distribution with a mean value $\overline{v(s)} = v_0 e^{-\gamma s}$; this result is in accord with (15·8·10).

FOURIER ANALYSIS OF RANDOM FUNCTIONS

15 · 13 *Fourier analysis*

When dealing with a random function of time $y(t)$, it is very often convenient to consider its frequency components obtained by Fourier analysis. This has the advantage that it is commonly easier to focus attention on the amplitudes and phases of simple sinusoidally varying functions than on the very complicated variation in time of the random function $y(t)$. In addition, suppose that $y(t)$ is a quantity which is used as the input to a linear system (e.g., an electrical voltage which is used as input to an electrical circuit involving resistors, inductors, and capacitors); then it is quite easy to discuss what happens to each frequency component passing through the system, but it would be very difficult to analyze the situation without resolving $y(t)$ into Fourier components.

The quantity $y(t)$ has statistical properties described by a representative ensemble similar to that illustrated in Fig. 15·5·1 for the function $F(t)$. We want to represent $y(t)$ within the very large time interval $-\Theta < t < \Theta$ in terms of a superposition of sinusoidally varying functions of time. (We can ultimately go to the limit where $\Theta \to \infty$.) To avoid possible convergence difficulties, we shall then try to find the Fourier representation of the modified function $y_\Theta(t)$ which is identical with $y(t)$ in the entire domain $-\Theta < t < \Theta$, but which vanishes otherwise. Thus $y_\Theta(t)$ is defined by

$$y_\Theta(t) \equiv \begin{cases} y(t) & \text{for } -\Theta < t < \Theta \\ 0 & \text{otherwise} \end{cases} \qquad (15\cdot 13\cdot 1)$$

We make use of the result (A·7·14) that the complex exponential function satisfies the ("orthogonality") property

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega(t-t')}\,d\omega = \delta(t-t') \qquad (15\cdot 13\cdot 2)$$

where $\delta(t-t')$ is the Dirac δ function. *For any one system of the ensemble,* we can then write the function $y_\Theta(t)$ in the form

$$\begin{aligned} y_\Theta(t) &= \int_{-\infty}^{\infty} dt'\,\delta(t-t')\,y_\Theta(t') \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dt' \int_{-\infty}^{\infty} d\omega\, e^{i\omega(t-t')}\,y_\Theta(t') \end{aligned}$$

or

$$y_\Theta(t) = \int_{-\infty}^{\infty} C(\omega)\,e^{i\omega t}\,d\omega \qquad (15\cdot 13\cdot 3)$$

where

$$C(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} y_\Theta(t')\,e^{-i\omega t'}\,dt'$$

or

$$C(\omega) = \frac{1}{2\pi}\int_{-\Theta}^{\Theta} y(t')\,e^{-i\omega t'}\,dt' \qquad (15\cdot 13\cdot 4)$$

and where we have used the definition (15·13·1) in the last step. The relation (15·13·3) is the desired "Fourier integral" representation of the function $y_\Theta(t)$ in terms of a superposition of complex exponential functions of different frequencies ω; the coefficient $C(\omega)$ is then determined by (15·13·4). Since $y(t)$ is real, its complex conjugate satisfies the relation

$$y^*(t) = y(t) \qquad (15\cdot 13\cdot 5)$$

Hence it follows by (15·13·4) that

$$C^*(\omega) = C(-\omega) \qquad (15\cdot 13\cdot 6)$$

In any one system k of the ensemble, the function $y^{(k)}(t)$ can thus be represented by its corresponding Fourier coefficient $C^{(k)}(\omega)$ given by (15·13·4).

15 · 14 *Ensemble and time averages*

There are two types of averages that are of interest. The first of these is the ordinary statistical average of y *at a given time* over all systems of the ensemble. This *ensemble* average, which we denote interchangeably by $\bar{y}$ or $\langle y \rangle$, is defined by

$$\overline{y(t)} \equiv \langle y(t) \rangle \equiv \frac{1}{N}\sum_{k=1}^{N} y^{(k)}(t) \qquad (15\cdot 14\cdot 1)$$

where $y^{(k)}(t)$ is the value assumed by $y(t)$ in the kth system of the ensemble and where N is the very large total number of systems in the ensemble.

The second average of interest is the average of y for a *given system of the ensemble* over some very large time interval 2Θ (where $\Theta \to \infty$). We shall denote this *time* average by $\{y\}$ and define it for the kth system of the ensemble by

$$\{y^{(k)}(t)\} \equiv \frac{1}{2\Theta}\int_{-\Theta}^{\Theta} y^{(k)}(t+t')\,dt' \qquad (15\cdot 14\cdot 2)$$

(In more pictorial terms illustrated by Fig. 15·5·1, the ensemble average is taken vertically for a given t, while the time average is taken horizontally for a given k.)

Note that the operations of taking a time average and taking an ensemble average commute. Indeed

$$\begin{aligned} \langle\{y^{(k)}(t)\}\rangle &= \frac{1}{N}\sum_{k=1}^{N}\left[\frac{1}{2\Theta}\int_{-\Theta}^{\Theta} y^{(k)}(t+t')\,dt'\right] \\ &= \frac{1}{2\Theta}\int_{-\Theta}^{\Theta}\left[\frac{1}{N}\sum_{k=1}^{N} y^{(k)}(t+t')\right]dt' = \frac{1}{2\Theta}\int_{-\Theta}^{\Theta}\langle y(t+t')\rangle\,dt' \end{aligned}$$

or

$$\langle\{y^{(k)}(t)\}\rangle = \{\langle y(t)\rangle\} \qquad (15\cdot 14\cdot 3)$$

Consider now a situation which is "stationary" with respect to y. This means that there is no preferred origin in time for the statistical description of y, i.e., the same ensemble ensues when all member functions $y^{(k)}(t)$ of the ensemble are shifted by arbitrary amounts in time. (In an equilibrium situation this would, of course, be true for all statistical quantities.) For such stationary ensembles there is an intimate connection between ensemble and time averages if one assumes that (with the possible exception of a negligible number of exceptional systems in the ensemble) the function $y^{(k)}(t)$ for each system of the ensemble will in the course of a sufficiently long time pass through all the values accessible to it. (This is called the "ergodic" assumption.) One can then imagine that one takes, for example, the kth system of the ensemble and subdivides the time scale into very long sections (or intervals) of magnitude 2Θ, as shown in Fig. 15·14·1. Since Θ is very large, the behavior of $y^{(k)}(t)$ in each such section will then be independent of its behavior in any other section. Some large number of M such sections should then constitute as good a representative ensemble of the statistical behavior of y as the original ensemble. Hence the time average should be equivalent to the ensemble average.

More precisely, in such a stationary ensemble the time average of y taken over some very long time interval Θ must be independent of the time t. Furthermore, the ergodic assumption implies that the time average must be the

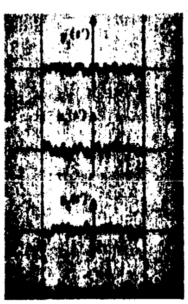

Fig. 15·14·1 The time dependence of $y^{(k)}(t)$ in the kth member of a stationary ergodic ensemble. The time scale is shown broken up into sections of very long duration 2Θ. These sections are shown rearranged vertically in the bottom part of the figure to form another representative ensemble equivalent to the original one. (Here $y_j(t') = y^{(k)}(2j\Theta + t')$, with $-\Theta < t' < \Theta$.)

same for essentially all systems of the ensemble. Thus

$$\{y^{(k)}(t)\} = \{y\} \qquad \text{independent of } k \tag{15·14·4}$$

Similarly, it must be true that in such a stationary ensemble the ensemble average of y must be independent of time. Thus

$$\langle y(t)\rangle = \langle y\rangle \qquad \text{independent of } t \tag{15·14·5}$$

The general relation (15·14·3) leads then immediately to an interesting conclusion. By taking the ensemble average of (15·14·4), one gets simply

$$\langle\{y^{(k)}(t)\}\rangle = \{y\}$$

Furthermore, taking the time average of (15·4·5) gives simply

$$\{\langle y(t)\rangle\} = \langle y\rangle$$

Hence (15·14·3) yields, *for a stationary ergodic ensemble*, the important result

$$\{y\} = \langle y\rangle \tag{15·14·6}$$

15 · 15 *Wiener-Khintchine relations*

Consider the random function $y(t)$, which is stationary. Its correlation function is then, by definition,

$$K(s) \equiv \langle y(t)y(t+s)\rangle \tag{15·15·1}$$

and is independent of t since $y(t)$ is stationary. Note that $K(0) = \langle y^2\rangle$ gives the dispersion of y if $\langle y\rangle = 0$.

The correlation function of y can, like any other function of time and like y itself, also be expressed as a Fourier integral. Analogously to (15·13·3) one can then write

▶
$$K(s) = \int_{-\infty}^{\infty} J(\omega)\, e^{i\omega s}\, d\omega \tag{15·15·2}$$

where the coefficient $J(\omega)$ is called the "spectral density" of y. From (15·15·2) it follows that $J(\omega)$ can conversely be expressed in terms of $K(s)$. One thus obtains, analogously to (15·13·4),

▶
$$J(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} K(s)\, e^{-i\omega s}\, ds \tag{15·15·3}$$

Remark This follows explicitly from (15·15·2) by multiplying both sides of that relation by $e^{-i\omega' s}$ and then integrating over s. Thus

$$\begin{aligned}\int_{-\infty}^{\infty} ds\, K(s)\, e^{-i\omega' s} &= \int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} d\omega\, J(\omega)\, e^{i(\omega-\omega')s} \\ &= 2\pi \int_{-\infty}^{\infty} d\omega\, J(\omega)\, \delta(\omega - \omega') \\ &= J(\omega')\end{aligned}$$

which is identical to (15·15·3) if one puts $\omega' = \omega$.

The correlation function $K(s)$ is real and satisfies the symmetry property (15·8·5). Thus

$$K^*(s) = K(s) \qquad \text{and} \qquad K(-s) = K(s) \tag{15·15·4}$$

Hence it follows by (15·15·3) that $J(\omega)$ is also real and satisfies similar symmetry properties, i.e.,

$$J^*(\omega) = J(\omega) \qquad \text{and} \qquad J(-\omega) = J(\omega) \tag{15·15·5}$$

Remark **The proofs are immediate since, by virtue of (15·15·3) and (15·15·4),**

$$J^*(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} K(s)\, e^{i\omega s}\, ds = \frac{1}{2\pi}\int_{-\infty}^{\infty} K(s)\, e^{-i\omega s}\, ds = J(\omega)$$

and $$J(-\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} K(s)\, e^{i\omega s}\, ds = \frac{1}{2\pi}\int_{-\infty}^{\infty} K(s)\, e^{-i\omega s}\, ds = J(\omega)$$

where we have changed the variable of integration from s to $-s$ in the second set of integrals.

Note that (15·15·2) implies the particularly important result that

$$\langle y^2\rangle = K(0) = \int_{-\infty}^{\infty} J(\omega)\, d\omega = \int_0^{\infty} J_+(\omega)\, d\omega \tag{15·15·6}$$

where

$$J_+(\omega) \equiv 2J(\omega) \tag{15·15·7}$$

is the spectral density for positive frequencies.

The Fourier integrals (15·15·2) and (15·15·3) are known as the Wiener-Khintchine relations. They can also be written in explicitly real form by putting $e^{\pm i\omega s} = (\cos \omega s \pm i \sin \omega s)$ and noting that, by virtue of (15·15·4) and (15·15·5), the part of the integrand involving $\sin \omega s$ is odd and leads to a vanishing integral. Thus (15·15·2) and (15·15·3) become

$$K(s) = \int_{-\infty}^{\infty} J(\omega) \cos \omega s\, d\omega = 2\int_0^{\infty} J(\omega) \cos \omega s\, d\omega \tag{15·15·8}$$

$$J(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} K(s) \cos \omega s\, ds = \frac{1}{\pi}\int_0^{\infty} K(s) \cos \omega s\, ds \tag{15·15·9}$$

It is of interest to express $K(s)$ and $J(\omega)$ directly in terms of the Fourier coefficients $C(\omega)$ of the original random function $y(t)$. If $y(t)$ is stationary and ergodic, $K(s)$ is independent of time, and the ensemble average can be replaced by a time average over any system of the ensemble. Hence (15·15·1) can be written

$$K(s) = \langle y(0)y(s)\rangle = \{y(0)y(s)\}$$

or

$$K(s) = \frac{1}{2\Theta}\int_{-\Theta}^{\Theta} dt'\, y(t')y(s + t')$$

where we have used the definition (15·14·2). By replacing $y(t')$ by the modified function $y_\Theta(t')$ of (15·13·1), this becomes

$$K(s) = \frac{1}{2\Theta}\int_{-\infty}^{\infty} dt'\, y_\Theta(t')y_\Theta(s + t') \tag{15·15·10}$$

(Replacement of the function $y(s + t')$ by $y_\Theta(s + t')$ in the integrand is permissible since it causes an error only of the order of s/Θ, which is negligible in the assumed limit $\Theta \to \infty$). By using the Fourier expansion (15·13·3), the last expression can then be written

$$\begin{aligned}
K(s) &= \frac{1}{2\Theta}\int_{-\infty}^{\infty} dt' \int_{-\infty}^{\infty} d\omega'\, C(\omega')\, e^{i\omega' t'} \int_{-\infty}^{\infty} d\omega\, C(\omega)\, e^{i\omega(s+t')} \\
&= \frac{1}{2\Theta}\int_{-\infty}^{\infty} d\omega' \int_{-\infty}^{\infty} d\omega\, C(\omega')C(\omega)\, e^{i\omega s} \int_{-\infty}^{\infty} dt'\, e^{i(\omega'+\omega)t'} \\
&= \frac{1}{2\Theta}\int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d\omega'\, C(\omega')C(\omega)\, e^{i\omega s}[2\pi\, \delta(\omega' + \omega)] \\
&= \frac{\pi}{\Theta}\int_{-\infty}^{\infty} d\omega\, C(-\omega)C(\omega)\, e^{i\omega s} \\
&= \frac{\pi}{\Theta}\int_{-\infty}^{\infty} d\omega\, |C(\omega)|^2\, e^{i\omega s} \qquad \text{by (15·13·6)}
\end{aligned}$$

or $$K(s) = \int_{-\infty}^{\infty} J(\omega)\, e^{i\omega s}\, ds$$

where

▶ $$J(\omega) = \frac{\pi}{\Theta}|C(\omega)|^2 \tag{15·15·11}$$

Hence one also obtains

$$\langle y^2\rangle = K(0) = \frac{\pi}{\Theta}\int_{-\infty}^{\infty} |C(\omega)|^2\, d\omega \tag{15·15·12}$$

The relation (15·15·11) provides an expression for the spectral density $J(\omega)$ in terms of the Fourier coefficient $C(\omega)$ of any system in the ensemble. It also shows explicitly that $J(\omega)$ can never be negative.

15 · 16 *Nyquist's theorem*

Suppose that an electrical resistor R is connected across the input terminals of a linear amplifier which is tuned so as to pass (angular) frequencies in the range between ω_1 and ω_2. The fluctuating current $I(t)$ due to the random thermal motion of electrons in the resistor gives rise to a random output signal (or "noise") in the amplifier. The interactions responsible for this random current can be represented by an effective fluctuating emf $V(t)$ in the resistor. If this emf $V(t)$ is expressed in terms of Fourier components, then one can write

$$\langle V^2\rangle = \int_0^{\infty} J_+(\omega)\, d\omega \tag{15·16·1}$$

where $J_+(\omega)$ is the spectral density of the emf $V(t)$. Since $J_+(\omega)$ is intrinsically nonnegative, it provides an appropriate measure of the magnitude of