closer to the physics of the device. It is not difficult, however, to convert the one equivalent circuit into the other.

3.2
NOISE FIGURE

3.2a. Spot Noise Figure—Overall Noise Figure†

In order to characterize the noise of an amplifier stage, one uses this stage in front of a main receiver of effective bandwidth B_{eff}, the effective bandwidth being so chosen that it is small in comparison with the bandwidth B of the input circuit of the amplifier stage under test, say $B_{eff} < \frac{1}{8}B$. Instead of a signal source, a conductance g_s is connected between the input terminals of the amplifier stage under test; g_s is at the reference temperature T_0. A saturated thermionic diode is connected in parallel with g_s (Fig. 3.7). So much current is now passed through the saturated diode that the output noise power of the receiver is doubled.

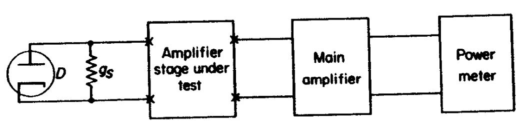

Fig. 3.7. Circuit for measuring the spot noise figure of an amplifier stage.

Let it first be assumed that the main receiver gives a negligible contribution to this output noise power. If the power is doubled for $I_d = I_{eqs}$, then we call I_{eqs} the equivalent input saturated diode current at the source side of the amplifier stage. We may, then, for a small frequency interval Δf, represent the noise of the amplifier by a current generator $\sqrt{2qI_{eqs}\Delta f}$ in parallel with g_s.

We now replace this current generator by an equivalent current generator $\sqrt{F(f)\cdot 4kT_0 g_s \Delta f}$, where T_0 is the reference temperature. Then

$$F(f)\cdot 4kT_0 g_s \Delta f = 2qI_{eqs}\Delta f \quad \text{or} \quad F(f) = \frac{q}{2kT_0}\frac{I_{eqs}}{g_s} \tag{3.15}$$

The amplifier stage then gives $F(f)$ times as much noise output power as the thermal noise of g_s at the reference temperature T_0. This quantity $F(f)$ is called the *spot noise figure* or *narrow band* noise figure, since the bandwidth B_{eff} over which the measurement takes place is small in comparison with the bandwidth B of the input circuit of the amplifier stage. The reference temperature T_0 is usually taken as 290°K (room temperature).

†H. T. Friiss, *Proc. I.R.E.*, 32, 419 (1944); A. van der Ziel, *Noise*, Prentice-Hall, Inc., Englewood Cliffs, N. J., 1954.

If we now tune the main receiver to different frequencies and determine the spot noise figure at each frequency we find that the spot noise figure $F(f)$ is frequency dependent. It is usually a minimum near the center of the passband of the amplifier stage and increases toward the edges of the passband.

Now suppose we have the amplifier stage connected to a main receiver of comparable bandwidth that is again supposed to give a negligible contribution to the output noise power of the receiver. In that case the noise for any frequency interval Δf again can be represented by a current generator $\sqrt{F(f)\cdot 4kT_0 g_s \Delta f}$ in parallel with g_s. Hence, if $g(f)$ is the signal transfer function of the combination, then the total output noise power P_{tot} is

$$P_{tot} = \int_0^\infty F(f)\cdot 4kT_0 g_s |g(f)|^2\, df \tag{3.16}$$

Of this the part

$$P_s = \int_0^\infty 4kT_0 g_s |g(f)|^2\, df \tag{3.16a}$$

comes from the thermal noise of g_s. The *overall noise figure* F_{av} of the system is now defined as

$$F_{av} = \frac{P_{tot}}{P_s} = \frac{\int_0^\infty F(f)|g(f)|^2\, df}{\int_0^\infty |g(f)|^2\, df} \tag{3.17}$$

so that F_{av} is the average value of $F(f)$ averaged over the frequency response of the combination.

If the main receiver gives a noticeable contribution to the output noise power of the receiver, one obviously must correct for this contribution. It is shown in Chapter 4 how this can be done. In some cases the correction may be so large that it seriously deteriorates the accuracy of the noise figure measurement.

3.2b. Noise Temperature of Receivers and Amplifier Stages

We saw in the previous section that the noise of an amplifier stage or amplifier could be represented by an equivalent current generator $\sqrt{F(f)\cdot 4kT_0 g_s \Delta f}$ in parallel with the source conductance g_s. We now write

$$F(f)\cdot 4kT_0 g_s \Delta f = 4kT_0 g_s \Delta f + [F(f) - 1]\cdot 4kT_0 g_s \Delta f \tag{3.18}$$

The first term can be interpreted as the thermal noise of the source at the temperature T_0, so that the second term is the noise of the amplifier or amplifier stage. We now write

$$[F(f) - 1]\cdot 4kT_0 g_s \Delta f = 4kT_{ne} g_s \Delta f \quad \text{or} \quad T_{ne} = T_0[F(f) - 1] \tag{3.19}$$

The parameter T_{na} is called the *equivalent noise temperature* of the amplifier or amplifier stage.

The advantage of the equivalent noise temperature is that *noise temperatures are additive*. If the source is not at room temperature, but has an equivalent noise temperature T_{ns} instead, and the amplifier has an equivalent noise temperature T_{na}, then the equivalent noise temperature T_{eq} of source plus amplifier is

$$T_{eq} = T_{ns} + T_{na} \tag{3.20}$$

and the noise of all the sources combined can be represented by an equivalent current generator $\sqrt{4kT_{eq}g_s\,\Delta f}$ in parallel with the source conductance g_s.

The introduction of the equivalent noise temperature must be modified if the quantum correction to Nyquist's theorem becomes significant (Section 5.1); this problem is more fully discussed in Chapter 7.

3.2c. Calculation of the Noise Figure in a Simple Case

In calculating the noise figure F of an amplifier stage one draws the equivalent circuit, includes all noise sources, calculates the mean square value $\overline{v_o^2}$ of the output noise voltage, and defines

$$F = \frac{\overline{v_o^2}}{\text{contribution of the source noise to } \overline{v_o^2}} \tag{3.21}$$

Figure 3.8 shows the equivalent circuit of an amplifier stage in which e_n and i_i are uncorrelated. Introducing

$$\overline{e_n^2} = 4kT_0R_n\,\Delta f \qquad \overline{i_i^2} = 4kT_0g_n\,\Delta f \tag{3.22}$$

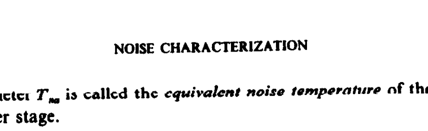

Fig. 3.8. Simple amplifier stage in which the noise sources e_n and i_i are uncorrelated.

we have

$$\overline{v_o^2} = \frac{4kT_0g_s\Delta f}{(g_s + g_i)^2} + \frac{4kT_0g_n\Delta f}{(g_s + g_i)^2} + 4kT_0R_n\Delta f \tag{3.22a}$$

so that from the definition of F in Eq. (3.21)

$$F - 1 = \frac{g_n}{g_s} + \frac{R_n(g_i + g_s)^2}{g_s} = 2R_ng_i + R_ng_s + \frac{g_n + R_ng_i^2}{g_s} \tag{3.23}$$

Considered as a function of g_s, $F - 1$ has a minimum value

$$F_{min} - 1 = 2R_ng_i + 2\sqrt{R_ng_n + R_n^2g_i^2} \tag{3.23a}$$

for

$$g_s = (g_s)_{opt} = \sqrt{g_i^2 + \frac{g_n}{R_n}} \tag{3.23b}$$

We thus see that F, considered as a function of g_s, is a hyperbola and that the minimum value F_{min} of F can be reached by properly coupling the source to the input circuit.

In the above calculation we ignored the noise of the load impedance in the output of the stage. We shall see in Section 3.3 that in cascaded amplifiers the noise of the output load is always counted as belonging to the next stage.

3.2d. Short-Circuit Noise Resistance—Open-Circuit Noise Conductance

According to Eq. (3.23) the noise figure of an amplifier stage can be written in the form

$$F = A + Bg_s + \frac{C}{g_s} \tag{3.24}$$

This can be shown to be the case as long as one deals with lumped-circuit networks (Chapter 7). Since B has the dimension of a resistance and C the dimension of a conductance, we write

$$B = R_{n0} \qquad C = g_{n0} \tag{3.24a}$$

where R_{n0} and g_{n0} are the noise resistance for short-circuited input and the noise conductance for open input, respectively. F then has minimum value

$$F_{min} = A + 2\sqrt{g_{n0}R_{n0}} \quad \text{for} \quad g_s = (g_s)_{opt} = \sqrt{\frac{g_{n0}}{R_{n0}}} \tag{3.24b}$$

F_{min} is a good measure for the noisiness of the amplifier stage for intermediate values of g_s, R_{n0} is a good measure for large values of g_s, and g_{n0} is a good measure for small values of g_s. This is an important distinction when the source conductance g_s must satisfy certain constraints.

3.3
FRIISS' FORMULA—NOISE MEASURE

3.3a. Friiss' Formula

Having defined the noise figure F of a single stage, it is important to know how to calculate the noise figure of a full amplifier if the noise figures of the individual stages can be defined and are known. This leads to the *Friiss' formula*, named after the man who first solved the problem.†

†H. T. Friiss, *Proc. I.R.E.*, 32, 419 (1944).

In order to formulate this formula, we first must divide the amplifier into individual "stages." One thereby uses the convention that the interstage networks belong to the input of the next stage. Only with this convention does Friiss' formula hold.

We further need to define the available gain of an amplifier stage. Let a current generator i_s be connected in parallel with the source conductance g_s. Then the power available at the source, usually called the *available power* and defined as the power fed into a matched load, is

$$P_{av} = \frac{1}{4}\frac{\overline{i_s^2}}{g_s} \tag{3.25}$$

If i_s represents thermal noise of the source, so that $\overline{i_s^2} = 4kT_0 g_s\,\Delta f$,

$$P_{av} = kT_0\,\Delta f \tag{3.25a}$$

If the amplifier without load has an open circuit voltage v_o and an output conductance $g_o > 0$, then the available power P_o at the output is

$$P_o = \frac{1}{4}\overline{v_o^2}g_o \tag{3.26}$$

The *available gain* G_{av} is now defined as

$$G_{av} = \frac{P_o}{P_{av}} = g_o g_s \frac{\overline{v_o^2}}{\overline{i_s^2}} \tag{3.27}$$

With this convention and these definitions the following theorem holds: "If a number of amplifier stages are coupled one behind the other (cascade connection) and for the given coupling to the source and between the stages the individual noise figures $F_1, F_2, F_3, \ldots$ and available gains $G_{av1}, G_{av2}, G_{av3}, \ldots$ are defined, then the noise figure F of the combination is

$$F = 1 + F_1 - 1 + \frac{F_2 - 1}{G_{av1}} + \frac{F_3 - 1}{G_{av1}G_{av2}} + \cdots\text{"} \tag{3.28}$$

This equation is known as *Friiss' formula*. It holds for spot noise figures and under the condition that each stage has a positive output conductance g_o.

We prove the formula for a two-stage amplifier. We already know that the available thermal noise power of a conductance g at the temperature T_0 is $kT_0\,\Delta f$. Therefore, if the first stage has an available gain G_{av1} and a noise figure F_1, the available output noise power of that stage is $G_{av1}\cdot F_1 kT_0\,\Delta f$. If g_o is the output conductance of the first stage, then the noise of the first stage can be represented by an equivalent current generator $\sqrt{G_{av1}\cdot F_1\cdot 4kT_0 g_o\,\Delta f}$ in parallel with g_o. But if the second stage has a noise figure F_2 for the given interstage coupling, then the noise of the second stage minus the thermal noise of g_o can be represented by a noise current generator $\sqrt{(F_2 - 1)\cdot 4kT_0 g_o\,\Delta f}$ in parallel with g_o. The sum of the two (quadratic addition, since the noises are independent!) must equal $\sqrt{FG_{av1}\cdot 4kT_0 g_o\,\Delta f}$,

as is found by representing the noise of the *two* stages by an equivalent current generator in parallel with g_o. Hence

$$FG_{av1} = F_1 G_{av1} + (F_2 - 1) \quad \text{or} \quad F = 1 + (F_1 - 1) + \frac{(F_2 - 1)}{G_{av1}} \tag{3.28a}$$

in agreement with Friiss' formula. In the same way the formula is proved for more stages.

3.3b. Noise Measure†

Sometimes it happens that the noise figure F_1 and the available gain G_{av1} of an amplifier stage are both close to unity. In that case we shall see that the quantity

$$M = \frac{F_1 - 1}{1 - 1/G_{av1}} \tag{3.29}$$

provides a good measure for the noisiness of the stage. It is appropriately called *noise measure*.

To prove Eq. (3.29) we observe that more stages are needed if the available gain of a stage is close to unity. Of course, by adding them we also add more noise. The question is "How much more noise?" To answer this question we couple the individual stages in such a way that each has the same noise figure F_1 and the same available gain G_{av1}. We then have from Eq. (3.28)

$$F = 1 + F_1 - 1 + \frac{F_1 - 1}{G_{av1}} + \frac{F_1 - 1}{G_{av1}^2} + \frac{F_1 - 1}{G_{av1}^3} \cdots$$

which approaches

$$F = 1 + \frac{F_1 - 1}{1 - 1/G_{av1}} = 1 + M \tag{3.29a}$$

for a large number of stages, so that M is indeed a good measure for the noise.

Often G_{av1} is sufficiently large, so that $M \simeq F_1 - 1$. In such cases the noise measure is not needed. However, if $G_{av1} > 1$ but close to unity, the noise measure is quite significant. For $G_{av1} < 1$ the amplifier stage attenuates the signal and adds noise so that one can better do without it. Therefore the noise measure only makes sense for $G_{av1} > 1$.

The noise measure has the following interesting property: *If two stages of noise measures M_1 and M_2 are coupled one behind the other, then the lowest noise measure M of the combination is obtained if the one with lowest noise measure is put in front.*

Let the stages have noise figures F_1 and F_2, available gains G_{av1} and G_{av2},

†H. A. Haus and R. B. Adler, *Circuit Theory of Linear Noisy Networks*, John Wiley & Sons, Inc., New York, 1959.

and noise measures M_1 and M_2; then the noise figures of the two combinations can be denoted by F_{12} (first 1 and then 2) and F_{21} (first 2 and then 1). We now require

$$F_{12} < F_{21} \quad \text{or} \quad (F_1 - 1) + \frac{(F_2 - 1)}{G_{av1}} < (F_2 - 1) + \frac{(F_1 - 1)}{G_{av2}}$$

from which it follows that

$$(F_1 - 1)\left(1 - \frac{1}{G_{av2}}\right) < (F_2 - 1)\left(1 - \frac{1}{G_{av1}}\right)$$

so that $F_{12} < F_{21}$ means indeed $M_1 < M_2$ as needed to be proved.

3.3c. Example in Which Friiss' Formula Is Not Useful

Consider two amplifier stages coupled together. If the output conductance g_o of the first stage is zero, then the available gain of that stage is infinite, but so is the noise figure F_2 of the second stage. To evaluate the noise figure F of the combination, one could make up the limit

$$\lim_{g_o \to 0} \left(\frac{F_2 - 1}{G_{av1}}\right)$$

but rather than doing this we shall follow the simpler approach and calculate the noise figure F by inspection.

To do so we turn to Fig. 3.9. It is seen from this figure that the second stage gives a contribution

$$\frac{4kT_0 g_n \Delta f + 4kT_0 g_L \Delta f}{(g_i + g_L)^2} + 4kT_0 R_n \Delta f \tag{3.30}$$

to $\overline{v_g'^2}$. The noise voltage v_g at the input of the first stage gives the following contribution to $\overline{v_g'^2}$:

$$\frac{g_m^2}{(g_i + g_L)^2}\overline{v_g^2} \tag{3.31}$$

Therefore the noise of the second stage can be represented by an emf $\sqrt{\overline{v_{eq}^2}}$ in series with the input of the first stage, where

$$\overline{v_{eq}^2} = 4kT_0\Delta f\left[\frac{g_n + g_L}{(g_i + g_L)^2} + R_n\right]\frac{(g_i + g_L)^2}{g_m^2} = 4kT_0\Delta f R_n' \tag{3.32}$$

or

$$R_n' = \frac{g_n + g_L + R_n(g_i + g_L)^2}{g_m^2} \tag{3.32a}$$

We thus may apply the theory of the previous section, provided that the noise resistance R_n of the first stage is replaced by

$$R_n'' = R_n + R_n' = R_n + \frac{g_n + g_L}{g_m^2} + R_n\frac{(g_i + g_L)^2}{g_m^2} \tag{3.33}$$

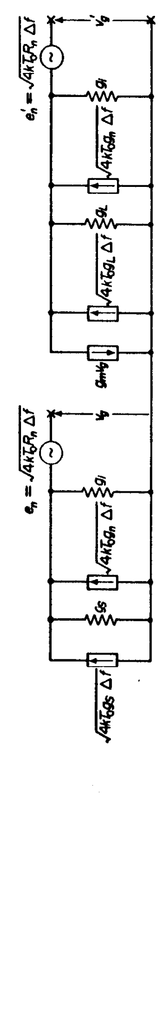

Fig. 3.9. Example of a two-stage amplifier for which Friiss' formula is not useful.

Here the first term gives the contribution of the first device, the second term gives the contribution of the interstage network, and the third term gives the contribution of the second device. By substituting Eq. (3.33) into Eq. (3.23) the effect of the second stage on the noise figure F can be evaluated.

3.4 NOISE FIGURE CONSIDERATIONS FOR NEGATIVE CONDUCTANCE AMPLIFIERS

Friiss' formula was derived under the assumption that the output conductance of each stage was positive. There are a few cases, however, where the output conductance of the stage can be negative; one of the most notable examples is the tunnel diode amplifier. In such a case a more detailed discussion is needed. There are now two approaches to this problem:

1. Extend the concepts of noise figure and available gain so that Friiss' formula can be generalized to cover these cases. This method was used by Haus and Adler† and will be briefly discussed here.
2. Solve the few problems for which this is significant by inspection of the equivalent circuit. This approach will be developed with the help of a few examples.

3.4a. Exchangeable Power, Exchangeable Gain, and Noise Figure

The available power P_{av} of a signal source consisting of a current generator of complex amplitude i_s in parallel with an admittance Y_s, having a positive real part g_s, is defined as the power *delivered* into a matched load Y_s^*, where the asterisk denotes the complex conjugate of Y_s:

$$P_{av} = \frac{1}{8}\frac{i_s i_s^*}{g_s} \tag{3.34}$$

If $g_s < 0$ we define the *exchangeable power* P_{ex} by the definition

$$P_{ex} = \frac{1}{8}\frac{i_s i_s^*}{g_s} \tag{3.34a}$$

It is negative, and its magnitude represents the power *extracted* from a matched load Y_s^*. It is an extension of the available power concept.

The available power gain G_{av} for an amplifier with a source of positive conductance g_s and a positive output conductance g_o is defined as

$$G_{av} = \frac{(P_{av})_{out}}{(P_{av})_{source}} \tag{3.35}$$

If the input source consists again of a current generator i_s in parallel with a source admittance $Y_s = g_s + jb_s$ and the output is represented by a current generator i_o with an output admittance $Y_o = g_o + jb_o$, then

$$G_{av} = \frac{g_s}{g_o}\frac{i_o i_o^*}{i_s i_s^*} \tag{3.35a}$$

If g_s or g_o are negative, the right-hand side of Eq. (3.35a) defines the *exchangeable gain* G_{ex}:

$$G_{ex} = \frac{(P_{ex})_{out}}{(P_{ex})_{in}} = \frac{g_s}{g_o}\frac{i_o i_o^*}{i_s i_s^*} \tag{3.35b}$$

It is an extension of the available gain concept.

We now have to define the *exchangeable thermal noise power* P_{Nex} of a source at temperature T_0. For a negative source conductance g_s at the temperature T_0 we formally define the noise by a short-circuit current generator $\sqrt{\overline{i_{ns}^2}}$, where

$$\overline{i_{ns}^2} = 4kT_0\,|g_s|\,\Delta f \tag{3.36}$$

which is the logical extension of Nyquist's theorem. Therefore the exchangeable noise power of this source is

$$P_{Nex} = \frac{\overline{i_s^2}}{4g_s} = -kT_0\,\Delta f \tag{3.36a}$$

For a source with positive source conductance the available noise power $P_{Nav} = kT_0\,\Delta f$. It is a drawback of the above treatment that the cases $g_s > 0$ and $g_s < 0$ have a different sign for P_{Nav} and P_{Nex}. We shall see that this causes trouble for the extension of Friiss' formula.

The noise figure for $g_s > 0$ and $g_o > 0$ can be defined as

$$F = 1 + \frac{(P_{Nav})_{out},\ \text{due to circuit only}}{(P_{Nav})_{out},\ \text{due to source only}} \tag{3.37}$$

and therefore for the case of arbitrary g_s and g_o it would be logical to define the *exchangeable noise figure* F_{ex} as

$$F_{ex} = 1 + \frac{(P_{Nex})_{out},\ \text{due to circuit only}}{(P_{Nex})_{out},\ \text{due to source only}} \tag{3.37a}$$

But if one does this, it turns out that the second term is always positive and then Friiss' formula does not hold for the case in which the first stage has $g_o < 0$.

To remedy this situation, Haus and Adler proposed the following definition for the exchangeable noise figure:

$$F_{ex} = 1 + \frac{(P_{Nex})_{out},\ \text{due to circuit only}}{G_{ex}kT_0\,\Delta f} \tag{3.37b}$$

This corresponds to eliminating the minus sign in Eq. (3.36a).

We now see that with this definition we have for $g_s > 0$ and $g_o < 0$ that G_{ex} and $(P_{Nex})_{out}$ are both negative so that $(F_{ex} - 1) > 0$. But for $g_s < 0$ and $g_o > 0$, G_{ex} is negative and $(P_{Nex})_{out}$ is positive so that $F_{ex} - 1$ is negative.

†H.A. Haus and R. B. Adler, *Circuit Theory of Linear Noisy Networks*, John Wiley & Sons, Inc., New York, 1959.