CHAPTER 26

Statistical Fluctuations in Nuclear Processes

It is well recognized that we can never measure any physical magnitude exactly, i.e., with no error. Progressively more elaborate experimental or theoretical efforts result only in reducing the possible error of the determination. In reporting the result of any measurements it is therefore obligatory to specify also the probability that the result is in error by some specified amount, since *a gamble on relative correctness is always involved in all physical determinations.* The theory of statistics and fluctuations, summarized here, describes the mathematical procedure involved in the reduction of data, particularly data of the type encountered in nearly every measurement in nuclear physics.

Nuclear processes, in common with all microscopic processes, are random in ultimate character. Because of the relatively large energies released in nuclear processes, it is possible to study single random events. The application of statistical theory to such measurements is therefore doubly important because it contributes to our understanding of nuclear processes and it gives insight into the statistical distributions which describe other random processes whose individual events are not observable. The exponential decay distribution is an example of a result derivable solely from probability considerations (Chap. 15), without detailed knowledge of the mechanism involved.

In any series of measurements, the frequency of occurrence of particular values is expected to follow some "probability distribution law," or "frequency distribution." There are about a half dozen distributions which are used most often in the statistical appraisal and interpretation of nuclear data. We begin by discussing the four most fundamental of these frequency distributions. Later the very useful generalized Poisson distribution (Sec. 3) and the generalized interval distribution (Chap. 28, Sec. 2) will be considered.

The theory presented here is called *efficient statistics* for it extracts the maximum amount of statistical information from the data. A recent development termed *inefficient statistics* can extract a major portion, but not all, of the information by very much simpler calculations. It does this by making reasonable approximations in the efficient theory. It is thus necessary to understand the efficient theory presented here and in Chaps. 27 and 28 in order to be able to use the inefficient theory wisely. For this reason, only the efficient theory is treated here; however, the reader will find some useful inefficient statistics in Appendix G.

1. *Frequency Distributions*

a. The Binomial Distribution. The binomial distribution is the fundamental frequency distribution governing random events. The other frequency distributions can be derived from it.† Historically, it was the first probability distribution to be enunciated theoretically. Bernoulli, early in the eighteenth century, showed that if p is the probability that an event will occur, and $q = 1 - p$ is the probability that it will not occur, then in a random group of z independent trials the probability P_x that the event will occur x times is represented by that term in the binomial expansion of $(p + q)^z$ in which p is raised to the x power. Thus the expansion of $(p + q)^z$, which is always equal to unity, represents the sum of the individual probabilities of observing $x = z$ events, $x = (z - 1)$ events, . . . , $x = 0$ events, as follows:

$$(p + q)^z = p^z + zp^{z-1}q + \frac{z(z-1)}{2!} p^{z-2}q^2 + \cdots + q^z$$
$$= p^z + zp^{z-1}(1 - p) + \frac{z(z-1)}{2!} p^{z-2}(1 - p)^2 + \cdots + (1-p)^z$$
$$= P_z + P_{z-1} + P_{z-2} + \cdots + P_0 = 1$$

Any individual term in this binomial expansion can be written as

$$P_x = \frac{z!}{x!(z-x)!} p^x(1 - p)^{z-x} \tag{1.1}$$

which is the general form of the binomial distribution. The binomial distribution, Eq. (1.1), contains the two independent parameters p and z and rigorously applies to those phenomena in which the total number of trials z and the number of successes x are both *integers.*

It therefore describes the fluctuations in counting α rays from radioactive bodies, provided that p, which is equivalent to the probability $\lambda\,\Delta t$ that a particular atom will decay during an observation of short duration Δt, is constant. Like the normal and Poisson distributions, to be considered next, it represents the true probability when the total amount of radioactive material is essentially unaltered during the

† Representative treatises containing detailed proofs of many of the statements in this chapter include:

T. C. Fry, "Probability and Its Engineering Uses," D. Van Nostrand Company, Inc., New York, 1928.

R. A. Fisher, "Statistical Methods for Research Workers," Oliver & Boyd, Ltd., Edinburgh and London, 1930.

S. S. Wilks, "Mathematical Statistics," Princeton University Press, Princeton, N.J., 1943.

P. G. Hoel, "Introduction to Mathematical Statistics," 2d ed., John Wiley & Sons, Inc., New York, 1954.

N. Arley and K. R. Buch, "Introduction to the Theory of Probability and Statistics," John Wiley & Sons, Inc., New York, 1950.

period of the observations. The tests must, therefore, be made in a time interval Δt which is very short compared with the half-period of the radioactive substance. But under this restriction of small p, Poisson's distribution is a satisfactory approximation to the binomial distribution.

Applications of the binomial distribution to the tossing of coins and the throwing of dice are doubtless familiar to the reader. Here, it applies rigorously because p is constant. Thus the chance of throwing three, two, one, or zero aces in three throws of a single die (or in one throw of three dice) is

$$\begin{aligned}(\tfrac{1}{6} + \tfrac{5}{6})^3 &= (\tfrac{1}{6})^3 + 3(\tfrac{1}{6})^2(\tfrac{5}{6}) + 3(\tfrac{1}{6})(\tfrac{5}{6})^2 + (\tfrac{5}{6})^3 \\ &= \tfrac{1}{216} + \tfrac{15}{216} + \tfrac{75}{216} + \tfrac{125}{216} \\ &= P_3 + P_2 + P_1 + P_0 = 1\end{aligned}$$

Note that the chance of getting no ace in three throws is $\frac{125}{216} = 58$ per cent, although the average number of aces is $pz = 0.50$.

The binomial distribution is a special case of the multinomial distribution describing processes in which several results having fixed probabilities $p_1, p_2, \ldots, p_s$ are possible. The separate probabilities are then given by terms of the expansion $(p_1 + p_2 + \cdots + p_s)^z$, where $p_1 + p_2 + \cdots + p_s = 1$.

b. The Normal Distribution. The normal distribution† is an analytical approximation to the binomial distribution when z is very large. It is applicable to distributions in which the observed variable is not confined to integer values but can take on any value from $-\infty$ to $+\infty$. The normal distribution thus generally applies to a continuously variable observed magnitude, such as the distance separating two spectral lines, while the binomial and Poisson distributions are applied to discontinuous variables, such as particle counting rates, which take on successive whole-number integral values. The statistical theory of errors (D19) is ordinarily based on the normal distribution.

Near the center of the distribution curve the binomial distribution, for large z and constant average value $m = pz$, approaches identity with the normal distribution, which states that the probability dP_x that x will lie between x and $x + dx$ is

$$dP_x = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-m)^2/2\sigma^2}\, dx \tag{1.2}$$

where $e = 2.7183$ is the base of the natural system of logarithms, m is the true value of the quantity whose measured values are x, and σ is the *standard deviation*, a parameter which describes the breadth of the distribution of *deviations* $(x - m)$ from the mean. The standard deviation is discussed in detail in Sec. 2, but for the present it may be regarded simply as one of the two parameters, m and σ, of the normal distribution.

Figure 1.1 illustrates the general form of the normal distribution, drawn for a mean value of $m = 100$ and a standard deviation of $\sigma = 10$.

† The normal-distribution curve is sometimes erroneously referred to as the Gaussian error curve, but its derivation by Gauss (1809) was antedated by those of Laplace (1774) and DeMoivre (1735).

The ordinates are normalized so that the total area under the curve is unity. Thus the area included between any two abscissas x_1 and x_2 is the probability that a single measurement of x will lie between x_1 and x_2, while a very large number of measurements of x would have a mean value of m. The correspondence between Fig. 1.1 and Eq. (1.2) lies in the relationships

$$dP_x = dA = y\,dx \qquad \int_{-\infty}^{\infty} dP_x = A = 1 \tag{1.3}$$

in which y is the ordinate of Fig. 1.1 and dA is an element of area. The coefficient $1/\sigma\sqrt{2\pi}$ in Eq. (1.2) normalizes the area to unity, as given by Eq. (1.3).

Differentiation of Eq. (1.2) shows that the points of maximum slope, at which $d^2y/dx^2 = 0$, fall at the points $(x - m) = \pm\sigma$, where the slope

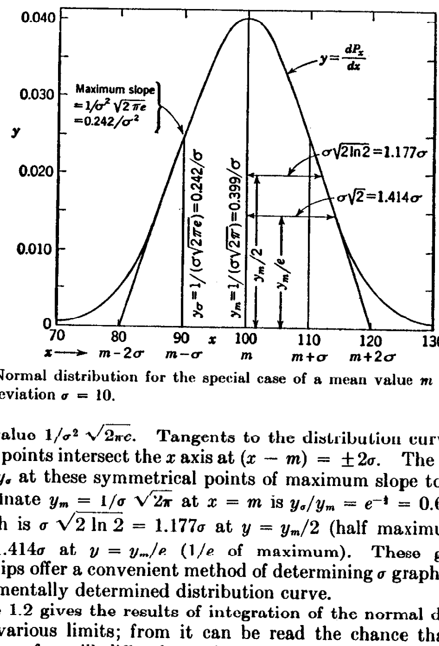


Fig. 1.1 Normal distribution for the special case of a mean value $m = 100$ and a standard deviation $\sigma = 10$.

has the value $1/\sigma^2\sqrt{2\pi e}$. Tangents to the distribution curve at these inflection points intersect the x axis at $(x - m) = \pm 2\sigma$. The ratio of the ordinate y_σ at these symmetrical points of maximum slope to the maximum ordinate $y_m = 1/\sigma\sqrt{2\pi}$ at $x = m$ is $y_\sigma/y_m = e^{-\frac{1}{2}} = 0.6065$. The half width is $\sigma\sqrt{2\ln 2} = 1.177\sigma$ at $y = y_m/2$ (half maximum) and is $\sigma\sqrt{2} = 1.414\sigma$ at $y = y_m/e$ ($1/e$ of maximum). These geometrical relationships offer a convenient method of determining σ graphically from an experimentally determined distribution curve.

Figure 1.2 gives the results of integration of the normal distribution between various limits; from it can be read the chance that a single observation of x will differ from the mean value m by more than any arbitrary assigned amount. Figure 1.2 has many other uses which will be referred to later.

c. The Poisson Distribution. Poisson's distribution *describes all random processes whose probability of occurrence is small and constant.* It therefore has wide and diverse applicability and describes the statistical fluctuations in such random processes as the number of soldiers kicked and killed yearly by cavalry horses, the disintegration of atomic nuclei, the emission of light quanta by excited atoms, and the appearance of

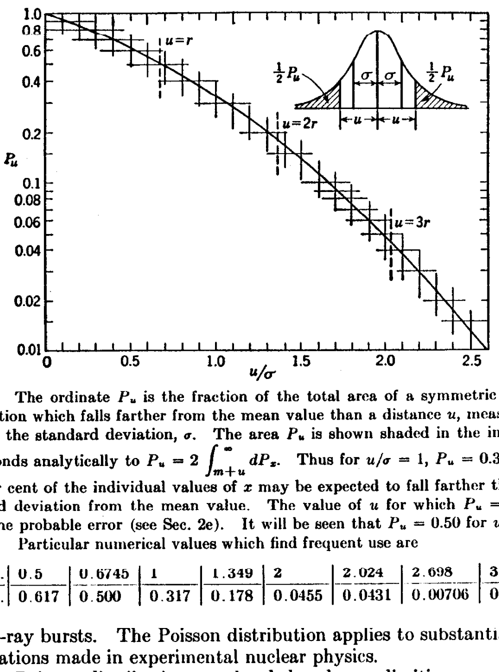


Fig. 1.2 The ordinate P_u is the fraction of the total area of a symmetric normal distribution which falls farther from the mean value than a distance u, measured in units of the standard deviation, σ. The area P_u is shown shaded in the inset and corresponds analytically to $P_u = 2\int_{m+u}^{\infty} dP_x$. Thus for $u/\sigma = 1$, $P_u = 0.317$, and 31.7 per cent of the individual values of x may be expected to fall farther than one standard deviation from the mean value. The value of u for which $P_u = 0.50$ is called the probable error (see Sec. 2e). It will be seen that $P_u = 0.50$ for $u = r = 0.6745\sigma$. Particular numerical values which find frequent use are

u/σ.....	0.5	0.6745	1	1.349	2	2.024	2.698	3
P_u......	0.617	0.500	0.317	0.178	0.0455	0.0431	0.00706	0.00272

cosmic-ray bursts. The Poisson distribution applies to substantially all observations made in experimental nuclear physics.

The Poisson distribution can be deduced as a limiting case of the binomial distribution, for those random processes in which the probability of occurrence is very small, $p \ll 1$, while the number of trials z becomes very large and the mean value $m = pz$ remains fixed. Then in Eq. (1.1) $m \ll z$ and $x \ll z$, and so we can write, approximately,

$$\frac{z!}{(z-x)!} \simeq z^x$$
$$(1-p)^{z-x} \simeq e^{-p(z-x)} \simeq e^{-pz}$$

and in the limiting case of small probability p, Eq. (1.1) approaches

$$P_x = \frac{z^x p^x}{x!} e^{-pz} = \frac{m^x}{x!} e^{-m}$$

which is the Poisson distribution.

A much clearer feeling for the statistical principles underlying the Poisson distribution is obtained by deriving this frequency distribution from first principles. Specializing the general conditions under which the Poisson distribution holds to the readily visualized case of a radioactive disintegration, we would write the following necessary and sufficient conditions:

1. The chance for an atom to disintegrate in any particular time interval is the same for all atoms in the group (all atoms identical).
2. The fact that an atom has disintegrated in a given time interval does not affect the chance that other atoms may disintegrate in the same time interval (all atoms independent).
3. The chance for an atom to disintegrate during a given time interval is the same for all time intervals of equal size (mean life long compared with the total period of observation).
4. The total number of atoms and the total number of equal time intervals are large (hence statistical averages significant).

Let a be the average rate of appearance of particles from such a random process; then the average number of events in a time interval t is at. Then in a short time interval, dt, such that $a\,dt \ll 1$, the quantity $a\,dt$ is simply the probability $P_1\,(dt)$ of observing one particle in the time dt. As dt decreases without limit, the probability of observing two or more particles in the time dt becomes vanishingly small in comparison with the probability of observing one particle, that is, $P_1\,(dt) \gg P_2\,(dt) \gg P_3\,(dt)$ . . . . The probability of observing no particle in dt is

$$P_0\,(dt) = 1 - P_1\,(dt) = 1 - a\,dt$$

We may now write the probability of observing x particles in the time $(t + dt)$ as the combined probabilities of $(x-1)$ particles in t and one in dt, and of x particles in t and none in dt; thus

$$\begin{aligned} P_x(t+dt) &= P_1\,(dt)\,P_{x-1}(t) + P_0\,(dt)\,P_x(t) \\ &= a\,dt\,P_{x-1}(t) + (1 - a\,dt)\,P_x(t) \end{aligned} \tag{1.4}$$

Rewriting Eq. (1.4) in differential form, we have

$$\begin{aligned} \frac{dP_x(t)}{dt} &= \frac{P_x(t+dt) - P_x(t)}{dt} \\ &= a[P_{x-1}(t) - P_x(t)] \end{aligned} \tag{1.5}$$

The solution (B19) of Eq. (1.5) is

$$P_x(t) = \frac{(at)^x}{x!} e^{-at} \tag{1.6}$$

as can be verified by differentiation. Now if the equal time intervals are chosen of length t, then the average number of particles per interval is $at = m$; substituting this in Eq. (1.6), we have the usual form of the Poisson distribution

$$P_x = \frac{m^x}{x!} e^{-m} \tag{1.7}$$

in which P_x is the probability of observing x events when the average for a large number of tries is m events. Although m may have any positive value, x is restricted to integer values only. It is easy to show that Eq. (1.7) correctly leads to

$$\sum_{x=0}^{x=\infty} P_x = 1$$

It must be noted that, in contrast with the two previous frequency distributions, *the Poisson distribution has but one parameter m*. The binomial distribution with parameters p and z becomes identical with the Poisson distribution when $p \to 0$ in such a way that $zp = m$. The normal distribution, near the center of the distribution, approaches equality with Poisson's distribution when m is large, so that the histogram of Poisson's discontinuous distribution approaches the continuous normal distribution (compare Figs. 1.1 and 1.3).

Following the numerical evaluation† of a particular value of P_x, other neighboring values may be computed quickly by using the following exact relationships, which can be derived easily from Eq. (1.7)

$$P_{x-1} = \frac{x}{m} P_x \tag{1.8}$$

$$P_{x+1} = \frac{m}{x+1} P_x \tag{1.9}$$

$$P_0 = e^{-m} \tag{1.10}$$

The Poisson distribution is slightly asymmetric, favoring low values of x. Thus substitution of $x = m$ in Eq. (1.8) shows that if the mean value is an integer *the probability of observing one less than the mean value is the same as the probability for the mean value.*

In computations with Eq. (1.7) it is often convenient to use Stirling's approximation to the factorial

$$x! = \sqrt{2\pi x}\, x^x e^{-x} \left(1 + \frac{1}{12x} + \cdots\right) \tag{1.11}$$

in which neglect of the final parentheses involves a negative error of only

† Extremely useful tables of the individual terms P_x and especially of the cumulated terms $\sum_{x}^{\infty} P_x$ for Poisson distributions with $m = 0.001$ to 100 have been published by E. C. Molina (M49).

0.8 per cent for $x = 10$, and 0.08 per cent when $x = 100$. Then Eq. (1.7) becomes, to a good approximation, for $x > 10$

$$P_x \simeq \frac{1}{\sqrt{2\pi x}} \left(\frac{m}{x}\right)^x e^{-(m-x)} \tag{1.12}$$

The probability of actually observing the mean value m in a series of observations on a random process of constant average value is surprisingly small. This is seen by substitution of $x = m$ in Eq. (1.12) which gives the following values

$m =$	10	100	1,000
$P_m =$	0.127	0.040	0.013

The Poisson distribution must always be represented by a histogram, since x must assume whole-number values only. Figure 1.3 illustrates the Poisson distribution for $m = 100$; the slight asymmetries should be noted, as well as the similarity with the symmetric normal distribution

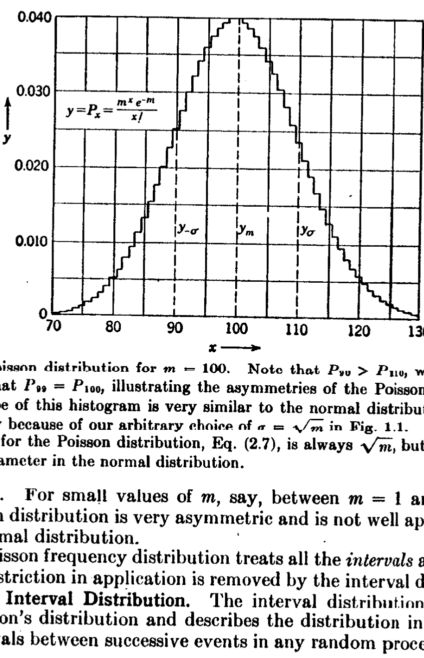


Fig. 1.3 Poisson distribution for $m = 100$. Note that $P_{90} > P_{110}$, whereas $P_{70} < P_{130}$; also that $P_{99} = P_{100}$, illustrating the asymmetries of the Poisson distribution. The envelope of this histogram is very similar to the normal distribution shown in Fig. 1.1 only because of our arbitrary choice of $\sigma = \sqrt{m}$ in Fig. 1.1. The standard deviation, σ for the Poisson distribution, Eq. (2.7), is always $\sqrt{m}$, but σ is an independent parameter in the normal distribution.

of Fig. 1.1. For small values of m, say, between $m = 1$ and $m = 10$, the Poisson distribution is very asymmetric and is not well approximated by the normal distribution.

The Poisson frequency distribution treats all the *intervals* as *independent;* this restriction in application is removed by the interval distribution.

d. The Interval Distribution. The interval distribution is derived from Poisson's distribution and describes the distribution in size of the time intervals between successive events in any random process in which the mean rate has the constant value of a events per unit time (M12, R20). From Eq. (1.7) or (1.10) the probability that there will be no events in a time interval t, during which time there should be at events on the average, is†

$$P_0 = \frac{(at)^0}{0!} e^{-at} = e^{-at} \tag{1.13}$$

The probability that there will be an event in the time interval dt is simply $a\,dt$. The combined probability that there will be no events during the time interval t, but one event between time t and $t + dt$, is $ae^{-at}\,dt$. Hence, in a random distribution which follows the Poisson distribution and has a constant average interval of $1/a$, the probability dP_t that the duration of a particular interval will be between t and $t + dt$ is

$$dP_t = ae^{-at}\,dt \tag{1.14}$$

We see at once that *small time intervals have a higher probability* than large time intervals between the randomly distributed events.

If the data concern a large number N of intervals, then the number of intervals greater than t_1 but less than t_2 is

$$\begin{aligned} n &= N \int_{t_1}^{t_2} ae^{-at}\,dt \\ &= N(e^{-at_1} - e^{-at_2}) \end{aligned} \tag{1.15}$$

where a is the average number of events per unit time. Equations (1.14) and (1.15) are the general differential and integral forms of the interval distribution for randomly spaced events.

Two limiting cases are of special interest. Letting $t_2 \to \infty$, we find that *the number of intervals greater than any duration t is Ne^{-at}*, in which at is simply the average number of events in the interval t. Because the average interval is $\bar{t} = 1/a$, we note that the fraction of the intervals which are longer than the average is $n/N = e^{-1} = 0.37$.

Letting $t_1 \to 0$, Eq. (1.15) shows that *the number of intervals shorter than any duration t is $N(1 - e^{-at})$*. Examples of the usefulness of the interval distribution in α-ray counting experiments and in cosmic-ray-burst observations will be given in Chap. 27, Sec. 4. A generalization of the interval distribution, giving the frequency distribution of intervals which contain any predetermined number of random events, is derived in Chap. 28, Sec. 2.

Problems

"The reader is, however, advised that the detailed working of numerical examples is essential to a thorough grasp, not only of the technique, but of the principles by which an experimental procedure may be judged to be satisfactory and effective." (R. A. Fisher, in the preface to the first edition of his "The Design of Experiments," Oliver & Boyd, Ltd., Edinburgh and London, 1935.)

† That factorial zero equals unity follows from the gamma functions:

$$n! = \Gamma(n+1) \qquad \Gamma(1) = 1 \qquad \Gamma(0+1) = 1$$

1. From elementary probability arguments and consideration of the number of combinations and permutations of z things taken x at a time, "derive" the binomial distribution.

2. In 1693 (hence pre-Bernoulli and pre-binomial distribution), Samuel Pepys propounded the following question to his friend Isaac Newton, who prepared a lengthy response and engaged the tax accountant George Tollet in a protracted controversy over the answer:

"A has 6 dice in a box, with which he is to fling a six, B has in another box 12 dice, with which he is to fling 2 sixes, C has in another box 18 dice, with which he is to fling 3 sixes. Question—whether B and C have not as easy a task as A at even luck."

(*a*) Assuming Pepys meant *exactly* one, two, and three sixes for the three contestants, what are their chances of succeeding on a single throw?

(*b*) Assuming he meant *at least* one, two, and three sixes, in what direction will this modify their chances of success?

3. With a simultaneous throw of six dice, calculate the probabilities of obtaining just zero, one, two, three, four, five, and six sixes. Show that the sum of these probabilities is unity when the solutions are obtained using the binomial distribution. Compare with these the probabilities for the same events as given by the Poisson distribution. Point out the reasons for these differences.

4. On June 5, 1951, Dom DiMaggio had a batting average of 0.359, had been at bat 189 times in 44 games, and had hit safely at least once in each of his last 25 consecutive games.

(*a*) What is the probability that he will hit safely if he is at bat four times in the baseball game on June 6, 1951?

(*b*) What is the probability of his hitting safely in 26 consecutive games, if he is at bat four times in each game?

(*c*) Explain concisely why the odds in (*a*) and (*b*) are so different.

5. Consider the chances of a bomber pilot surviving a series of statistically identical raids, in which the chance of being shot down is always 5 per cent.

(*a*) From an original group of 1,000 such pilots, how many should survive 1, 5, 10, 15, 20, 40, 80, and 100 raids? Plot the survival curve, with the number of flights as abscissa.

(*b*) Estimate the mean life of a pilot in number of raids.

(*c*) In a single raid of 100 planes, what are the chances that zero, one, five, or ten planes will be lost?

6. Calculate and plot a normal distribution having a mean value of 10 and a standard deviation of 3.

7. Show that the sum of the probabilities $P_x = e^{-m}m^x/x!$ of all possible positive values $x = 0, 1, 2, \ldots$ is unity for the Poisson distribution.

8. In any Poisson distribution, show analytically that the probability of observing one less than the mean value is the same as the probability for the mean value.

9. In any Poisson distribution,

(*a*) Show that the ratio of the probability P_{2m} of observing twice the mean value to the probability P_m of observing the mean value is

$$P_{2m}/P_m = m^m m!/(2m)!$$

or when $m \gg 1$,

$$P_{2m}/P_m = 0.707(0.824)^{2m}$$

(*b*) Compare P_{2m}/P_m for $m = 2, 10, 100$.

10. Calculate and plot a Poisson distribution with a mean value of 10 and values of x from 0 to 20. Compare with the normal distribution for $m = 10$, $\sigma = 3$.

11. The average background of a certain α-ray counter is 20 α rays per hour. In how many hours out of 200 would you expect to observe only 10 α rays?

12. In an α-ray counting experiment, on a source of constant average intensity, a total of 19,278 α rays are counted in 51 hr of continuous observations. The time of arrival of each α ray is recorded on a tape, so that the number of α rays recorded in each successive 1-min interval can be determined.

(*a*) What is the average number of α rays per 1-min interval?

(*b*) In how many of the total number of 1-min intervals would you expect to observe no α rays?

(*c*) In how many 1-min intervals should one observe one α ray?

(*d*) In how many 1-min intervals should one observe six α rays?

13. A Geiger-Müller counter having a resolving time of 300 μsec is placed in a plane parallel beam of 5-Mev photons from a pulsed generator. The counter has a cylindrical cathode 2 cm in diameter and 10 cm long, is placed with its axis perpendicular to the beam, and has an absolute efficiency of 2 per cent for 5-Mev photons. Each pulse of photons from the generator is 50 μsec in duration, and the repetition rate is 120 pulses per second. Under these conditions of operation, the counter displays an average counting rate of 3,600 counts per minute. During each pulse, what is the flux in photons per second per square centimeter at the counter position?

14. In a random distribution having an average interval $\bar{t}$, show by application of the interval distribution that the average value of the *absolute* deviations from the mean interval $\bar{t}$ is

$$|t - \bar{t}|_{av} = \frac{2}{e}\bar{t} = 0.7358\bar{t}$$

15. The average background of a certain α-ray counter is 30 α rays per hour.

(*a*) What fraction of the intervals between successive counts will be longer than 5 min?

(*b*) What fraction of the intervals will be longer than 10 min?

(*c*) What fraction of the intervals will be shorter than 30 sec?

16. A certain radioactive sample contains a mixture of an α-ray emitter and a β-ray emitter. The two substances are assumed to be independent. Using a particular pair of counters, the observed activities are A α counts per minute and B β counts per minute.

(*a*) What is the combined probability that a particular interval between two successive α rays will have a duration between t and $t + dt$ and will also contain exactly x β rays?

(*b*) Show that the probability $M(x)$ of observing just $x (x = 0, 1, 2, \ldots)$ β rays in the time interval between *any* two successive α-ray counts is

$$M(x) = \frac{R^x}{(R + 1)^{x+1}}$$

where $R = B/A$.

(*c*) What is the probability of observing just one α ray in the time interval between *successive β rays* if $A = 100$ and if $B = 500$ counts per minute?

17. Derive an interval distribution governing the output of a scale-of-2 circuit if the input receives randomly distributed pulses at an average rate of a pulses per minute. Specifically,

(*a*) Show that the number n of observed intervals of length between t_1 and t_2 min, when N is the total number of scale-of-2 intervals studied, is

$$\frac{n}{N} = (at_1 + 1)e^{-at_1} - (at_2 + 1)e^{-at_2}$$

(*b*) Derive an expression for the fraction n/N of the intervals which will be longer than t.

(*c*) Derive an expression for the fraction of the intervals which will be shorter than t.

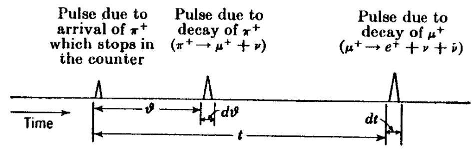
18. A scale-of-2 counter gave 292 pulses in 11 hr.

(*a*) What are the duration of the average interval and the average rate of the statistical process (scale of 1)?

(*b*) Compute the number of scale-of-2 intervals expected to be longer than 7.5 min. (Four were observed.)

(*c*) Compute the number of scale-of-2 intervals expected to be shorter than 5 sec. (One was observed.)

19. An interval distribution is to be derived, which will describe the distribution of time intervals between the arrival of π^+ mesons in a counter and the decay of the daughter μ^+ meson. The events to be considered are:



If λ_1 and λ_2 are the decay constants of the π^+ and μ^+ meson,

(*a*) What is the probability that the π^+ meson will decay in the time interval between ϑ and $\vartheta + d\vartheta$?

(*b*) What is the probability that a μ^+ meson will decay in the time interval between t and $t + dt$ after the arrival of its parent π^+ meson?

(*c*) What is the most probable time interval t_0 between arrival of π^+ and decay of μ^+? Use a mean life of 0.02 μsec for π^+, and of 2.0 μsec for μ^+.

2. *Statistical Characterization of Data*

a. Mean Value. In any *finite* series of measurements we can never find the *exact* value of the true mean value m, which corresponds to the infinite population of (i.e., an infinite amount of) data. Although the mean value† is constant, our individual measurements should be distributed about this mean value in a manner given by the particular frequency distribution which describes the process being studied. For the

† The *mean* value (i.e., the average value) is to be distinguished from the *modal* value (i.e., the most probable value) and from the *median* value (i.e., the value which is as frequently exceeded as not). Only for a symmetric distribution are the mean, mode, and median coincident. For an asymmetric distribution of numbers such as: 2, 3, 5, 6, 7, 8, 8, 9, 9, 9, 11, 11, 12, the mean value is 7.69, the mode is 9, and the median is 8.

four frequency distributions discussed in the previous section, it can be shown that our best approximation to m is simply the arithmetic average $\bar{x}$ of the n separate measurements, $x_1, x_2, x_3, \ldots, x_n$; that is,

$$m \simeq \bar{x} = \frac{1}{n}\sum_{1}^{n} x_i \tag{2.1}$$

The "expectation value" of any statistical variable is synonymous with the mean value obtained for this variable in a large number of trials. Thus the *expectation value* of x is m.

b. Standard Deviation and Variance. The breadth of the statistical fluctuations of our individual readings about the true mean value is expressed quantitatively by the fundamentally important parameter, the *standard deviation* σ. For a particular mean value m, a small σ gives a sharply peaked distribution, whereas a large σ gives a broad, flattened distribution. In any case, the significance of the standard deviation as descriptive of the spread of the data is best seen in the normal distribution. Figure 1.2 shows that, in the normal distribution, about 32 per cent of a large series of individual observations must deviate from the mean value by more than $\pm\sigma$ and consequently that 68 per cent of the individual observations should lie within the band $(\bar{x} \pm \sigma)$.

For any frequency distribution, the standard deviation (often abbreviated S.D.) is defined as the square root of the average value of the square of the individual deviations $(x - m)$, for a large number of observations. Thus

$$\sigma^2 = \sum_{x=-\infty}^{x=\infty} (x - m)^2 P_x \tag{2.2}$$

or, in terms of a large series of n measurements of x,

$$\sigma^2 = \frac{1}{n}\sum_{i=1}^{i=n} (x_i - m)^2 \tag{2.3}$$

The square of the S.D. is thus seen to be simply the second moment of the frequency distribution taken about the mean. The quantity σ^2 is usually called the *variance*. As is suggested by the form of Eq. (2.3), σ occasionally is called the *root-mean-square error*.

We can now use Eq. (2.2) for the derivation of analytical expressions for the S.D. of the various distributions.

For the binomial distribution, with mean value $m = zp$, the square of the standard deviation is

$$\sigma^2 = \sum_{x=0}^{x=z} (x - zp)^2 P_x = \sum_{x=0}^{x=z} \frac{(x - zp)^2 z!\, p^x (1 - p)^{z-x}}{x!(z - x)!}$$

Upon expansion and summation, this expression reduces to simply

$$\sigma^2 = zp(1 - p) \tag{2.4}$$

or, because the mean value of x is $m = zp$, we have also

$$\sigma^2 = m(1 - p) \tag{2.5}$$

Note especially that, for the binomial distribution, the variance σ^2 is always less than the mean value m $(\simeq \bar{x})$.

For the normal distribution, the evaluation of Eq. (2.2) by integration gives, of course,

$$\sigma^2 = \int_{-\infty}^{+\infty} (x - m)^2\, dP_x = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} (x - m)^2 e^{-(x-m)^2/2\sigma^2}\, dx = \sigma^2 \tag{2.6}$$

because the S.D. is simply one of the two independent parameters of the normal distribution and therefore may have any value.

For the Poisson distribution, however, the S.D. has a definite value in terms of the mean value, which is the only parameter of the Poisson distribution. From Eq. (2.2), we find on expansion that

$$\sigma^2 = \sum_{x=0}^{x=\infty} \frac{(x - m)^2 m^x}{x!} e^{-m} = m \tag{2.7}$$

Hence for the Poisson distribution, the S.D. of the distribution of individual observations is simply $\sqrt{m}$. This result is, of course, in agreement with the S.D. of the binomial distribution, Eq. (2.5), in the limiting case for $p \ll 1$.

For the interval distribution governing randomly distributed events occurring at an average rate a, hence with average interval $\bar{t} = 1/a$, Eqs. (1.14) and (2.2) lead to

$$\sigma^2 = \int_0^{\infty} \left(t - \frac{1}{a}\right)^2 a e^{-at}\, dt = \left(\frac{1}{a}\right)^2 \tag{2.8}$$

Thus the S.D. is just equal to the average interval $1/a$.

Table 2.1 now summarizes the properties of the four frequency distributions which apply to random processes having a constant average value.

c. Estimate of Standard Deviation from a Finite Series of Observations. In a finite series of n observations, we can never know m exactly. Hence we can never determine σ exactly, as implied in Eq. (2.2) which applies to the infinite population of data. Our best approximation to the S.D. of the distribution, in terms of our finite number n of observations, can be shown to be

$$\sigma^2 \simeq \frac{n}{n-1}\left[\frac{1}{n}\sum_{1}^{n} (x_i - \bar{x})^2\right] = \frac{1}{n-1}\sum_{1}^{n} (x_i - \bar{x})^2 \tag{2.9}$$

Table 2.1. Summary of Frequency Distributions Describing Random Processes

For ease of reference we include two generalizations which are to be described in Sec. 3 of this chapter and in Chap. 28, Sec. 2.

Frequency distribution	Distribution	Average value	Standard deviation	Distribution is approximation to binomial when
Binomial	$P_x = \frac{z!}{x!(z-x)!} p^x (1-p)^{z-x}$	$\bar{x} = zp$	$\sqrt{zp(1-p)}$	
Normal	$dP_x = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-m)^2/2\sigma^2} dx$	$\bar{x} = m$	σ	$z \gg 1$
Poisson	$P_x = \frac{m^x}{x!} e^{-m}$	$\bar{x} = m$	$\sqrt{m}$	$p \ll 1;\ pz = m$
Interval	$dP_t = ae^{-at}\,dt$	$\bar{t} = \frac{1}{a}$	$\frac{1}{a}$	
Generalized Poisson, this chapter, Sec. 3	$P_u = aP_x - bP_y + cP_z + \cdots$	$u = ax + by + cz + \cdots$	$\sigma = \sqrt{a^2x + b^2y + c^2z + \cdots}$	
Generalized interval, Chap. 28, Sec. 2	$dP_t = \frac{a^s t^{s-1} e^{-at}}{(s-1)!} dt$	$\bar{t} = \frac{s}{a}$	$\frac{\sqrt{s}}{a}$	

This practical expression for the (S.D.)² differs from Eq. (2.3) only in its denominator $(n - 1)$ and in the use of $\bar{x}$ in place of m. The term $(n - 1)$ is to be correlated with the view that the dispersion among the data is associated with the number of "degrees of freedom." From n independent observations of x we are provided originally with n independent equations. We reduce this number by one when we compute $\bar{x}$ and hence have only $(n - 1)$ independent data from which to compute σ. It can be seen readily that, in the special case in which only one observation is made, $\bar{x} = x$, and σ is indeterminate. The latter condition is correctly given by Eq. (2.9) but could not be obtained from Eq. (2.3) directly.

In the theory of mathematical statistics the so-called "sample variance" s^2 is defined as

$$s^2 \equiv \frac{1}{n} \sum_{1}^{n} (x_i - \bar{x})^2 \tag{2.10}$$

It can be shown quite generally that the *expectation value* $E[s^2]$ for the *sample variance* of n observations is

$$E[s^2] = \frac{n-1}{n} \sigma^2 \tag{2.11}$$

This is the formal basis for our Eq. (2.9) which we shall use hereafter without further explicit reference to the sample variance.

d. Standard Deviation of the Mean Value (Standard Error). If our n individual measurements of x exhibit, say, an approximately normal distribution about the mean value $\bar{x}$, then Fig. 1.2 shows that some 68 per cent of our individual observations have fallen within the central band $\bar{x} \pm \sigma$. This means that *one* additional single observation, if made, would have a 68 per cent chance of lying within $\bar{x} \pm \sigma$. In recognition of this probability interpretation, the "*standard deviation of the distribution*" σ, as determined from Eq. (2.3) or (2.9), can be called more precisely the "*standard deviation of a single observation.*"

Obviously, if we, or another observer, were to repeat our entire experiment of n observations, we should expect the new mean value to have much greater than a 68 per cent chance of falling within $\bar{x} \pm \sigma$. Therefore, in reporting our mean value $\bar{x}$, we wish to assign to it a S.D. of the mean value $\sigma_{\bar{x}}$ such that there is approximately a 68 per cent chance that some new mean value $\bar{x}_2$ will lie within the band $(\bar{x} \pm \sigma_{\bar{x}})$. Obviously, $\sigma_{\bar{x}}$ is smaller than σ.

It is well known in the theory of errors that a series of k mean values, $\bar{x}_1, \bar{x}_2, \bar{x}_3, \ldots, \bar{x}_k$, each based on n observations, will tend to exhibit a normal distribution about their grand average $\bar{\bar{x}}$. This is true if n is sufficiently large, even if the parent population x_i is not normally distributed but is, for example, an asymmetric Poisson distribution. In general, *the distribution of mean values tends to be much more nearly normal than the parent population.* This is the justification for a theoretical derivation of the relationship between σ and $\sigma_{\bar{x}}$, based on a normal distribution of $\bar{x}$. Then it can be shown that in a large series of k measurements of the mean value $\bar{x}$, each based on n measurements of x, the grand average $\bar{\bar{x}}$ approaches the true mean m and that the S.D. of $\bar{x}$ depends upon n in the following way

$$\sigma_{\bar{x}}^2 = \frac{1}{k} \sum_{j=1}^{j=k} (\bar{x}_j - m)^2 = \frac{\sigma^2}{n} \tag{2.12}$$

The result of a single series of n measurements of x is then to be reported as $(\bar{x} \pm \sigma_{\bar{x}})$, where

$$\bar{x} = \frac{1}{n} \sum_1^n x_i$$

$$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}} \simeq \sqrt{\frac{1}{n(n-1)} \sum_1^n (x_i - \bar{x})^2} \tag{2.13}$$

Then a repetition of the series of n measurements would, in general, give a different mean value, but the chance that the new mean value would lie within $(\bar{x} \pm \sigma_{\bar{x}})$ is 68 per cent. The S.D. of the mean value $\sigma_{\bar{x}}$ is often called the *standard error*.

The validity of Eq. (2.12) or (2.13) is almost self-evident for the Poisson distribution. Suppose a total of $v = \sum_1^n x_i$ random events are observed. Then by Eq. (2.7) the S.D. in this single observation is $\sqrt{\sum_1^n x_i}$, so that the result would be reported as

$$v \pm \sigma = \sum_1^n x_i \pm \sqrt{\sum_1^n x_i} \tag{2.14}$$

and the *fractional S.D.* would be

$$\text{F.S.D.} = \frac{\sigma}{v} = \frac{1}{\sum_1^n x_i} \sqrt{\sum_1^n x_i} = \frac{1}{\sqrt{\sum_1^n x_i}} \tag{2.15}$$

Suppose now that a zealous assistant was present, while you tallied only

the total number $\sum_1^n x_i$ events, and that he broke the data into n contiguous and equal intervals, recording

$$x_1 + x_2 + x_3 + \cdots + x_n = \sum_1^n x_i$$

Then he would obtain

$$\bar{x} = \frac{1}{n} \sum_1^n x_i$$

$$\sigma = \sqrt{m} \simeq \sqrt{\bar{x}}$$

$$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}} = \frac{1}{n} \sqrt{\sum_1^n x_i}$$

and the result of the measurements would be reported as

$$\bar{x} \pm \sigma_{\bar{x}} = \frac{1}{n} \left(\sum_1^n x_i \pm \sqrt{\sum_1^n x_i} \right) \tag{2.16}$$

which has the same *fractional* S.D. as Eq. (2.15). In fact, Eq. (2.16) could have been obtained directly from Eq. (2.14) by simply dividing by the number of classifications n into which the data were subdivided. In either case, and in general, the observation of a total of $\sum_1^n x_i$ randomly distributed events has a *fractional* S.D. of $1 / \sqrt{\sum_1^n x_i}$. Thus the S.D. in counting 100 random events is 10 per cent, and one must count 10,000 events to reduce the S.D. to 1 per cent. *No mere method of treating the same total data can ever reduce the magnitude of the fractional uncertainty due purely to randomicity.*

e. Probable Error. A result quoted as $\bar{x} \pm \sigma_{\bar{x}}$ implies that the chance that the average value $\bar{x}$ differs from the true mean value m by more than $\sigma_{\bar{x}}$ is 0.317, if the error distribution is normal. While the S.D. has a definite statistical value and a basic significance in the principal frequency distributions, it is becoming increasingly common to fail to use it in reporting physical results. Instead, a quantity derived from the S.D. and called the probable error is often given. Its wide adoption rests on its easily visualized interpretation and perhaps also on the fact that, of all the common types of error specification, the probable error has the least value and hence makes the data look best.

The probable error is, by definition, exactly as likely to be exceeded as not. The probable error is ordinarily derived from the S.D. From Fig. 1.2 it can be seen that the particular error r which has exactly a 0.50

chance of being exceeded in a *normal distribution* is

$$r = 0.6745\sigma \qquad \text{and } r_{\bar{x}} = 0.6745\sigma_{\bar{x}} \tag{2.17}$$

Similarly, for a normal distribution, the chance that the actual error $(m - \bar{x})$ exceeds $r_{\bar{x}}$, $2r_{\bar{x}}$, $3r_{\bar{x}}$, etc. (without regard to sign) is given in the following table:

Chance that $\lvert m - \bar{x} \rvert$ is greater than	$r_{\bar{x}}$	$2r_{\bar{x}}$	$3r_{\bar{x}}$	$4r_{\bar{x}}$
Is	0.500	0.178	0.043	0.0071

Intermediate values may be read from Fig. 1.2. It is customary therefore to regard $3r_{\bar{x}}$ (or $2\sigma_{\bar{x}}$) as equivalent to the *limit of error*, though this is clearly arbitrary and unreal rigorously. Moreover, the specification of a physical result as $\bar{x} \pm r_{\bar{x}}$ is exact *only* for a symmetric normal distribution.

Any asymmetry in the actual distribution will result in the probable positive error differing from the probable negative error for single observations; that is, ordinates of the distribution curve at $x = 0$, $(m - r)$, m, $(m + r)$, ∞, no longer divide the errors (area) into four equal parts (quartiles). Of course, an analogous objection can often be made to the lack of significance of the plus and minus sign if used with the S.D. of an asymmetric distribution. The only rigorous interpretation of S.D. of an asymmetric distribution is as root-mean-square error, Eqs. (2.9) and (2.13), and not as a plus or minus value having symmetric probabilities of being exceeded. It is only because the asymmetry of the Poisson distribution becomes small, and because this distribution approaches the normal distribution in the vicinity of the mean value when $m \gg 1$, that probable error can have any exact significance.

Graphical integration of Poisson distributions shows that the asymmetry is of the order of 10 to 4 per cent for m = 10 to 100 and vanishes as $m \to \infty$. The asymmetry for $m \geq 10$ does not invalidate Eq. (2.17), but for $m < 10$ much more significance attaches to the S.D. than to the probable error. The general dependence of r on $\sigma = \sqrt{x}$ for the Poisson distribution is given in the following table (R25):

x	20	60	100	200	400	1,000	∞
$\frac{r}{\sigma} = \frac{r}{\sqrt{x}}$	0.575	0.613	0.628	0.640	0.647	0.660	0.6745

It will be noted that use of the conventional expression $r = 0.6745\sigma$ even for the Poisson distribution results in a conservative estimate of the probable error and is a safe procedure to follow.

If the mean value of an asymmetric distribution is estimated from a very large number n of observations, then the probable error of the mean value can have a true "plus-or-minus" significance, because the distribution of mean values is always more nearly normal than the parent population.

f. Dimensions of Statistical Parameters. From consideration of Eq. (2.7) or (1.7), it is evident that both x and σ must be *dimensionless* quantities, since σ has the same dimensions as x and $\sqrt{x}$. It is generally true that all such quantities in the distribution functions and in other statistical expressions are without dimensions. For example, $\bar{x}$ may be physically the average number of counts per minute, but statistically the time unit chosen is only an arbitrary interval or classification by which the data have been taken. It is to be regarded statistically as dimensionless. This can be visualized by considering time intervals measured off on a chronograph tape, in which case the particular interval used for classification might equally well be one second, or an equivalent length of tape, or even an equivalent mass of tape. The interval itself does not have the dimensions of time, length, or mass but is always statistically dimensionless, as are all the other basic statistical quantities.

While the interval distribution Eq. (1.14) contains the rate a in events per unit time, it always occurs in the product at or $a\,dt$, which is again dimensionless.

Problems

1. Prove analytically that the standard deviation is $\sqrt{m}$ for any Poisson distribution.

2. In the interval distribution for single randomly distributed events, show that the standard deviation is just equal to the average interval, that is, $\sigma = 1/a$.

3. In computing the standard deviation σ of a series of observations x_i, the arithmetic often can be greatly simplified by referring the individual readings to some arbitrary value x_0 (usually chosen as a round number near $\bar{x}$). Then if, as usual,

$$\bar{x} = \frac{1}{n}\sum_{1}^{n} x_i \qquad \text{and} \qquad \sigma^2 \simeq \frac{1}{n-1}\sum_{1}^{n}(x_i - \bar{x})^2$$

show that

$$(a)\ \bar{x} = x_0 + \frac{1}{n}\sum_{1}^{n}(x_i - x_0)$$

$$(b)\ \sigma^2 \simeq \left[\frac{1}{n-1}\sum_{1}^{n}(x_i - x_0)^2\right] - \left[\frac{n}{n-1}(\bar{x} - x_0)^2\right]$$

4. In successive 30-min intervals, the number of α rays observed on a certain counter are 13, 9, 16, 9, 14, 11, 17, 12, 7, 12, 15.

(*a*) Compute the average rate in α rays per hour.

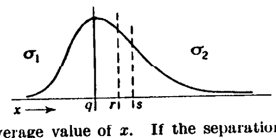
(*b*) If a single additional 30-min observation is made, what are its probable value, standard deviation, and probable error?

(*c*) What is the probable error of the mean value determined in (*a*)?

(*d*) Compare (*c*) with the value expected if the data follow the Poisson distribution.

5. Calculate and plot a Poisson distribution having a mean value of 1.2 for values of x from 0 to 6. What is the standard deviation? What can be said concerning probable error in such an asymmetric distribution?

6. Consider an asymmetric normal distribution having a *modal* (most probable) value of $x = q$, a standard deviation σ_1 for $x < q$, and of σ_2 for $x > q$. The *median* value $x = r$ divides the distribution into two equal areas. The *mean*



value $x = s$ is the average value of x. If the separation $(r - q)$ between the mode and median is small compared with σ_2, show that

$$(\text{median-mode}) = r - q = \sqrt{\frac{\pi}{8}}\,(\sigma_2 - \sigma_1) = 0.627(\sigma_2 - \sigma_1)$$

$$(\text{mean-mode}) = s - q = \sqrt{\frac{2}{\pi}}\,(\sigma_2 - \sigma_1) = 0.798(\sigma_2 - \sigma_1)$$

HINT: Does the assumption $q = 0$ result in any loss of generality?

7. Show that

$$\sum_1^n (x_i - \bar{x})^2 = \sum_1^n x_i^2 - \frac{1}{n}\left(\sum_1^n x_i\right)^2$$

This form is useful when x_i contains one or at most two digits.

3. *Composite Distributions*

Most measurements or calculations in physics involve more than one source of error or of statistical fluctuations. The joint effect of simultaneous but independent sources of statistical fluctuations is now to be considered.

a. Generalized Poisson Distribution. Superposition of Several Independent Random Processes. The complete generalization of the Poisson distribution is usually required in nuclear problems, because several types of radiation will actuate most detection instruments simultaneously. Thus, in ionization-chamber measurements of α rays, there will be present a background composed of α and β rays from radioactive contamination of the walls of the instrument, of cosmic rays, and of γ rays from the earth and the surrounding building materials. If each of several such processes is itself random, the resulting over-all fluctuations may be derived (E24).

Let $x, y, z, \ldots$ be the average number of particles from the several independent random processes, in the time interval chosen. Let them respectively produce specific effects (such as ion pairs) of $a, b, c, \ldots$ per particle. Then the average effect on the instrument is

$$u = ax + by + cz + \cdots \tag{3.1}$$

Generalization of Eq. (2.7) shows (E24) that the square of the S.D. of a single observation of u is given by

$$\sigma^2 = a^2x + b^2y + c^2z + \cdots \tag{3.2}$$

Equations (3.1) and (3.2) are applied to differential measurements by noting that instrumentally subtracted effects correspond simply to negative values of the appropriate coefficient $a, b, c, \ldots$ in Eq. (3.2) and leave the fluctuations unchanged.

In Eqs. (3.1) and (3.2), dimensions may be associated with $a, b, c, \ldots$ and u, but not with $x, y, z, \ldots$. In this case, both the mean value u and the S.D. σ have the dimensions of $a, b, c, \ldots$.

Suppose that a certain ionization chamber receives in unit time an average of $x = 100$ α rays, each producing $a = 10^5$ ion pairs, and also $y = 10^4$ β rays, each producing $b = 10^3$ ion pairs. Then the total average ionization produced is

$$u = 10^5 \times 100 + 10^3 \times 10^4 = 2 \times 10^7 \text{ ion pairs}$$

However, the standard deviation in u is

$$\begin{aligned}\sigma &= \sqrt{(10^5)^2 \times 100 + (10^3)^2 \times 10^4} = \sqrt{10^{12} + 10^{10}} \\ &= \sqrt{1.01 \times 10^{12}} = 1.005 \times 10^6 \text{ ion pairs}\end{aligned}$$

or 5 per cent of u. Thus the α rays produce only half the total ionization but, because of their small number and their large ionization per particle, they account for 99.5 per cent of the statistical fluctuations in the combined ionization effects.

Let this chamber and a second identical ionization chamber be connected in a differential circuit such that an electrometer reads the difference of the ionization in the two chambers. If the second chamber also receives in unit time an average of $z = 100$ α rays, each producing 10^5 ion pairs, then $c = -10^5$ ion pairs because the instrument subtracts cz from $ax + by$. Then the net average differential ionization will be

$$u = 10^5 \times 100 + 10^3 \times 10^4 - 10^5 \times 100 = 10^7 \text{ ion pairs}$$

However, the S.D. in this differential reading is increased to

$$\begin{aligned}\sigma &= \sqrt{(10^5)^2 \times 100 + (10^3)^2 \times 10^4 + (-10^5)^2 \times 100} \\ &= \sqrt{2.01 \times 10^{12}} = 1.41 \times 10^6 \text{ ion pairs}\end{aligned}$$

or 14 per cent of the net u. Note that the differential circuit does not decrease the fluctuations in the total ionization. In fact, *σ has the same value whether the two ionization chambers are connected to oppose each other or to supplement each other.*

In single-particle counting apparatus, such as a Geiger-Müller counter, $a = b = 1$, because the counter discharges once whether the initiating ray is an α ray or a β ray. Thus the $x = 100$ α rays and $y = 10^4$ β rays, if acting in a Geiger-Müller counter, would produce a total of

$$u = 1 \times 10^2 + 1 \times 10^4 = 1.01 \times 10^4$$

counts with a S.D. of

$$\sigma = \sqrt{1 \times 10^2 + 1 \times 10^4} = 1.005 \times 10^2 \text{ counts}$$

or only 1 per cent of u.

We shall return later to Eq. (3.2) for the discussion of the statistics of scaling circuits (Chap. 28).

b. Propagation of Errors. The laws for the propagation of errors are rigorous for the S.D. and can, in fact, be inferred from Eqs. (3.1) and (3.2). In the present section we use probable error merely as a symbol for 0.6745σ, or for $0.6745\sigma_{\bar{x}}$, as required by the context.

Where a physical magnitude is to be obtained from the *summation or* the *differences* of independent observations on two or more physical quantities, the final probable error R of the derived magnitude is obtained from

$$R^2 = r_1^2 + r_2^2 + \cdots \tag{3.3}$$

where $r_1, r_2, \ldots$ are the *absolute* values of the probable errors in the mean values of the several quantities, expressed, of course, in the same units. Thus,

$$(100 \pm 3) + (6 \pm 4) = (106 \pm 5)$$

while

$$(100 \pm 3) - (105 \pm 4) = -(5 \pm 5)$$

The arithmetic of subtraction may be further illustrated by the problem of measuring a counting rate due to some radiation source. A separate measurement must always be made of the natural-background counting rate of the instrument when the source is absent. Suppose that in a time t_b a total of bt_b background counts is recorded. Then the average background rate B and its probable error would be

$$B = (bt_b \pm 0.67\sqrt{bt_b})\frac{1}{t_b}$$
$$= b \pm 0.67\sqrt{\frac{b}{t_b}} \tag{3.4}$$

We note at once that the statistical uncertainty in our evaluation of the background rate depends inversely on the square root of the duration of our observation. Suppose that we now bring a radioactive source near the counter, increasing the true average counting rate to $(S + B)$, where S is due to the source and B to the background. Let $(s + b)$ be the observed counting rate over a period t_s, during which a total of $(s + b)t_s$ counts is recorded. Then our best estimate of $(S + B)$ and its probable error is

$$S + B = [(s + b)t_s \pm 0.67\sqrt{(s + b)t_s}]\frac{1}{t_s}$$
$$= s + b \pm 0.67\sqrt{\frac{s}{t_s} + \frac{b}{t_s}} \tag{3.5}$$

Subtracting Eq. (3.4) from Eq. (3.5) in order to obtain the average counting rate due to the source, we obtain, by Eq. (3.3),

$$S = s \pm 0.67\sqrt{\frac{s}{t_s} + \frac{b}{t_s} + \frac{b}{t_b}} \tag{3.6}$$

It will be noted that the background uncertainty enters twice, once for

its fluctuation during the measurement of $(s + b)$ and once for its fluctuation during the measurement of b alone. In the design of counting experiments, it is clear from Eq. (3.6) that the uncertainty in s, measured for a fixed time t_s, can always be reduced by prolonging the independent background measurements t_b.

On the other hand, if a physical magnitude Y is to be obtained by *multiplication* or *division* of results of several independent observations on two or more physical magnitudes $y_1, y_2, \ldots$, the *fractional* probable error R/Y in the resulting value of Y depends upon the fractional probable errors $r_1/y_1, r_2/y_2, \ldots$ in the measurement of $y_1, y_2, \ldots$ and is given by

$$\left(\frac{R}{Y}\right)^2 \simeq \left(\frac{r_1}{y_1}\right)^2 + \left(\frac{r_2}{y_2}\right)^2 + \cdots + \left(\frac{r_n}{y_n}\right)^2 \tag{3.7}$$

or its equivalent

$$R \simeq Y\sqrt{\left(\frac{r_1}{y_1}\right)^2 + \left(\frac{r_2}{y_2}\right)^2 + \cdots + \left(\frac{r_n}{y_n}\right)^2} \tag{3.8}$$

Thus we have, for example,

$$(100 \pm 0.3)(6 \pm 0.4) = 600 \pm 600\sqrt{(\tfrac{0.3}{100})^2 + (\tfrac{0.4}{6})^2} = 600 \pm 40$$
$$(100 \pm 3)(100 \pm 4) = 10^4 \pm 10^4\sqrt{(\tfrac{3}{100})^2 + (\tfrac{4}{100})^2} = 10{,}000 \pm 500$$
$$\frac{100 \pm 40}{10 \pm 3} = 10 \pm 10\sqrt{(\tfrac{40}{100})^2 + (\tfrac{3}{10})^2} = 10 \pm 5$$

Equation (3.7) is a good approximation if the fractional errors are small, that is, if $n(r_i/y_i)^2 \ll 1$, as often happens.

c. Significance of the Difference of Two Means. Especially in nuclear physics, many experiments have to be statistically designed as optimum compromises between maximum resolution and maximum intensity. It often happens that the statistical fluctuations in the natural background of a detecting instrument may be comparable with the average value of some feeble radiation effect which is to be measured. In such cases, special care must be taken in interpreting the results of the measurements, and standard tests of the "significance of the difference of means" may need to be applied to the data.

Let m_u and m_v be the true mean values of two independent populations of normally distributed data, while $(\bar{u} \pm \sigma_{\bar{u}})$ and $(\bar{v} \pm \sigma_{\bar{v}})$ are measured values of samples from the two populations, each measurement being based on a sufficient number of observations so that the uncertainty in $\sigma_{\bar{u}}$ and in $\sigma_{\bar{v}}$ is small. Then our best estimate of the difference, $(m_u - m_v)$, of the two means is

$$m_u - m_v \simeq (\bar{u} \pm \sigma_{\bar{u}}) - (\bar{v} \pm \sigma_{\bar{v}})$$
$$= (\bar{u} - \bar{v}) \pm \sqrt{\sigma_{\bar{u}}^2 + \sigma_{\bar{v}}^2} \tag{3.9}$$

It can be shown† that $(\bar{u} - \bar{v})$ is normally distributed about the true

† See P. G. Hoel, "Introduction to Mathematical Statistics," p. 109, John Wiley & Sons, Inc., New York, 1954, or S. S. Wilks, "Mathematical Statistics," p. 98, Princeton University Press, Princeton, N.J., 1943, on the problems of significance.

mean value $(m_u - m_v)$ with a standard deviation of

$$\sigma_{(\bar{u}-\bar{v})} = \sqrt{\sigma_{\bar{u}}^2 + \sigma_{\bar{v}}^2} \tag{3.10}$$

If, for example, the true mean value is $(m_u - m_v) = 0$, then from Fig. 1.2 there is about a 32 per cent chance that the absolute value of $(\bar{u} - \bar{v})$ will be numerically greater than the standard deviation of its own measurement, $\sigma_{(\bar{u}-\bar{v})}$. Similarly, because Fig. 1.2 shows $P_u = 0.045$ for $u = 2\sigma$, there is only about a 5 per cent chance that the observed absolute value of $(\bar{u} - \bar{v})$ would exceed $2\sigma_{(\bar{u}-\bar{v})}$ if $(m_u - m_v) = 0$.

It is customary but arbitrary in the theory of errors to reject any hypothesis which falls below a "*significance level*" of 5 per cent. Thus, the hypothesis being tested is usually rejected if it predicts that the observation made was so unusual that it should occur less than 5 per cent of the time. Accordingly, an observation of a difference of at least twice the S.D. (or three times the probable error) between two mean values would be said to be "*significant*" and would lead to rejection of any tentative hypothesis that the two true mean values were identical.

For example, suppose that a radiation-safety monitor is searching for β-ray contamination, using an ionization chamber whose natural background has an average value of 10 α rays (10^5 ion pairs per α ray) plus 100 β rays (10^3 ion pairs per β ray) per minute. What is the minimum number of additional β rays per minute which can just be detected in a 30-sec inspection, using the conventional significance level of 5 per cent? From Eq. (3.1), the average background ionization per 30-sec interval is

$$u = ax + by = 10^5 \times 5 + 10^3 \times 50 = 5.5 \times 10^5 \text{ ion pairs}$$

while the S.D. of u is, by Eq. (3.2),

$$\sigma = \sqrt{a^2x + b^2y} = \sqrt{(10^5)^2 \times 5 + (10^3)^2 \times 50} = 2.24 \times 10^5 \text{ ion pairs}$$

Making the valid and simplifying assumption that the additional β-ray activity cz which is just detectable will not alter σ appreciably, we require $cz = 2\sigma$ for the 5 per cent significance level. Taking $c = b = 10^3$ ion pairs per β ray, we find that $z = 2\sigma/c = 2 \times 2.24 \times 10^5/10^3 = 450\ \beta$ rays in 30 sec, or 900 β rays/min, as the least amount of β-ray activity which can be "detected" in 30 sec with this instrument.

Evidently, instruments designed for the detection of small activities should have small fluctuations in the background. In the example cited, the major portion of the statistical fluctuations is due to the α rays. Another ionization chamber, having no appreciable α-ray background but the same total average background due entirely to 1,100 β rays/min, would have $\sigma = \sqrt{(10^3)^2 \times 550} = 2.34 \times 10^4$ ion pairs for 30-sec readings. Such a chamber could therefore detect an addition of 2σ ion pairs, or 47 β rays in 30 sec, or an average activity of about 100 β rays/min in a 30-sec observation. Although both ionization chambers considered here have the same average background, their "useful sensitivities" to small sources differ by a factor of 9 (!) because of the important effects of fluctuations in the background.

This numerical example illustrates a broad general principle which is too often overlooked in discussions of the relative sensitivity of various types of detecting equipment. A measure of goodness, or of effective relative sensitivity, is the instrument's response to some small standard source, divided not by the average background but by the magnitude of the fluctuations of the background in unit time. The mere ratio of response divided by average background is meaningless.

In principle, a huge background would be perfectly acceptable if it could be absolutely steady in value. It is the inevitable increase in the absolute value of the statistical fluctuations with increasing background which directs instrument designers to seek low backgrounds.

Problems

1. Two measured quantities and their probable errors are $a = 50 \pm 4$ and $b = 30 \pm 3$. Find the values, with probable error, of the quantities ab, a/b, $(a - b)$, $(a + b)$.

2. A counter has a background of 90 counts per minute as determined from a 1-hr observation. A small sample, tentatively thought to be nonradioactive, is placed near the counter for 5 min. During this time 475 counts are recorded.

(*a*) On a basis of this evidence, is the sample radioactive?

(*b*) If in a period of 20 min 1,900 counts were recorded with the sample present, would it be judged as radioactive?

3. Using a counter having a very accurately measured average background of 120 counts per minute, what must be the duration of an observation of a radioactive source having a constant average activity of about 240 counts per minute if the activity of the source is to be measured with a probable error of 2 per cent?

4. The rate of emission of β rays from a single radioactive substance, for example, P^{32}, is being observed by counting the particles emitted during accurately measured time intervals of equal duration t. The background of the counter is first observed for a time t and is 3,000 counts. Then the source is brought up, and the counting rate rises to 7,000 counts in a time t.

(*a*) From these two observations alone, what is the fractional probable error (in per cent) of the observed counting rate due to the β rays?

(*b*) Why must t be much shorter than the half-period of the radioactive substance for the calculation in (*a*) to be valid?

5. The radioactivity of a long-lived substance emitting β rays is to be measured, using a Geiger-Müller counter. The background of the counter is such that a total of 3,200 counts are recorded in a total running time of t_b min. With the source in position, a total of 3,200 counts are recorded in t_s min.

(*a*) Show that the per cent probable error in the measurement of the source strength, in terms of the observed quantities t_b and t_s, is

$$\left(\frac{67}{\sqrt{3{,}200}}\right)\left[\frac{\sqrt{t_b^2 + t_s^2}}{(t_b - t_s)}\right]$$

(*b*) What is the per cent probable error if $t_b/t_s = 2$?

(*c*) What is the per cent probable error if $t_b/t_s = 10$?

6. Two Geiger-Müller counters are exposed to the same radiation to determine whether they have the same absolute sensitivity.

(*a*) In the first trial, counter 1 gives a total of 900 counts in the same time counter 2 gives 940 counts. Can this be considered a "statistically significant" difference?

(*b*) If counter 2 gave 990 counts instead of the 940, would this be a "statistically significant" difference?

7. In successive 10-min intervals, the background of a counter is 1,020; 970; 990; 1,040; 950; 1,010; and 980. A radioactive source of long half-period is brought up to the counter, and the increased counting rate, for successive 10-min intervals, is 3,060; 3,100; 2,980; 3,010; 2,950; 3,030. Calculate the average values and probable errors for (*a*) the background, (*b*) the background and source, and (*c*) the source alone.

8. A time T is available in which to measure the counting rate s due to a radiation source, using an instrument whose background counting rate b is not known accurately and must be measured during part of T. Show that maximum accuracy is obtained in the measurement of s by using a time αT for observing the source, and $(1-\alpha)T$ for observing the background, where

$$\text{"background time"} = 1 - \alpha = \frac{1}{1 + \sqrt{(b+s)/b}}$$

Plot α vs. $\log (b/s)$ for $0.01 \le (b/s) \le 10$. What is the limiting value of α for very weak sources? For very strong sources?

9. The background b of a counter is to be measured and then the counter is to be used to measure the activity s of a source, all in a fixed time T. If the true mean values are $b = 30$ counts per minute (cpm) and $s = 300$ counts per minute, and if $T = 20$ min, what is the probable error of s in counts per minute when T is divided between background and source measurements such that (*a*) the same total number of counts are recorded for background as with the source in position, (*b*) one-half the time available is used for background, and (*c*) the optimum division of time is utilized? *Ans.*: (*a*) 9.5 counts per minute; (*b*) 4.0 counts per minute; (*c*) 3.6 counts per minute.

10. Measurements are made with a γ-ray counter on a source of substantially constant average activity.

(*a*) A total count (source plus background) of 8,000 is observed in 10 min. Then, with the source removed, 10 min gave a total of 2,000 background counts. What is the average source strength in counts per minute? What is the standard deviation in this value?

(*b*) If the total time to make measurements is fixed, what is the optimum fraction of time to spend measuring background in part (*a*)?

11. A choice is to be made between two somewhat similar α-ray counters. One is distinguished especially by its low background, the other by its high efficiency.

(*a*) If the average background of a counter is B counts per hour, and the calibration constant or "sensitivity" is S counts per hour per micromicrocurie of, say, radon, show that the fractional standard deviation in the measurement of A $\mu\mu c$ in a time T is

$$\frac{\sigma}{SA} = \sqrt{\frac{SA + B}{TS^2A^2}}$$

(*b*) What is the fractional standard deviation for very weak sources ($A \to 0$)? For very strong sources ($A \to \infty$)?

(*c*) The two instruments which are available have $B_1 = 10$, $S_1 = 100$ and $B_2 = 150$, $S_2 = 250$. For very weak sources, should the instrument with the low background or the one with the high sensitivity be used? Which instrument is preferable for strong sources?

(*d*) What is the particular source strength A_0, in micromicrocuries, for which these two instruments give the same fractional statistical error of measurement in any fixed time T?

12. A large group of atoms, whose number is exactly N at $t = 0$, undergoes radioactive decay with decay constant λ and mean life $\tau = 1/\lambda$.

(*a*) State the probability that a given atom has survived at time t.

(*b*) State the probability that a given atom has decayed between $t = 0$ and $t = t$.

(*c*) What is the expectation value $\bar{n}$ of the number of survivors at time t (i.e., the mean number of survivors for many such groups of N similar atoms)?

(*d*) If $t = \tau$, which of the distribution laws studied describe(s) the fluctuation of the number of survivors n about $\bar{n}$?

(*e*) What is the standard deviation of n about $\bar{n}$?

(*f*) What is the probability that a given atom will survive through τ and decay between τ and $\tau + \Delta t$?

(*g*) What is the expectation value $\overline{\Delta n}$ of the number of atoms decaying between τ and $\tau + \Delta t$?

(*h*) If $\Delta t = \frac{1}{10}\tau$, which of the distribution laws studied describe(s) the fluctuation about $\overline{\Delta n}$ of the number of atoms, Δn, decaying between τ and $\tau + \Delta t$?

(*i*) What is the standard deviation of Δn about $\overline{\Delta n}$?

(*j*) At $t = 0$ we have 100 groups of N atoms, each of the above type, which we shall call A, and 100 groups of N atoms, each of a second type B. Observations between τ and $\tau + \Delta t$ result in the following

Type of atom	Mean Δn	S.D. of Δn about mean
A	$\overline{\Delta n}_A$	σ_A
B	$\overline{\Delta n}_B$	σ_B

If $\delta = |\overline{\Delta n}_A - \overline{\Delta n}_B|$, how large a value may δ have without seriously upsetting the hypothesis that types A and B are actually the same atoms?

13. The radium content of an unknown sample is to be determined on an absolute basis by comparison with the γ-ray activity of a radium standard. If A is the observed activity of the unknown and B is the observed activity of the radium standard, then the best value of the ratio A/B is the quantity sought from the measurements. A standardized technique is used, such that each individual measurement of A or of B has a fractional standard deviation of 0.5 per cent.

(*a*) If only one measurement of A and one of B are made, what is the fractional standard deviation of A/B?

(*b*) If three measurements of A are made, what is the fractional S.D. of the average activity $\bar{A}$ of the sample?

(*c*) If three measurements of A and n measurements of B are made, what is the fractional S.D. in the average ratio $\bar{A}/\bar{B}$?

(*d*) It is desired to make enough measurements n on the standard so that no appreciable statistical error is introduced in the final ratio $\bar{A}/\bar{B}$ by uncertainty in the activity $\bar{B}$ of the standard. Again, three measurements are made on A. What is the minimum number n of measurements of B such that the fractional S.D. in $\bar{A}/\bar{B}$ will not exceed 1.2 times the fractional S.D. in $\bar{A}$?