Undergraduate experiment on noise thermometry[a)]

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An absolute temperature scale noise thermometer is described and Boltzmann's constant is measured. The apparatus involves an easily constructed temperature probe and standard electronic instruments. The design considerations for the experiment are a useful introduction to low noise electronics.

I. INTRODUCTION

Stochastic phenomena have been matters of research interest for many decades, and are of central importance in many branches of physics. Because of the relative sophistication of the concepts, there are few experiments accessible to undergraduate students which provide an introduction to this fascinating field. One exception is electrical noise, both wanted and unwanted, which is encountered by nearly everyone who makes physical measurements. Our research program in turbulence required us to learn more about noise and its characteristics and we were interested to note that in spite of enormous literature, there are few compact and practical accounts of electrical noise designed for science students. We set out to design an advanced undergraduate laboratory experiment which would help one learn about electrical noise, its measurement, and how to read noise characteristics specified in instruction manuals of measuring instruments. We were fortunate to have three undergraduates who were willing to try these experiments and to assess their usefulness as actual lab assignments.

In considering various experiments, we were attracted by the subject of thermal noise in a resistor. The derivation of the Nyquist formula is within the reach of an undergraduate (see, for example, Kittel[1] and Reif[2]); the experiment measures the fundamental energy kT, where k is Boltzmann's constant and T the absolute temperature; and the specification of either k or T determines the other quantity in an absolute way, free of adjustable parameters. There is considerable literature describing noise devices as thermometers,[3-7] and with Josephson junctions at hand it is possible to do noise thermometry at ultralow temperatures.[8] This recent research interest is another motivation to have a laboratory experiment in thermal noise. While such a lab is not new,[9] the one we describe requires little construction, makes use of commercially available electronic equipment, and gives good results: Boltzmann's constant can be found to better than 2%. The students are also introduced to some of the difficulties in making sensitive electrical measurements.

The plan of this paper is as follows. We describe Johnson noise and the method of observing it with an amplifier in Sec. II. In Sec. III we discuss the considerations involved in making a practical measurement with available equipment. The apparatus and experimental method is discussed in Sec. IV, and typical results in Sec. V. Some alternate experiments using a white-noise generator are described in Sec. VI. This also forms an introduction to those useful devices.

II. JOHNSON NOISE

The discovery of electrical noise in resistors was reported in 1928 in two notable papers published together in the Physical Review by Johnson,[10] who did the experiments, and Nyquist,[11] who did the theory. They showed that the mean square voltage $\langle V^2 \rangle$ thermally generated in a resistance R at frequency f is given by

$$d\langle V^2 \rangle = 4kTRdf. \tag{1}$$

The papers demonstrated that the noise generated is independent of the material of the resistor and that it is independent of frequency, or "white." Using Eq. (1), Johnson was able to determine k to within 8% of the accepted value. Looked at another way, Eq. (1) shows that the maximum noise power generated per unit frequency interval to a matched load will be $d\langle V^2/4R \rangle/df = kT$. The factor of 4 arises because the power delivered to a load R_L is $\langle I^2 \rangle R_L = \langle V^2 \rangle R_L/(R + R_L)^2$ which is a maximum when $R = R_L$.

The voltage from Eq. (1) needs amplification to be observed. When amplified by a bandpass amplifier, the total mean square voltage of the output is

$$\langle V_0^2 \rangle = \int_0^\infty 4kTR|G(f)|^2\, df, \tag{2}$$

where $|G(f)|$ is the magnitude of the voltage gain, e.g., $|G(f)| = [1 + (2\pi f/RC)]^{-1/2}$ for a simple RC lowpass filter.

Equation (2) can be made to look like Eq. (1) by the introduction of the "equivalent noise bandwidth" (ENBW):

$$\Delta f = \int_0^\infty \frac{|G(f)|^2}{G_0^2}\, df, \tag{3}$$

where G_0 is the voltage gain at some reference frequency f_0. Clearly only the product $G_0^2\Delta f$ is physically significant, so the choice of f_0 is not important. By convention, f_0 is usually chosen to be either the central frequency of the amplifier's pass band, or the frequency at which G is a maximum. Thus we may write the input voltage to the amplifier as

$$\langle V_i^2 \rangle = \langle V_0^2 \rangle / G_0^2 = 4kTR\Delta f. \tag{4}$$

The bandwidth of an amplifier is usually specified as being the frequency range over which $|G(f)|^2 \geq 1/2$ of the maximum value of $|G|^2$. In general this bandwidth is not the same as Δf, e.g., for a simple lowpass filter $\Delta f = (\pi/2)$ (bandwidth).

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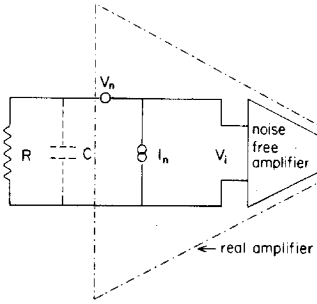


Fig. 1. Noise equivalent circuit for an amplifier with a source resistance and (shown in dashed lines) a shunting capacitance.

Noise thermometry is based on Eq. (4), which may be rewritten in a form to suggest the measurement of kT:

$$kT = \langle V_0^2 \rangle / (4RG_0^2 \Delta f). \qquad (4a)$$

III. PRACTICAL CONSIDERATIONS

In practice, noise thermometry is not as easy as suggested in the previous section. A classical discussion of a noise thermometer has been given by Garrison and Lawson.[3] Current discussions have been given by Kamper[5] and by Actis, Cibrario, and Crovini.[6] Our discussion is tailored to the aims discussed in the Introduction. One of the principle difficulties is the presence of extraneous noise sources, as they add a background noise level to the Johnson noise which must be eliminated or independently measured. The most important of these noise sources are V_n (voltage noise in the amplifier), I_n (current noise in the amplifier), V_x (excess noise in the resistor), and V_P (electrical interference, or "pick-up"). These noise sources can be considered as uncorrelated and hence statistically independent. Thus the mean squares of the voltages add and the total noise V_T in the system will be of the form

$$\langle V_T^2 \rangle = a_1 \langle V_R^2 \rangle + a_2 \langle V_n^2 \rangle + a_3 \langle R^2 I_n^2 \rangle + a_4 \langle V_x^2 \rangle + a_5 \langle V_P^2 \rangle, \qquad (5)$$

where the a_n are constants and V_R is now the Johnson noise.

A. Amplifier noise (Refs. 12–14)

The amount of noise generated by an amplifier is dependent on its design. The usual procedure when discussing amplifier noise is to pretend that all the noise is generated at the input of the amplifier. The amplifier is assumed to be noise-free and the noise is assumed to be generated by fictitious voltage and current sources at the amplifier's input. This is shown schematically in Fig. 1. Such a circuit is called a noise equivalent circuit. The "noise-free" amplifier in this circuit sees an input voltage of

$$\langle V_i^2 \rangle = \langle V_n^2 \rangle + 4kTR\Delta f + \langle I_n^2 \rangle R^2. \qquad (6)$$

This is called the noise equivalent input. The quantities $\langle V_n^2 \rangle$ and $\langle I_n^2 \rangle$ are frequency and bandwidth dependent. This dependence can be written explicitly as

$$\langle V_n^2 \rangle = \int_0^\infty \left[v(f) \frac{|G(f)|}{G_0} \right]^2 df \qquad (7)$$

and

$$\langle I_n^2 \rangle = \int_0^\infty \left[i(f) \frac{|G(f)|}{G_0} \right]^2 df, \qquad (8)$$

where $v(f)$ and $i(f)$ are the Fourier transforms of what the voltage and current noise would be if the amplifier had an infinite pass band [i.e., $G(f) = G_0$ for all f]. For many amplifiers, $v(f)$ is a decreasing function of frequency and $i(f)$ is an increasing function.

The noise in amplifiers is normally specified by the *noise figure*, NF, which is defined as

$$\mathrm{NF} = 10 \log_{10} \left(\frac{\langle V_i^2 \rangle}{4kTR\Delta f} \right). \qquad (9)$$

The quantity NF is a double-valued function of frequency because $v(f)$ decreases and $i(f)$ increases with f; and a double-valued function of R because $\langle V_i^2 \rangle$ is quadratic in R [Eq. (6)]. Manufacturers of low-noise amplifiers usually display the NF of their products by plotting contours of constant NF for $T = 290$ K in the frequency resistance plane. These contours are found by measuring the NF at a few points and using Eq. (6) to fit the data.[15,16] Figure 2 shows the NF contours of the amplifier used in the present experiment.

For any frequency there is a source resistance for which NF is a minimum. This resistance is called the *optimum source* resistance, R_{opt}. Hence, R_{opt} is defined as the resistance where $\partial NF/\partial R = 0$. So, by combining Eqs. (6) and (9)

$$R_{\mathrm{opt}} = (\langle V_n^2 \rangle / \langle I_n^2 \rangle)^{1/2}. \qquad (10)$$

Similarly, for any resistance there is a frequency for which NF is a minimum. This frequency can be called the *optimum frequency*, f_{opt}. Hence f_{opt} is defined as the frequency where $\partial \mathrm{NF}/\partial f = 0$. Unfortunately, f_{opt} cannot be expressed in a simple form such as R_{opt} is in Eq. (10). In general, the curves of R_{opt} and f_{opt} do not coincide (see Fig. 2).

Amplifier noise affects the choice of the resistor used in the noise thermometer. From Eqs. (6) and (10) it can be

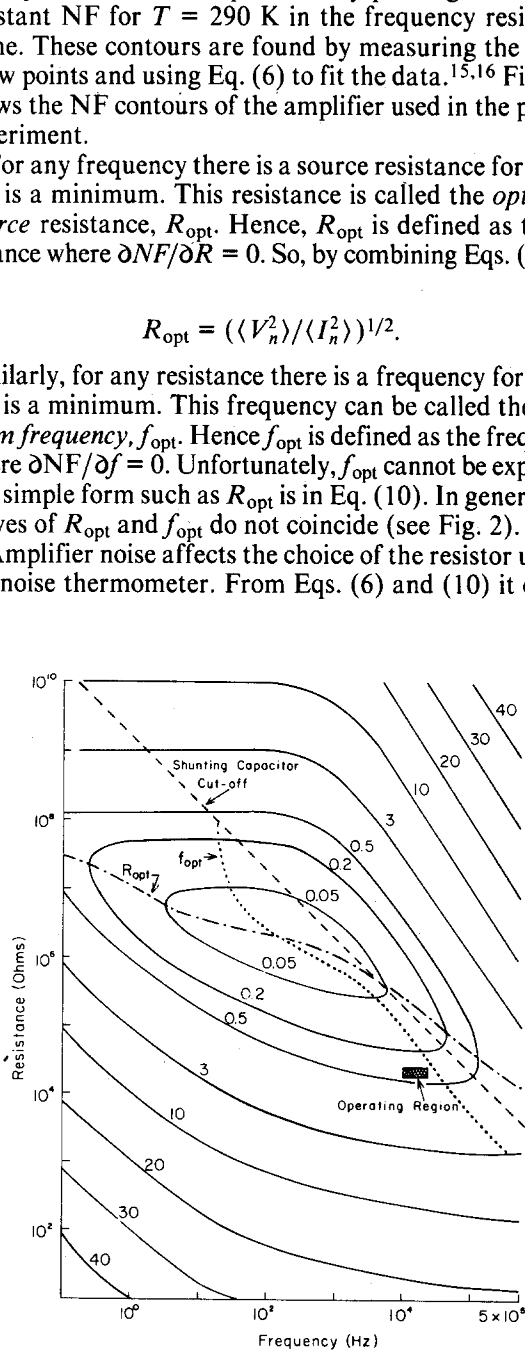


Fig. 2. Noise figure contours for the amplifier we used (PAR 113) showing the approximate region we operated in and the approximate location of the cut-off due to the shunting capacitor. The curves of optimum resistance and optimum frequency are also shown. The numbers in the figure are the NF in db.

seen that we have three principle cases depending on R/R_{opt}:

$$\langle V_i^2 \rangle \approx 4kTR\Delta f + \langle I_n^2 \rangle R^2 \qquad (R \gg R_{\mathrm{opt}}), \tag{11a}$$

$$\langle V_i^2 \rangle = \langle V_n^2 \rangle + 4kTR\Delta f + \langle I_n^2 \rangle R^2 \qquad (R \approx R_{\mathrm{opt}}), \tag{11b}$$

and

$$\langle V_i^2 \rangle \approx \langle V_n^2 \rangle + 4kTR\Delta f \qquad (R \ll R_{\mathrm{opt}}). \tag{11c}$$

For a given frequency, the least extraneous noise (minimum NF) is achieved with $R \approx R_{\mathrm{opt}}$, but this requires a knowledge of both $\langle V_n^2 \rangle$ and $\langle I_n^2 \rangle$. In the other two cases, only one of the two extraneous noise terms is needed. $\langle I_n^2 \rangle$ can be difficult to measure, especially if R_{opt} is large, in which case capacitive shunting effects become important (see discussion below). $\langle I_n^2 \rangle$ can be found by measuring $\langle V_i^2 \rangle$ as a function of R and looking for the quadratic term. However, $\langle V_n^2 \rangle$ is easier to measure: if the input of the amplifier is shorted ($R = 0$), then $\langle V_i^2 \rangle = \langle V_n^2 \rangle$.

Similarly, for a given resistance the minimum NF is achieved with $f \approx f_{\mathrm{opt}}$. Since $\partial \mathrm{NF}/\partial f = 0$ at f_{opt}, the NF for constant R will be a slowly varying function of f. Thus a pass band chosen near f_{opt} will contain a near white noise spectrum.

The result of these considerations is that if both $\langle V_n^2 \rangle$ and $\langle I_n^2 \rangle$ are known, then the best choice of R and f is for $R \approx R_{\mathrm{opt}}$ and $f \approx f_{\mathrm{opt}}$. Otherwise, the next best choice is $R \ll R_{\mathrm{opt}}$ and $f \approx f_{\mathrm{opt}}$ so that only $\langle V_n^2 \rangle$ need be known.

B. Shunt capacitance (Ref. 14)

In all circuits, there is some capacitance shunting the source resistance. This is shown by the dashed line in Fig. 1. The capacitance shunts both the resistor's noise and the current noise, but not the voltage noise. By shunting some of the current generated in the resistor by the noise voltage, the capacitance changes the ENBW from that given in Eq. (4) to

$$\Delta' f = \int_0^\infty \frac{|G(f)|^2}{G_0^2} \frac{df}{(1 + f^2/f_C^2)}, \tag{12}$$

where $f_C = (2\pi RC)^{-1}$. When the amplifier's upper half-power point is much less than f_C, then $\Delta' f \cong \Delta f$. If the upper half-power point is above f_C, then $\Delta' f < \Delta f$ and $\Delta' f \rightarrow 0$ as $f_C/f_0 \rightarrow 0$, where f_0 is the central frequency of the pass band.

As noted in Eq. (7) and (8), the ENBW is implicitly contained in $\langle I_n^2 \rangle$. Without any shunting effect, $\langle I_n^2 \rangle = \int_0^\infty i^2 |G|^2 G_0^{-2}\, df$. When capacitance is included, this becomes

$$\langle I_n^2 \rangle' = \int_0^\infty i^2 |G|^2 G_0^{-2} (1 + f^2/f_C^2)^{-1}\, df.$$

For low frequencies, many amplifiers have $i(f)$ constant; therefore

$$\langle I_n^2 \rangle' = \langle I_n^2 \rangle\, \Delta' f/\Delta f. \tag{13}$$

The total noise due to the resistor and amplifier is found by combining Eqs. (6), (12), and (13):

$$\langle V_i^2 \rangle = \langle V_n^2 \rangle + 4kTR\Delta' f + \langle I_n^2 \rangle R^2\, \Delta' f/\Delta f. \tag{14}$$

From this it can be seen that the last two terms fall off as the center frequency of the pass band is increased, but the first term does not. For a good NF, a frequency band and a resistor should be chosen such that $|G(f)|/G_0 \ll 1$ for $f \gtrsim f_C$, in which case $\Delta' f \simeq \Delta f$.

[There is an interesting exception to this rule. If $|G(f)| = G_0$ for all f, Johnson-Nyquist term in Eq. (14) becomes

$$\langle V_R^2 \rangle = 4kTR \int_0^\infty (1 + f^2/f_C^2)^{-1}\, df = 4kT/C. \tag{15}$$

Note that although the resistor is the sole source of the noise, the power is determined by the capacitor. Equation (15) also holds for an RLC circuit where the resistor in Fig. 1 is replaced by a series LR network. Moullin[17] confirmed this relationship in a series of experiments in which he varied both the resonant frequency and the Q of the circuit.]

C. Other bandwidth considerations

Several different electronic instruments are used in making noise measurements. It is important to choose the pass band (the frequency region where G/G_0 is significant) such that the oscillators, noise generators, voltmeters, and other instruments respond throughout the significant region. If the minimum and maximum operating frequencies of the instruments are f_{min} and f_{max}, respectively, then the following relation must hold:

$$\int_{f_{\mathrm{min}}}^{f_{\mathrm{max}}} |G|^2 G_0^{-2}\, df \approx \int_0^\infty |G|^2 G_0^{-2}\, df. \tag{16}$$

In other words, Eq. (3) involves an integration over all frequencies. However, real instruments cannot operate over all frequencies, so the integration must be truncated at f_{min} and f_{max}. In order that the truncation does not affect the results, Eq. (16) must hold.

D. Types of resistors

There are several different types of resistors. The most common are carbon resistors. These have a temperature dependent resistance, and produce "excess noise" (noise in excess of the Johnson noise) when a current flows.[18,19] Both these properties make them unsuitable for noise thermometry. However, precision low-noise resistors such as wire-wound or metal film resistors usually do not have such problems. Therefore, one of these latter types should be chosen and its resistance should be measured throughout the temperature range to be used in the thermometry experiments.

E. Types of voltmeters

ac voltmeters are nonlinear devices that convert ac signals into dc ones. This is done by combining a nonlinear detector with some kind of averaging. The action of such a meter can be mathematically represented by

$$V_0(t) = \left(\int_0^\infty x(t - s) g(s)\, ds \right)^a, \tag{17}$$

where $V_0(t)$ is the output voltage, x is the detected signal, g represents the effect of the filter, and the a takes into account the post-filtering signal manipulation that occurs in some meters. Equation (17) has the form of a convolution because in frequency space the Fourier transform of the detected signal is multiplied by a function that represents the action of the filter. For simple RC filtering (a common type), $g(s) = \exp(-s/RC)$. Table I gives x and a for various types of meters where $V_i(t)$ is the input voltage.

The quantity to be measured in these experiments is the

Table I. The values of x and a needed for the evaluation of Eq. (17) for various types of meters.

Type	$x(t)$	a
Mean square	$V_i^2(t)$	1
rms	$V_i^2(t)$	½
Half wave rectifier	$\begin{Bmatrix} V_i(t), V_i > 0 \\ 0, V_i < 0 \end{Bmatrix}$	1
Full wave rectifier	$\lvert V_i(t)\rvert$	1

mean square voltage. The ideal measuring instrument would be a voltmeter that responded to either the rms or mean square voltage of any waveform. However, the most common voltmeters respond to the mean of the half-wave rectified signal. If this type of meter is used, a correction must be applied to convert its scale to an rms scale.[20] The correction factor depends on the wave form being measured.

For a pure sine wave ($A \sin 2\pi ft$), an average responding (half wave rectifier) voltmeter will have a mean output of

$$f \int_0^{1/f} x \, dt = f \int_0^{1/2f} A \sin 2\pi ft \, dt = A/\pi \quad (18)$$

the rms of such a sine wave is

$$\left(f \int_0^{1/f} x^2 \, dt\right)^{1/2} = \left(f \int_0^{1/f} A^2 \sin^2 2\pi ft \, dt\right)^{1/2} = A/\sqrt{2}\,. \quad (19)$$

So the output of an average responding voltmeter must be calibrated (multiplied by a factor of $\pi/\sqrt{2}$) to give the rms voltage of a pure sine wave. However, the signal we want to measure is not a sine wave but random noise. By the central limit theorem[21,22] the noise is assumed to be Gaussian. This means that the probability of finding the instantaneous voltage between V and $V + dV$ is

$$P(V)dV = A^{-1}(2\pi)^{-1/2} e^{-(V^2/2A^2)} \, dV. \quad (20)$$

The rms voltage for such noise is

$$\left(\int_{-\infty}^{\infty} V^2 P(V) \, dV\right)^{1/2} = A \quad (21)$$

and the rectified average is

$$\int_0^{\infty} V P(V) \, dV = A(2\pi)^{-1/2}. \quad (22)$$

Thus the output of an average responding meter must be multiplied by $(2\pi)^{1/2}$ to give the rms value for noise. But the output has already been multiplied by $\pi/\sqrt{2}$ for the sine wave correction. To give the rms voltage for noise, the scale on an average responding meter must be multiplied by $(4/\pi)^{1/2}$ or 1.1284.

F. Accuracy versus measuring time

A noise signal will cause the output of any voltmeter to fluctuate. Such an output can be represented by

$$V_0 = \langle V_0 \rangle + v, \quad (23)$$

where $\langle v \rangle = 0$. The amplitude and rate of fluctuations of v depend on the type of meter, on the type of filtering incorporated in the meter, and on the noise itself. These considerations have been extensively studied[18,23] and it can be shown that for both average responding and rms meters with a simple RC filter of time constant $\tau = (RC)^{-1}$, that the accuracy of a single reading is

$$\alpha = \langle v^2 \rangle^{1/2} / \langle V_0 \rangle = (8\tau \Delta f)^{-1/2} \quad (24)$$

when $(2\pi\tau)^{-1} \ll$ (any frequency of the pass band of the noise) and Δf is the ENBW of the noise.[26] [This result is sometimes misquoted[14] as

$$\alpha = (2\tau \Delta f)^{-1/2} \quad (25)$$

which, however, is true for mean square meters.]

Further averaging can be done by taking the mean of n successive readings Δt apart, in which case Eq. (24) becomes

$$\alpha = (8n\tau \Delta f)^{-1/2} \quad (26)$$

when $\Delta t \gg \tau$.

G. Grounding and shielding

Proper shielding and grounding are important in any measurement of small signals. The most important principle is to avoid ground loops (multiple paths for ground currents). There should be only one current path between all shields and the grounding point. If more than one path exists, then a small induced emf can cause a large current to flow, which may add to the signal being processed. Sometimes ground loops are unavoidable, such as an amplifier that has a grounded input and a grounded power supply. In this case, a small (~10 Ω) resistor should be placed at a point in the loop where it does not interfere with the signal path. The resistor will prevent small emf's from generating large currents. Princeton Applied Research (PAR) and high fidelity component manufacturers do this in some amplifiers.

Morrison[24] has found three basic rules for shielding instruments:

(1) The local ground of any circuit should be connected to the shield enclosing it.

(2) This connection should be at the signal source.

(3) The number of shields should equal the number of signals being processed plus the number of power entrances.

We have tried to follow these rules in these experiments. We did this by (Fig. 4) connecting the shield, the signal source and all grounds to the amplifier. Since we had only one signal source, we had only one ground.

IV. APPARATUS AND METHOD

The temperature probe is shown in Fig. 3. A precision metal film resistor (20 kΩ Dale type RN 65C) is used to

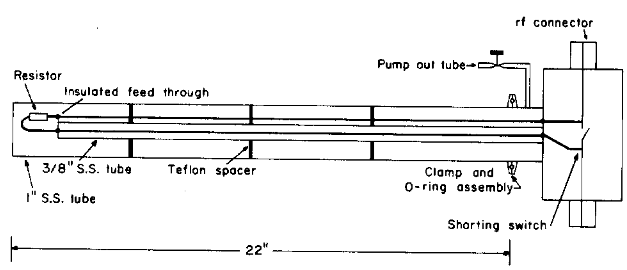
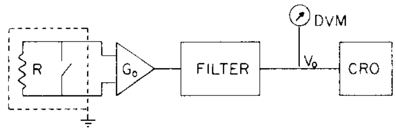


Fig. 3. Noise thermometry probe.

Fig. 4. Noise thermometry circuit.



minimize excess noise and to reduce the temperature dependence of the resistance. To shield the leads from one another, one of them is placed in a stainless steel tube. This tube, the other lead, and the resistor are placed in a grounded stainless-steel jacket. The jacket is pumped out and filled with ~0.3 atm of He exchange gas. This gas maintains thermal contact between the resistor and the jacket. At the top of the jacket the leads are attached to two coax sockets and a shorting switch. When the switch is open, $\langle V_0^2 \rangle = G_0^2 \langle V_i^2 \rangle$ can be measured (Sec. III A). When the switch is closed, $G_0^2 \langle V_n^2 \rangle$ can be measured. It is important that the capacitances of the probe and of the leads to the amplifier are low. To accomplish this, we used a low capacitance coax RG-114 A/V (Times Wire and Cable Company). Our total capacitance was <90 pF. This corresponds to a cut-off frequency [Eq. (12)] of f_c >90 kHz, which is well above our pass band (Fig. 6).

The electronics used in the thermometry are shown in Fig. 4. The amplifier is a PAR 113. It was used in the differential ac mode with a nominal gain of 10^4 and a 300 Hz–1000 kHz pass band. The filter was a General Radio (GR) 1952 with zero gain, set to provide a 10–20 kHz pass band. For this pass band, 130 kΩ < R_{opt} <250 kΩ. Our 20-kΩ resistor is much smaller than R_{opt}; therefore the $\langle I_n^2 \rangle$ term of Eq. (6) can be ignored as in Eq. (11c). At the same time, this pass band is high enough in frequency to filter out any interference associated with the 60-Hz line frequency and it is low enough to ignore the capacitance. Also, for 20 kΩ, f_{opt} = 30 kHz so the pass band is near f_{opt}. This filter was chosen because of its fast roll-off characteristics. However, the filter generates some noise because it is an active filter (it contains an amplifier). If the filter were placed before the amplifier, this noise would dominate the measurement. To avoid this, we placed the filter after the amplifier. The measurement of V_n includes this noise. The voltmeter was a Hewlett-Packard (HP) 3490A digital multimeter that was also used to measure R. Since this is an average responding meter, the readings had to be corrected by a factor of $(4/\pi)^{1/2}$ (Sec. III E).

An oscilloscope was used to detect electrical interference. Most of this interference we had was locked to the 60-Hz line voltage. The interference had to be eliminated before measurements could be made. This was done by twisting together the two coax leads between the probe and the amplifier, by careful grounding, and by trying various filter settings. The oscilloscope was also useful in showing whether the noise was symmetric and approximately Gaussian.

To measure $G_0^2 \Delta f$, a different arrangement is used as shown in Fig. 5. The amplifier, filter, and voltmeter are the same as before. However, this time the noise source (the resistor) is replaced with an oscillator and a voltage divider. The voltage divider was a GR 1455 AL four decade voltage divider set at an attenuation α = 0.0010. The voltmeter is used to alternatively read the input and output voltages, V_i and V_0, respectively. The oscillator was stepped from 0 to 50 kHz in 1-kHz steps. At each step, V_i and V_0 were measured. $G_0^2 \Delta f$ was found by a numerical integration:

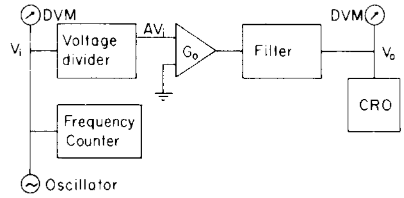


Fig. 5. Circuit for finding the equivalent noise bandwidth (EBNW).

$$G_0^2 \Delta f = \int_0^\infty (V_0/\alpha V_i)^2 \, df. \tag{27}$$

The integration could be truncated at 50 kHz because with the pass band chosen, $V_0 \ll \alpha V_i$ above 50 kHz. Numerical integrations often result in large errors depending on the sophistication of the method and the nature of the integrand. We used Simpson's rule, and estimate our error to be ~1%.

V. RESULTS

The following are typical results obtained from the apparatus. A plot of $(V_0/\alpha V_i)^2$ is shown in Fig. 6. From this, $G_0^2 \Delta f$ was found to be 9.278(1) × 10^{11} Hz where the error represents the error in the data, not the error due to the numerical integration. The Johnson-Nyquist noise was measured at three temperatures and Boltzmann's constant was calculated for each temperature. To insure temperature equilibrium between the resistor and temperature bath, the probe was inserted deep enough into the bath that the generated noise was independent of depth. The temperatures were independently measured with a National Bureau of Standards traceable, Rosemount platinum resistance thermometer (No. 146 MA-2000F). The results are shown in Fig. 7 and in Table II. The voltage readings are uncorrected (Sec. III E) and represent the mean and standard deviation of 75 successive measurements. These represent a relative error in voltage of ~0.7%. Our voltmeter had a time constant of ~8 ms. Using this in Eq. (26) results in a predicted error of 0.5%. However, because of the complicated filtering in the voltmeter, Eq. (26) is not valid but can be used as an approximation. Most of the error in our values of k come from the fluctuations of V_0 and V_n. The error in the numerical integration contributes to the systematic error in k. In Fig. 7, the experimental values of kT are plotted

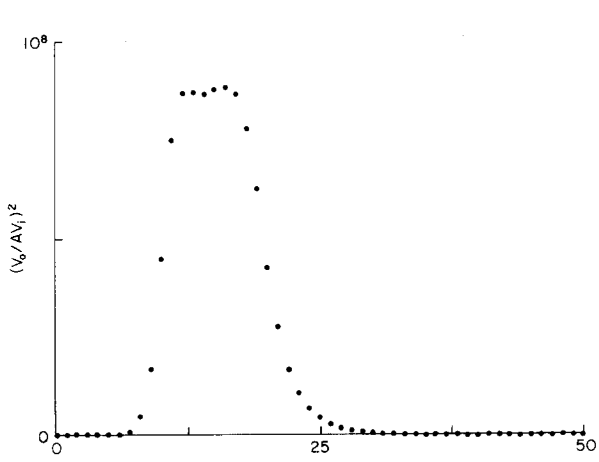


Fig. 6. Plot of data used in determining the ENBW.

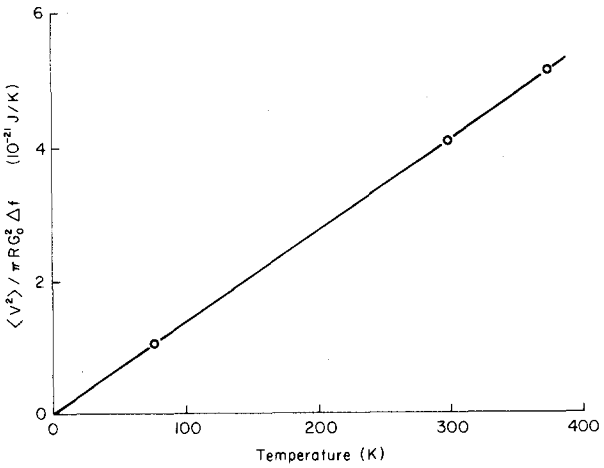


Fig. 7. Plot of the results of the noise thermometer. Note that the voltmeter correction has been included in the ordinate.

versus T. These points are then fitted to the function $(kT)_{\rm exp} = k(T - T_0)$. A least-squares fit results in $k = 1.369 \times 10^{-23}$ and $T_0 = 50$ mK. These numbers have a large error because only three points are being fitted. However, the fact that $T_0 = 50$ mK indicates that our method of measuring background noise was good. Also, our values of k are in good agreement with the accepted value of 1.3807×10^{-23} $\rm JK^{-1}$.[25]

VI. ALTERNATE METHODS

If a calibrated white-noise generator is available, there is an alternative method[15] of finding $G_0^2\Delta f$. The pass band of the noise generator must encompass the pass band of the system under test. To find $G_0^2\Delta f$, the oscillator and frequency counter of Fig. 5 are replaced by the noise generator. Then

$$G_0^2\Delta f = \Delta_g f (V_0/\alpha V_i)^2, \qquad (28)$$

where $\Delta_g f$ is the ENBW of the noise generator. In principle, $G_0^2\Delta f$ can be found by just two measurements, one of V_0 and one of V_i. Of course, V_0 and V_i are fluctuating quantities, so averages of several readings are needed for good accuracy.

An uncalibrated white noise generator can be calibrated by combining the above method with the previous oscillator method. First, the ENBW of a circuit is found with an oscillator, as in Eq. (27), and then the oscillator is replaced with a noise generator and $\Delta_g f$ is found from Eq. (28). Thus

$$\Delta_g f = \left(\frac{V_i}{V_0}\right)^2_{\text{noise gen}} \int_0^\infty \left(\frac{V_0}{V_i}\right)^2_{\text{oscillator}} df. \qquad (29)$$

Table II. Results from noise thermometer.

T(K)	V_0^2 (volts^2)	V_n^2 (volts^2)	$R(\Omega)$	$k(\rm JK^{-1})$
373.2(1)	3.287(47) $\times 10^{-4}$	3.00(4) $\times 10^{-5}$	20062(3)	1.369(22) $\times 10^{-23}$
297.9(1)	2.686(43) $\times 10^{-4}$	3.06(4) $\times 10^{-5}$	20030(3)	1.369(25) $\times 10^{-23}$
77.6(1)	0.922(12) $\times 10^{-4}$	3.01(4) $\times 10^{-5}$	20096(3)	1.367(26) $\times 10^{-23}$

We did this for an HP 3722A noise generator using the same amplifier and filter settings used to get the results in Sec. V. Using the average of 75 measurements of V_0 and V_i, we found $\Delta_g f = 51.3(9)$ kHz when the noise generator was set at 50 kHz. This is within the manufacturer's specifications.

VII. CONCLUSIONS

We believe that this experiment is a useful introduction to the properties of noise and to sensitive measurements. With the equipment we had on hand, students achieved accuracies of 1-2%. We also tried doing the experiment with less expensive equipment. As long as a low-noise amplifier was used, the results were good, being limited by the accuracy of the equipment. Using an analog meter to measure the voltages, we could read the temperature data directly without having to average a long series of readings.

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