical energy.

Table 5.24.3. Numerical values of some functions entering in the expressions for the shower functions under Approximation *A*, Eqs. (5.10.2) to (5.10.7). [From Rossi and Greisen (RB41.1).]

s	$H_\pi^{(\pi)}(s)$	$H_\gamma^{(\gamma)}(s)$	$H_\gamma^{(\pi)}(s)$	$H_\pi^{(\gamma)}(s)$
0.0	0.500	0.500	0.469	0.533
0.1	0.537	0.463	0.478	0.521
0.2	0.543	0.457	0.489	0.507
0.3	0.542	0.458	0.498	0.499
0.4	0.536	0.464	0.508	0.489
0.5	0.526	0.474	0.520	0.480
0.6	0.513	0.487	0.531	0.471
0.7	0.496	0.504	0.541	0.463
0.8	0.477	0.523	0.551	0.453
0.9	0.456	0.544	0.560	0.443
1.0	0.433	0.567	0.567	0.433
1.1	0.408	0.592	0.573	0.422
1.2	0.383	0.617	0.576	0.410
1.3	0.357	0.643	0.578	0.397
1.4	0.331	0.669	0.577	0.384
1.5	0.306	0.694	0.574	0.370
1.6	0.281	0.719	0.568	0.355
1.7	0.257	0.743	0.561	0.340
1.8	0.235	0.765	0.554	0.325
1.9	0.213	0.787	0.542	0.310
2.0	0.194	0.806	0.530	0.295
2.1	0.176	0.824	0.518	0.280
2.2	0.160	0.840	0.505	0.266
2.3	0.145	0.855	0.492	0.252
2.4	0.132	0.868	0.478	0.240
2.5	0.120	0.880	0.465	0.227
2.6	0.109	0.891	0.451	0.215
2.7	0.099	0.901	0.438	0.204
2.8	0.090	0.910	0.425	0.193
2.9	0.082	0.918	0.412	0.183
3.0	0.075	0.925	0.401	0.173
4.0	0.034	0.966	0.304	0.108
5.0	0.018	0.982	0.242	0.073

(e) *Track lengths.* For $E_0 \gg E \gg \epsilon_0$ the differential electron and photon track lengths, $p_0(E_0,E)$ and $g_0(E_0,E)$, are proportional to E_0/E^2, and the corresponding integral track lengths, $P_0(E_0,E)$ and $G_0(E_0,E)$, are proportional to E_0/E. Equations (5.9.1), (5.9.2), and (5.9.3) are the analytic expressions of p_0, g_0, and P_0 under Approximation *A*.

Under Approximation *B*, $P_0^{(\pi)}(E_0,E)$ is given for all values of E_0 and E by the formula of Tamm and Belenky, Eq. (5.19.5). The mathematical