radiation appeared to possess a split personality, sometimes behaving as if it were composed of waves and sometimes behaving as if it were composed of particles.

The experimental facts concerning the properties of electromagnetic radiation, as well as the interpretations of these properties in terms of the existence of *both* wave aspects and particle aspects, remain essentially unchanged today. However, as a consequence of the broader point of view provided by the development of the theory of quantum mechanics, the attitude of physicists concerning this situation is now very different from their initial attitude. The duality evident in the wave-particle aspects of electromagnetic radiation is no longer considered unusual because it is now known to be a general characteristic of all physical entities. Furthermore, this duality is no longer considered to represent a problem, since it is possible to reconcile the existence of both aspects with the aid of the theory of quantum mechanics. This theory will be discussed in detail later.

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EXERCISES

1. Electrons are emitted at thermal velocities from a heated cathode, are accelerated through a voltage drop V, and then move in a circle of radius R through a magnetic field of strength H perpendicular to their direction of motion. Derive an expression relating V, H, and R to e/m. Design equipment, based upon this expression, with which it would be possible to make a 1 percent measurement of e/m.

2. Prove the statement, made in section 6, that the average kinetic energy of a vibrating system is proportional to the square of the amplitude of its vibration. Do this for the case of a pendulum.

3. Derive an expression for the kinetic energy of the recoiling electron in the Compton scattering process.

4. Derive the Compton equation for the case $\theta = 180°$ by transforming to the CM frame, treating the collision, and then transforming back to the LAB frame.

5. List the experimental evidence, mentioned in section 9, for the properties of electromagnetic radiation that can be explained only by wave motion.

CHAPTER 4

The Discovery of the Atomic Nucleus

1. Thomson's Model of the Atom

That electrons are emitted from the metallic cathode of a cathode ray or photoelectric tube implies that the atoms of which the cathode is comprised contain electrons. If so, it is reasonable to assume that all atoms contain electrons. This assumption is attractive because it leads to the simple picture of a positively ionized atom as an atom from which one or more electrons have been removed. It agrees with the experimental observation that the charge of a singly ionized atom is equal in magnitude to the charge of a single electron, or that the charge of a doubly ionized atom is equal to the magnitude of the charge of two electrons, etc. Additional evidence for the existence of electrons in atoms was soon obtained from the experiments of Barkla and others (1909) concerning the scattering of X-rays by atoms. These experiments will be discussed in Chapter 14, but it is appropriate to mention at this point that the experiments provided an estimate of Z, the number of electrons in an atom. It was found that Z is roughly equal to $A/2$, where A is the chemical atomic weight of the atom in question. Another set of experiments which provided a measure of the number of electrons in an atom will be described in this chapter.

Since atoms are normally neutral, they must also contain positive charge equal in magnitude to the negative charge carried by their normal complement of electrons. Thus a neutral atom has a negative charge of magnitude

Ze, where *e* is the electronic charge, and also a positive charge of the same magnitude. That the mass of an electron is very small compared to the mass of even the lightest atom implies that most of the mass of the atom must be associated with the positive charge.

All these considerations naturally led to the question of the distribution of the positive and negative charges within the atom. Thomson proposed a tentative description, or *model*, of the composition of an atom, according to which the negatively charged electrons were located within a continuous

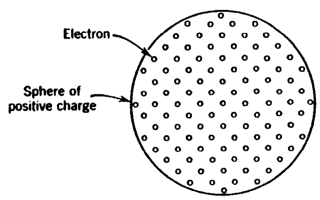

Figure 4–1. Thomson's model of the atom.

distribution of positive charge. The positive charge distribution was assumed to be spherical in shape with a radius of the order of 10^{-8} cm, which was known to be the order of magnitude of the radius of an atom. Owing to their mutual repulsion, the electrons would be uniformly distributed through the sphere of positive charge. This is illustrated in figure (4–1). In an atom in its lowest possible energy state, the electrons would be fixed at their equilibrium positions. In excited atoms (e.g., atoms in a material at high temperature), the electrons would vibrate about their equilibrium positions. Since the electromagnetic theory predicts that an accelerated charged body, such as a vibrating electron, emits electromagnetic radiation, it was possible to understand qualitatively the emission of such radiation by excited atoms on the basis of Thomson's model. However, a calculation of the spectrum of radiation which would be emitted showed that the model did not appear to be able to lead to quantitative agreement with the experimentally observed spectra.

Really conclusive proof of the inadequacy of Thomson's model was obtained in 1911 by Rutherford from the analysis of certain experiments involving the scattering of *alpha particles* by atoms. Rutherford's analysis showed that, instead of being spread throughout the atom, the positive charge is concentrated in a very small region, or *nucleus*, at the center of the atom. Rutherford's work comprised one of the most important developments in the subject of atomic physics, as well as the foundation of the subject of nuclear physics.

2. Alpha Particles

Alpha particles are doubly ionized helium atoms which are spontaneously emitted at very high velocities from several of the heaviest elements, such as U and Ra. This phenomenon will be discussed in detail later in this book; here we shall present only a few pertinent points which are required as background for a description of the experiments which led Rutherford to the discovery of the nucleus.

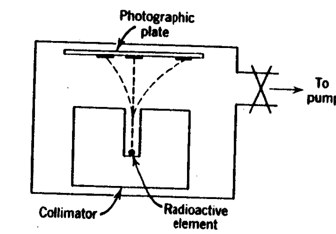

Figure 4–2. An apparatus used to study the types of radiation emitted by radioactive elements.

In 1896–98 Becquerel and Mme. Curie discovered that several of the heaviest elements spontaneously emit penetrating radiation capable of blackening a photographic plate. Soon the radiation emitted by these so-called *radioactive* elements was being investigated in many laboratories. With apparatus of the type indicated in figure (4–2), it was shown that the radiation consists of three separate components. Radiation is emitted from a radioactive element at the bottom of a channel in a lead block which collimates the radiation into a well-defined beam. With no magnetic field, a photographic plate exposed to the radiation shows, upon development, a darkened spot at a point in line with the collimating channel. When a magnetic field is applied to the region between the collimator and the plate, in a direction perpendicular to the plane of the figure, three darkened spots are seen on the plate after it is developed. The two spots which are deviated from the zero field position must be due to charged particles emitted by the radioactive element, and which are deflected, by the application of the magnetic field, in a direction depending on the sign of the charge. The undeviated spot is presumably due to uncharged radiation upon which the magnetic field would have no influence. The particles of which the positively charged component of the radiation are comprised are called *alpha particles*. The negatively charged particles are called *beta particles*, and the uncharged radiation is given the name *gamma radiation*.

In the experiments just described the chamber containing the collimator and plate is evacuated. If the experiment is repeated with the chamber at atmospheric pressure, the spot due to the alpha particles does not appear. Apparently a few centimeters of air is sufficient to stop the alpha particles.

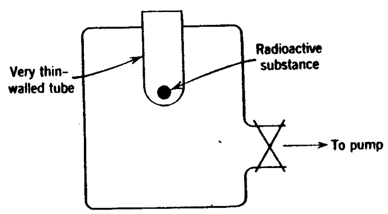

Figure 4–3. An apparatus used to prove that alpha particles are doubly ionized helium atoms.

Similar experiments establish that a foil of some dense substance several millimeters thick will stop the beta particles from reaching the plate, but that the intensity of the spot due to the gamma radiation is significantly decreased only when several centimeters of lead are placed between the collimator and the plate. We know now that the beta particles are high energy electrons, whereas the gamma radiation consists of electromagnetic radiation of the same nature as light, but of very much higher frequency.

The nature of the alpha particles was established in a series of measurements performed by Rutherford. Using apparatus basically similar to that used by Thomson, he measured the charge to mass ratio of the alpha particles as well as their velocity. These measurements showed that the alpha particles were emitted with a well-defined velocity in the range 1.4×10^9 to 2.2×10^9 cm/sec, the exact value depending upon the radioactive element from which they were emitted. Note that these velocities are roughly one-twentieth the velocity of light. The charge to mass ratio of the alpha particles was found to be one-half the value of that ratio for singly ionized hydrogen atoms. Thus, if their charge was equal in magnitude to a single electronic charge, the alpha particles would have a mass twice the mass of hydrogen atoms. Alternatively, if the alpha particles were assumed to have a charge twice the magnitude of an electronic charge, their mass would be four times the mass of hydrogen atoms (i.e., equal to the mass of helium atoms). Rutherford was inclined toward the

latter interpretation since it would mean that alpha particles were simply doubly ionized helium atoms.

By a simple experiment, using the apparatus shown in figure (4–3), Rutherford proved that this is indeed the case. A very thin-walled glass tube containing a radioactive substance was surrounded by a glass enclosure which was initially evacuated. Some of the alpha particles emitted by the radioactive substance were able to penetrate the thin-walled tube and enter the outer enclosure. After a time, a sensitive test of the contents of the outer enclosure showed that it contained a detectable amount of ordinary helium gas. This finding confirmed the argument that alpha particles are doubly ionized helium atoms—which can pick up two electrons and become neutral helium atoms during or after their passage through the thin-walled tubing.

3. The Scattering of Alpha Particles

Rutherford and his collaborators performed many experiments in the course of a program designed to determine the properties of alpha particles and of their interaction with matter. By far the most interesting of these were the experiments concerning the scattering of alpha particles upon

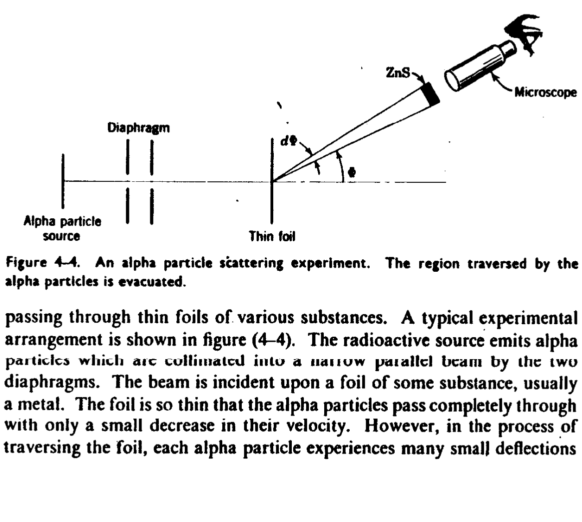

Figure 4–4. An alpha particle scattering experiment. The region traversed by the alpha particles is evacuated.

passing through thin foils of various substances. A typical experimental arrangement is shown in figure (4–4). The radioactive source emits alpha particles which are collimated into a narrow parallel beam by the two diaphragms. The beam is incident upon a foil of some substance, usually a metal. The foil is so thin that the alpha particles pass completely through with only a small decrease in their velocity. However, in the process of traversing the foil, each alpha particle experiences many small deflections due to the Coulomb force acting between its charge and the positive and negative charges of the atoms of the foil. Since the deflection of an alpha particle in passing through a single atom depends on the details of its trajectory through the atom, it is apparent that the net deflection in passing through the entire foil is different for different alpha particles in the beam. As a result, the beam emerging from the far side of the foil is divergent. A quantitative measure of its divergent character is obtained by measuring the number of alpha particles scattered into each angular range Φ to $\Phi + d\Phi$. This is accomplished with the aid of an alpha particle detector consisting of a layer of the crystalline compound ZnS and a microscope. ZnS has the useful property of producing a small flash of light when struck by an alpha particle. If observed by a microscope, the flash due to the incidence of a *single* alpha particle can be distinguished.† In the experiment an observer counts the number of light flashes produced per unit time as a function of the angular position of the detector.

4. Predictions of Thomson's Model

The deflection of an alpha particle in following a given trajectory through an atom can be calculated from Coulomb's law and the laws of mechanics, if the distribution of charges in the atom is given. Thus, for the charge distribution assumed in Thomson's model of the atom, it is possible to calculate the expected results of the experiment just described. It is

Figure 4-5. An alpha particle passing through a Thomson's model atom.

apparent from the discussion above that the calculation will not be able to predict the behavior of a single alpha particle, but only the *average* behavior of a large number of alpha particles. The average behavior, however, is precisely what is measured in the experiment.

Consider the traversal of an alpha particle through a single atom of the type discussed in Thomson's model; this is indicated schematically in figure (4-5). We shall estimate the order of magnitude of ϕ_{max}, the maximum possible deflection angle of the alpha particle. The calculation will

† This effect is used in the radium dials commonly found on wrist watches. Individual light flashes can be seen by a dark-adapted eye with the aid of a magnifying glass.

use classical mechanics, which is justifiable since $v/c \simeq 1/20$. The deflection is assumed to be due to the Coulomb forces acting between the charged alpha particle and the charges of the various parts of the atom. According to Coulomb's law, the force due to some region of the atom at a distance r from the instantaneous position of the alpha particle, and containing charge dq, is

$$d\mathbf{F} = \frac{2e\,dq}{r^2}\left(\frac{\mathbf{r}}{r}\right) \qquad (4\text{-}1)$$

where $(\mathbf{r}/r)$ is a vector of unit length directed from the location of the charge dq to the position of the alpha particle, and where $2e$ is the charge of the alpha particle.

Let us begin by evaluating the maximum deflection to be expected from the forces due to the electrons in the atom. At first glance, equation (4-1) would seem to imply that the electrons could give very large deflections to the alpha particle since $d\mathbf{F} \to \infty$ as $r \to 0$, so that the alpha particle would experience a very large force and consequently suffer a large deflection if it happened to pass very close to an electron. This is actually not the case because the mass of an electron is so very small compared to the mass of the alpha particle. We leave it as an exercise for the reader to show that, in the collision of a very massive body initially moving at velocity v, with a stationary body of small mass, the velocity of the light body after the collision cannot be greater than $2v$.† This result follows strictly from the conservation of momentum and energy and does not depend on the nature of the force between the two bodies. Consequently the maximum momentum which can be transferred to an electron is $p = m(2v)$, where m is the mass of an electron. This must be equal to the momentum lost by the alpha particle in the collision. Thus Δp_α, the maximum change in the momentum of the alpha particle due to a collision with an electron, is equal to $2mv$. To evaluate the *order of magnitude* of the maximum deflection angle ϕ_{max}, assume that the momentum change is directed perpendicular to the initial momentum p of the alpha particle as illustrated in figure (4-6). Then we have $\phi_{max} \sim \Delta p_\alpha/p_\alpha \sim 2mv/Mv$, where M is the mass of an alpha particle which is equal to $4 \times 1836m$. Consequently

$$\phi_{max} \sim \frac{2m}{4(1836)m} = \frac{2}{4(1836)} \sim 10^{-4} \text{ radians}$$

This is the order of magnitude of the maximum deflection angle due to a collision with a single electron in the atom; but it is clear that the

† We assume the binding energy of the electron to the atom to be small compared to the energy of the alpha particle. Then in the collision the electron will be essentially free.

maximum value of the total deflection angle (due to all the collisions with electrons in the atom) will be of the same order since the chance of having a very close collision with more than one or two of the electrons is negligible.

To estimate ϕ_{max} due to the forces arising from the positive charge, we integrate equation (4–1) over the positive charge distribution to obtain

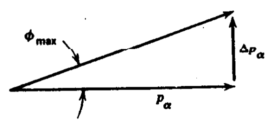

Figure 4–6. Illustrating the evaluation of the maximum deflection angle when a particle of momentum p_α looses momentum Δp_α.

the total force acting on the alpha particle at some instant when it is inside the atom. This gives

$$\mathbf{F} = \int \frac{2e\,dq}{r^2}\left(\frac{\mathbf{r}}{r}\right) \tag{4–2}$$

where r is the distance from the position of the alpha particle at that instant to a volume element containing the positive charge dq, and $(\mathbf{r}/r)$ is a vector of unit length directed from the volume element to the position of the alpha particle. It is clear that in evaluating the integral there will be cancellation from volume elements symmetrically disposed with respect to the position of the alpha particle. However, to estimate the *order of magnitude* of the maximum total force, we ignore such cancellation and take

$$F_{max} \sim \int \frac{2e}{r^2}\,dq$$

The order of magnitude of the integral can be evaluated by writing it as

$$F_{max} \sim 2e\overline{\left(\frac{1}{r^2}\right)}\int dq = 2eq\overline{\left(\frac{1}{r^2}\right)}$$

where $\overline{(1/r^2)}$ is the value of $1/r^2$ averaged over a region of space of atomic dimensions, and $q = Ze$ is the total positive charge of the atom. Now $\overline{(1/r^2)}$ must be of the order of $1/R^2$, where R is the radius of an atom. Consequently

$$F_{max} \sim 2Ze^2/R^2 \tag{4–3}$$

To estimate the maximum deflection angle ϕ_{max}, we evaluate Δp_α the

momentum given to the alpha particle by the force F_{max} acting for a time of the order of Δt, the transit time through the atom. The time Δt is approximately equal to R/v, where v is the velocity of the alpha particle, so

$$\Delta p_\alpha \sim F_{max}\,\Delta t \sim 2Ze^2/Rv$$

and we have

$$\phi_{max} \sim \Delta p_\alpha/p_\alpha \sim 2Ze^2/RMv^2 \tag{4–4}$$

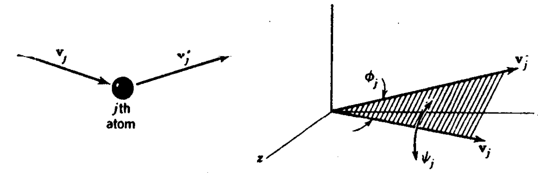

Figure 4–7. The velocity vectors before and after an alpha particle traverses the jth atom.

For atoms containing the largest positive charge, $Z \simeq 100$. The other quantities entering into equation (4–4) have the values

$$e \simeq 5 \times 10^{-10}\ \text{esu}$$
$$M \simeq 8 \times 10^{-24}\ \text{gm}$$
$$v \simeq 2 \times 10^{-9}\ \text{cm/sec}$$
$$R \simeq 10^{-8}\ \text{cm}$$

Consequently

$$\phi_{max} \sim \frac{2 \times 10^2 \times 2 \times 10^{-19}}{10^{-8} \times 8 \times 10^{-24} \times 4 \times 10^{18}} \sim 10^{-4}\ \text{radians}$$

We see that the positive charge can produce only a very small deflection because, since the charge $+Ze$ is distributed uniformly over a sphere of radius $\sim 10^{-8}$ cm, the very high velocity alpha particles never get close enough to a large enough quantity of charge that they experience a Coulomb force of the intensity required to produce a large deflection.

Now that we know something about the angle through which an alpha particle is deflected in the traversal of a single atom, let us see what we can learn about the angle through which the particle is deflected in the process of traversing the large number of atoms which lie along its path through the foil. Clearly, summing the results of the individual deflections is involved, and in this sum we must recognize the three dimensional character of the individual deflections. Consider figure (4–7), which indicates typical orientations of $\mathbf{v}_j$ and $\mathbf{v}_j'$, the velocity vectors before and after traversing the jth atom, and a coordinate system useful in describing these vectors. The coordinate system is chosen such that the z axis is in the direction of the beam of alpha particles incident upon the foil. Because the change in the momentum vector of the alpha particle in any single deflection is extremely small, the angles between all the velocity vectors and the initial velocity vector remain small and, for any j, both $\mathbf{v}_j$ and $\mathbf{v}_j'$ are nearly

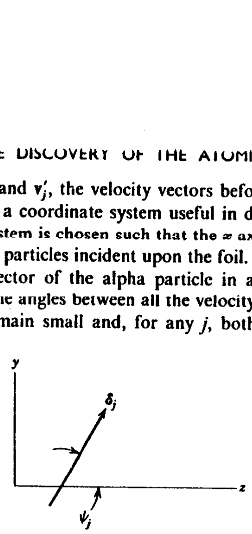

Figure 4-8. A two dimensional diagram representing the jth deflection of an alpha particle.

perpendicular to the yz plane. Consequently, the nature of the deflection can be specified by the angle ϕ_j measured between the two vectors in the plane which they define, and the angle ψ_j between that plane and the z axis. This allows us to represent the deflection in terms of a two dimensional diagram consisting of the y and z axes and a vector $\boldsymbol{\delta}_j$, as shown in figure (4-8). The length of the vector is equal to the magnitude of the angle ϕ_j, and the angle between the vector and the z axis is equal to the angle ψ_j. A moment's consideration will show that the relative orientation between the velocity vector after the Nth deflection and the initial velocity vector can be described in the same representation by a vector $\boldsymbol{\delta}$ where

$$\boldsymbol{\delta} = \sum_{j=1}^{N} \boldsymbol{\delta}_j$$

Thus the entire deflection process can be represented by a two dimensional diagram, a typical example of which is shown in figure (4-9). The net angle of deflection resulting from N individual deflections is equal to the length of the vector $\boldsymbol{\delta}$. Except for the constraint, $\phi_j \lesssim 10^{-4}$ radians, the magnitude of the individual deflection angles ϕ_j, as well as the orientation of the deflection planes specified by ψ_j, depends in an essentially fortuitous manner on both the exact trajectory of the alpha particle through the atom and on the exact location of the charges within the atom at the time the particle passes through. Thus the vectors $\boldsymbol{\delta}_j$ will have random directions and a distribution of lengths, as indicated in the figure.

The random nature of the $\boldsymbol{\delta}_j'$, and the fact that there are a large number of them, suggest that we apply the mathematical theory of statistics to the problem of determining the length of the vector $\boldsymbol{\delta}$. Now the diagram in figure (4-9) represents the deflections suffered by one alpha particle in its traversal of the foil. If we were to observe the traversals of a number of alpha particles and make similar diagrams for each of them, we could evaluate two characteristic properties of these diagrams: the *average*

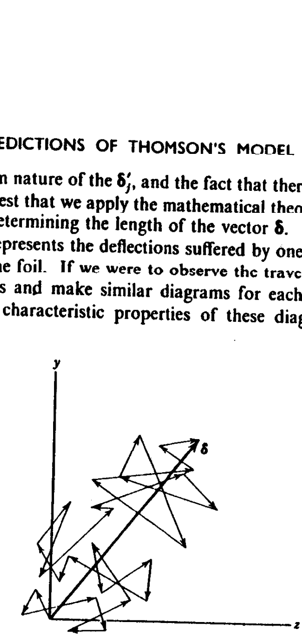

Figure 4-9. A two dimensional diagram representing the entire deflection process experienced by an alpha particle in passing through a foil.

length of the vector $\boldsymbol{\delta}$, and the *distribution* of the lengths of this vector. The theory of statistics makes predictions concerning both of these properties. These predictions are particularly easy to obtain because we have formulated the problem in such a way that it is identical with a well-known statistical problem called the "random walk."† According to the theory $(\overline{\delta^2})^{1/2}$, the root-mean-square average length of the net deflection vector $\boldsymbol{\delta}$, is related to the total number of individual deflections N and to $(\overline{\delta_j^2})^{1/2}$, the root-mean-square average length of the individual deflection vectors $\boldsymbol{\delta}_j$, by the equation

$$(\overline{\delta^2})^{1/2} = \sqrt{N}(\overline{\delta_j^2})^{1/2}$$

It is interesting to note that the average length of the vector $\boldsymbol{\delta}$ increases only as the square root of N because of the random directions of the $\boldsymbol{\delta}_j$. (If, on the other hand, the $\boldsymbol{\delta}_j$ were all pointing in the same direction, which is just the opposite of being randomly directed, the average length of $\boldsymbol{\delta}$

† See any textbook on the theory of statistics.

would clearly be proportional to the first power of N.) The theoretical predictions of the distribution of the lengths of the vector $\boldsymbol{\delta}$ can be stated as

$$\mathcal{N}(\delta)\, d\delta = \frac{2\mathcal{N}\delta}{\overline{\delta^2}}\, e^{-\delta^2/\overline{\delta^2}}\, d\delta$$

The quantity $\mathcal{N}(\delta)\, d\delta$ is the number of cases in which the length of the net deflection vector is within the range of values δ to $\delta + d\delta$; $\overline{\delta^2}$ is the average value of the square of the length of the net deflection vectors; and $\mathcal{N}$ is the total number of cases considered.

Since the lengths of the vectors $\boldsymbol{\delta}$ and $\boldsymbol{\delta}_i$ are equal, respectively, to the net deflection angle Φ and the individual deflection angle ϕ_i, the equations above can be immediately written in terms of the angles:

$$\mathcal{N}(\Phi)\, d\Phi = \frac{2\mathcal{N}\Phi}{\overline{\Phi^2}}\, e^{-\Phi^2/\overline{\Phi^2}}\, d\Phi \tag{4-5}$$

$$(\overline{\Phi^2})^{1/2} = \sqrt{N}(\overline{\phi_i^2})^{1/2} \tag{4-6}$$

The quantity $N(\Phi)\, d\Phi$ is the number of alpha particles scattered within the angular range Φ to $\Phi + d\Phi$; $\overline{\Phi^2}$ is the mean square scattering angle; $\overline{\phi_i^2}$ is the mean square scattering angle in a deflection from a single atom; $\mathcal{N}$ is the number of alpha particles passing through the foil; and N is the number of atoms traversed by an alpha particle in its passage through the foil. These equations, plus the restriction

$$\phi_i \lesssim 10^{-4} \text{ radians} \tag{4-7}$$

that the angle of deflection in passing through a single atom is very small, comprise the predictions of Thomson's model of the atom for the scattering of alpha particles traversing a thin foil.

5. Comparison with Experiment

These predictions were tested by Rutherford and his collaborators. In a typical experiment (Geiger and Marsden, 1909) alpha particles were scattered by an Au foil 10^{-4} cm thick. The average scattering angle $(\overline{\Phi^2})^{1/2}$ was found to be about $1° \simeq 2 \times 10^{-2}$ radians, and more than 99 percent of the alpha particles were scattered at angles less than about 3°. Now, the number of atoms traversed by the alpha particle is approximately equal to the thickness of the foil divided by the diameter of an atom. Thus $N \simeq 10^{-4}\text{ cm}/10^{-8}\text{ cm} = 10^4$. Knowing N and $(\overline{\Phi^2})^{1/2}$, we can use

equation (4-6) to evaluate the average deflection angle in traversing a single atom. We obtain

$$(\overline{\phi_i^2})^{1/2} = \frac{(\overline{\Phi^2})^{1/2}}{\sqrt{N}} \simeq \frac{2 \times 10^{-2}}{10^2} = 2 \times 10^{-4} \text{ radians}$$

This is in reasonable agreement with the restriction stated in equation (4-7). Accurate measurements of $\mathcal{N}(\Phi)\, d\Phi$ were difficult to obtain because of the small angles involved. However, the available data were in agreement with equation (4-5), using 1° for $(\overline{\Phi^2})^{1/2}$, for angles less than 2° or 3°.

On the other hand, the angular distribution of the small number of particles scattered at angles large compared to 1° was observed to be in marked disagreement with equation (4-5). This equation predicts that $\mathcal{N}(\Phi)\, d\Phi$ will decrease *very* rapidly with increasing Φ. For instance, if we evaluate the predicted fraction of alpha particles scattered at angles greater than 90°, we find

$$\frac{\mathcal{N}(\Phi > 90°)}{\mathcal{N}} = \frac{\int_{90°}^{180°} \mathcal{N}(\Phi)\, d\Phi}{\mathcal{N}}$$

Using the experimental value of 1° for $(\overline{\Phi^2})^{1/2}$, we obtain the impressively small number $\mathcal{N}(\Phi > 90°)/\mathcal{N} = e^{-(90)^2} \simeq 10^{-3500}$. But experimentally it was found that $\mathcal{N}(\Phi > 90°)/\mathcal{N} \simeq 10^{-4}$. In general, the number of scattered alpha particles was observed to be very much larger than the predicted number for all scattering angles greater than a few degrees. The existence of a small, but non-zero, probability for scattering at large angles was totally inexplicable in terms of Thomson's model of the atom, which could only lead to the *multiple small angle scattering* process described in section 4.

6. Rutherford's Model of the Atom

Experiments utilizing foils of various thicknesses showed that the number of large angle scatterings was proportional to N, the number of atoms traversed by the alpha particle. This is just the dependence on N that would arise if there was a small probability that an alpha particle could be scattered through a large angle in traversing a *single* atom. Since this cannot happen in Thomson's model of the atom, Rutherford proposed a new model (1911).

In Rutherford's model of the structure of the atom, all the positive charge *of the atom, and consequently essentially all its mass, is assumed to be concentrated in a small region called the* nucleus.

From symmetry considerations the nucleus was assumed to be located at the center of the atom, but its exact location played no role in Rutherford's work.

The arguments of section 4 indicate that a large angle scattering would indeed be possible in the traversal of a single atom of such structure if the alpha particle happened to pass very near the nucleus, provided that the dimensions of the nucleus are small enough. We can get a preliminary idea of the dimensions required from equation (4–4) since, as a review of its derivation will demonstrate, it can be used to estimate the scattering angle of an alpha particle passing near (or through) the nucleus if we interpret R to be the nuclear radius. In the computation directly below equation (4-4), we used $R \sim 10^{-8}$ cm and obtained $\phi \sim 10^{-4}$ radians. Consequently, to obtain a large angle scattering, $\phi \sim 1$ radian, the value of R must be $\sim 10^4$ times smaller; that is, $R \sim 10^{-12}$ cm. As will be indicated later in this chapter, and discussed in detail in Chapter 16, 10^{-12} cm is, in fact, a good estimate of the radius of the atomic nucleus.

7. Predictions of Rutherford's Model

Rutherford made a detailed calculation of the angular distribution to be expected for the scattering of alpha particles from atoms of the type proposed in his model. The calculation was concerned only with scattering at angles greater than several degrees. Consequently, as we have seen above, the scattering due to the atomic electrons can be ignored. The scattering is then due to the repulsive Coulomb force acting between the positively charged alpha particle and the positively charged nucleus. Furthermore, the calculation considered only the scattering from heavy atoms and thus permitted the assumption that the mass of the nucleus is so large compared to the mass of the alpha particle that the nucleus does not recoil (remains fixed in space) during the scattering process. It was also assumed that the alpha particle does not actually penetrate the nuclear region, so that the alpha particle and the nucleus act like point charges as far as the Coulomb force is concerned. We shall see later that all these assumptions are quite valid except for the scattering of alpha particles from the lighter nuclei. The calculation employs classical mechanics since $v/c \sim 1/20$.

Figure (4-10) illustrates the scattering of a particle, of charge $+ze$ and mass M, in passing near a nucleus of charge $+Ze$. The nucleus is

fixed at the origin of the coordinate system. When the particle is very far from the nucleus, the Coulomb force on it will vanish, and so the particle approaches the nucleus along a straight line with constant velocity v. After the scattering, the particle will move off finally along a straight line with constant velocity v'. The position of the particle is specified by the radial coordinate r and the polar angle θ, which is measured from an axis drawn parallel to the initial direction of motion. The perpendicular

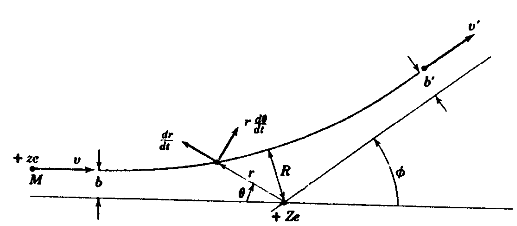

Figure 4–10. The scattering of a positively charged particle in passing near a nucleus.

distance from that axis to the line of initial motion is specified by the impact parameter b. The scattering angle ϕ is just the angle between the axis and a line drawn through the origin parallel to the line of final motion, and the perpendicular distance between these two lines is b'. We resolve the velocity of the particle at any point along its trajectory into a radial component of magnitude dr/dt and a tangential component of magnitude $r\,(d\theta/dt)$.

Now, the force acting on the particle is always in the radial direction. Consequently its angular momentum $Mr^2\,(d\theta/dt)$ has the constant value L. That is,

$$Mr^2 \frac{d\theta}{dt} = L \qquad (4\text{–}8)$$

Specifically, the initial angular momentum is equal to the final angular momentum, or

$$Mvb = Mv'b' = L \qquad (4\text{-}8')$$

Of course, the kinetic energy of the particle does not remain constant during the scattering. However, the initial kinetic energy must be equal to the final kinetic energy since the nucleus is assumed to remain stationary. Thus

$$\tfrac{1}{2}Mv^2 = \tfrac{1}{2}Mv'^2$$

From the last two equations we see that $v = v'$ and $b = b'$, as indicated in figure (4-10).

Next, applying Newton's law to the radial component of motion, we have

$$\frac{zZe^2}{r^2} = M\left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right] \tag{4-9}$$

The left-hand term is the Coulomb force. The first term in the bracket is the radial acceleration due to the change in r, and the second term is the (radially directed) centrifugal acceleration. To effect the fastest solution of this equation we transform from the coordinates r, θ to the coordinates u, θ, where

$$r = 1/u \tag{4-10}$$

Then

$$\frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = \frac{dr}{du}\frac{du}{d\theta}\frac{d\theta}{dt}$$

$$\frac{dr}{dt} = -\frac{1}{u^2}\frac{du}{d\theta}\frac{Lu^2}{M} = -\frac{L}{M}\frac{du}{d\theta}$$

$$\frac{d^2r}{dt^2} = \frac{d}{d\theta}\left[\frac{dr}{dt}\right]\frac{d\theta}{dt} = -\frac{L}{M}\frac{d^2u}{d\theta^2}\frac{Lu^2}{M}$$

$$\frac{d^2r}{dt^2} = -\frac{L^2u^2}{M^2}\frac{d^2u}{d\theta^2}$$

Substituting into equation (4-9), we have

$$-\frac{L^2u^2}{M^2}\frac{d^2u}{d\theta^2} - \frac{1}{u}\left(\frac{Lu}{M}\right)^2 = \frac{zZe^2u^2}{M}$$

$$\frac{d^2u}{d\theta^2} + u = -\frac{zZe^2M}{L^2} = -\frac{zZe^2M}{M^2v^2b^2} \tag{4-11}$$

The constant D, defined by

$$D \equiv \frac{zZe^2}{\frac{1}{2}Mv^2} \tag{4-12}$$

is a convenient parameter equal to the distance of closest approach to the nucleus in a *head-on collision* ($b = 0$), since D is the distance at which the potential energy zZe^2/D is equal to the initial kinetic energy $\frac{1}{2}Mv^2$; at this point the particle would come to a stop and then reverse its direction of motion. In terms of this parameter, equation (4-11) reads

$$d^2u/d\theta^2 + u = -D/2b^2 \tag{4-13}$$

This is a second order differential equation (i.e., involving a second derivative) for u as a function of θ. The *general solution* to this equation is

$$u = A\cos\theta + B\sin\theta - D/2b^2 \tag{4-14}$$

which contains two arbitrary constants, A and B. We may prove that equation (4-14) is, in fact, the solution to (4-13) by evaluating

$$du/d\theta = -A\sin\theta + B\cos\theta$$

and

$$d^2u/d\theta^2 = -A\cos\theta - B\sin\theta$$

and substituting into (4-13). This gives

$$-A\cos\theta - B\sin\theta + A\cos\theta + B\sin\theta - D/2b^2 \equiv -D/2b^2, \quad \text{Q.E.D.}$$

To determine A and B, we require that equation (4-14) conform to the *initial conditions:* $\theta \to 0$ as $r \to \infty$, and $dr/dt \to -v$ as $r \to \infty$. Thus

$$u = \frac{1}{r} = 0 = A\cos 0 + B\sin 0 - \frac{D}{2b^2}$$

or

$$A = \frac{D}{2b^2}$$

and

$$\frac{dr}{dt} = -\frac{L}{M}\frac{du}{d\theta} = -v = -\frac{L}{M}(-A\sin 0 + B\cos 0)$$

or

$$B = \frac{Mv}{L} = \frac{Mv}{Mvb} = \frac{1}{b}$$

Therefore the *particular solution* of the differential equation pertinent to the problem at hand is

$$u = \frac{D}{2b^2}\cos\theta + \frac{1}{b}\sin\theta - \frac{D}{2b^2}$$

or

$$\frac{1}{r} = \frac{1}{b}\sin\theta + \frac{D}{2b^2}(\cos\theta - 1) \tag{4-15}$$

This is the equation of a hyperbola in polar coordinates. We see that the trajectory of the particle moving under the influence of a repulsive inverse square Coulomb force is one of the conic sections. The trajectory of a particle moving under an attractive inverse square force (e.g., a satellite in the gravitational field of the earth) could be obtained from the derivation above by simply reversing the sign of the term on the left side of equation (4–9). The trajectory would then turn out to be one of the conic sections—exactly which one depending on the values of the parameters.

From the equation (4–15) of the trajectory of the particle, we may evaluate the scattering angle ϕ by letting $r \to \infty$. Then

$$0 = \frac{1}{b}\sin\theta' + \frac{D}{2b^2}(\cos\theta' - 1)$$

One solution to this equation is $\theta' = 0$. This is the polar angle for large r before scattering. The interesting solution is

$$\frac{2b}{D} = \frac{1 - \cos\theta'}{\sin\theta'} = \tan\frac{\theta'}{2}$$

which gives the polar angle for large r after scattering. It is apparent from figure (4–10) that for this solution $\phi = \pi - \theta'$. Consequently

$$\tan\left(\frac{\pi - \phi}{2}\right) = \frac{2b}{D}$$

which is equivalent to

$$\cot\frac{\phi}{2} = \frac{2b}{D} \qquad (4\text{–}16)$$

It will also be useful to evaluate R, the distance of closest approach of the particle to the center of the nucleus (the origin). From figure (4–10) it is apparent that the radial coordinate will assume this value when the polar angle is equal to $\theta'/2 = (\pi - \phi)/2$. Evaluating equation (4–15) for this angle, and using (4–16), we obtain

$$R = \frac{D}{2}\left[1 + \frac{1}{\sin(\phi/2)}\right] \qquad (4\text{–}17)$$

It is easy to verify that $R \to D$ as $\phi \to \pi$, and that $R \to b$ as $\phi \to 0$, as would be expected.

We see from equation (4–16) that, in the scattering of an alpha particle by a single nucleus, if the impact parameter is in the range b to $b + db$ the scattering angle is in the range ϕ to $\phi + d\phi$, where the relation between b and ϕ is given by the equation. Thus the problem of calculating the number of alpha particles scattered in the angular range ϕ to $\phi + d\phi$ in traversing the entire foil is equivalent to the problem of calculating the number which are incident, with impact parameter from b to $b + db$, upon the nuclei in the foil. Let the number of nuclei (i.e., the number of atoms) per cm³ in the foil be ρ, and let the foil be t cm thick. Consider a segment of the foil with a cross-sectional area of 1 cm², as shown in figure

(4–11). This diagram indicates a ring, of inner radius b and outer radius $b + db$, drawn around an axis passing through each nucleus. The area of each ring is $2\pi b\, db$. The number of such rings in the segment is ρt, where t is numerically equal to the volume of the segment. The probability $P(b)\, db$ that an alpha particle will pass through one of these rings is equal

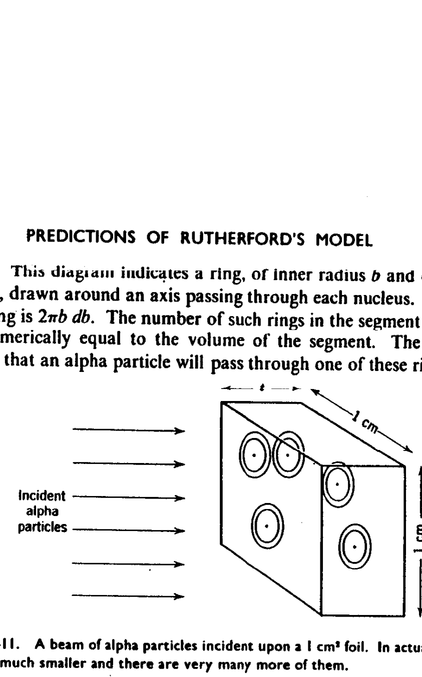

Figure 4–11. A beam of alpha particles incident upon a 1 cm² foil. In actuality the rings are very much smaller and there are very many more of them.

to the total area obscured by the rings, as seen by the incident alpha particles, divided by the total area of the segment. That is,

$$P(b)\, db = \rho t 2\pi b\, db$$

Now

$$b = \frac{D}{2}\cot\frac{\phi}{2}$$

so

$$db = -\frac{D}{2}\frac{\frac{1}{2}d\phi}{\sin^2(\phi/2)}$$

and

$$b\, db = -\frac{D^2}{8}\frac{\cos(\phi/2)\, d\phi}{\sin^3(\phi/2)} = \frac{D^2}{16}\frac{\sin\phi\, d\phi}{\sin^4(\phi/2)}$$

Thus

$$P(b)\, db = -\frac{\pi}{8}\rho t D^2 \sin\phi \frac{d\phi}{\sin^4(\phi/2)}$$

But $-P(b)\, db$ is equal to the probability that the incident particles will be scattered in the angular range ϕ to $\phi + d\phi$. (The minus sign compensates for the fact that db and $d\phi$ are of the opposite sign.) Using the notation employed in equation (4–5), this is

$$\frac{\mathcal{N}(\Phi)\, d\Phi}{\mathcal{N}} = -P(b)\, db = \frac{\pi}{8}\rho t D^2 \frac{\sin\Phi\, d\Phi}{\sin^4(\Phi/2)}$$

Evaluating D, we have

$$\mathcal{N}(\Phi)\,d\Phi = \mathcal{N}\frac{\pi}{8}\rho t\left(\frac{zZe^2}{\frac{1}{2}Mv^2}\right)^2\frac{\sin\Phi\,d\Phi}{\sin^4(\Phi/2)} \qquad (4\text{–}18)$$

This is the angular distribution derived by Rutherford for the scattering of alpha particles in passing through a foil composed of atoms with the nuclear structure proposed by his model. The quantity $\mathcal{N}$ is the number of alpha particles incident on the foil of thickness t and ρ nuclei per cm^3; $\mathcal{N}(\Phi)\,d\Phi$ is the number scattered at angles from Φ to $\Phi + d\Phi$; ze is the charge of the alpha particle; M is its mass; v is its velocity; and Ze is the charge of the nuclei. We note that, although the angular factor in this equation decreases rapidly with increasing angle, nevertheless the decrease is very much less rapid than the decrease predicted by the multiple small angle scattering angular distribution (equation 4–5).

8. Experimental Verification and the Determination of Z

Detailed experimental tests of equation (4–18) were performed by Geiger and Marsden within a few months after its derivation (1911). They investigated the following points.

1. The angular dependence was tested, using foils of Ag and Au, over the angular range 5° to 150°. Although $\mathcal{N}(\Phi)\,d\Phi$ varies by a factor of about 10^5 over this range, the experimental data remained proportional to the theoretical angular distribution to within a few percent.

2. The quantity $\mathcal{N}(\Phi)\,d\Phi$ was found indeed to be proportional to t for a range of thickness of about 10 for all the elements investigated.

3. Equation (4–18) predicts that the number of scattered alpha particles will be inversely proportional to the square of their kinetic energy, $\frac{1}{2}mv^2$. This was tested by using alpha particles from several different radioactive elements. The predicted energy dependence was observed over an energy variation of about a factor of 3.

4. Finally, the equation predicts $\mathcal{N}(\Phi)\,d\Phi$ to be proportional to $(Ze)^2$, the square of the nuclear charge. At the time, Z was not known for the various atoms, except that, as mentioned in section 1, X-ray scattering experiments had shown that Z was roughly equal to $A/2$, where A is the chemical atomic weight of the atom. The validity of equation (4–18) being assumed, the experimental data could be used to determine Z since everything else was known. The resulting values agreed with the law $Z \sim A/2$. Furthermore it was observed that, to within the accuracy of the experiment, the measured values of Z were equal to the chemical atomic number of the atoms in question, i.e., equal to the ordering number

of the atoms in the periodic table. This implies that the first atom, H, in the periodic table contains 1 electron, the second atom, He, contains 2 electrons, the third atom, Li, contains 3 electrons, etc., since Z is the number of electrons in the atom. This was shortly confirmed by very accurate determinations of Z using X-ray techniques, which will be described in Chapter 14.

9. The Size of the Nucleus

By evaluating R, the distance of closest approach, Rutherford was able to place limits on the size of the nucleus. As an example, in the scattering of Ra alpha particles ($v = 1.6 \times 10^9$ cm/sec) from Cu ($A = 63, Z = 29$) it was observed that the Rutherford scattering law was obeyed to the largest scattering angle investigated, $\Phi = 180°$. From equation (4–17) we see that R takes on its smallest value at that angle. So

$$R_{180°} = D = \frac{zZe^2}{\frac{1}{2}Mv^2}$$

$$R_{180°} = \frac{2 \times 29 \times (4.8 \times 10^{-10})^2}{\frac{1}{2}(4 \times 1.67 \times 10^{-24}) \times (1.6 \times 10^9)^2}$$

$$R_{180°} \simeq 1.7 \times 10^{-12}\ \text{cm}$$

Rutherford's derivation depends on the assumption that the force acting on the alpha particle is always strictly a Coulomb force between two point charges. This would not be true if the particle penetrated the nuclear region at its point of closest approach. The obvious implication is that the radius of the Cu nucleus is $\leqslant 1.7 \times 10^{-12}$ cm.†

The equation above shows that R decreases as Z decreases. The question arises: How much can R decrease before $R <$ nuclear radius? Departures were actually observed in the scattering from the very light (low Z) nuclei. Part of this was due to a violation, for the very light nuclei, of the assumption that the nuclear mass is large compared to the alpha particle mass. However, deviations remained even after the finite nuclear mass was taken into account in the theory. Figure (4–12) shows some data for the scattering of alpha particles, of various velocities, at a fixed large

† We may also conclude that the use of classical mechanics to calculate the angular distribution for scattering from a Coulomb field is valid, even for collisions in which R is as small as 10^{-12} cm. In Chapter 15 we shall find that quantum theory verifies this for the case of a Coulomb force, but that for any other force classical mechanics would give only an approximation in such collisions.

angle from an Al foil. The ordinate is the ratio of the observed number of scattered particles to the number predicted by the Rutherford theory (corrected for the finite nuclear mass). The abscissa is the distance of closest approach calculated from equation (4–17). These data imply that the radius of the Al nucleus is about 10^{-12} cm.

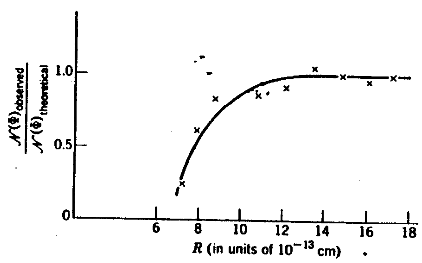

Figure 4–12. Some data obtained in the scattering of alpha particles from aluminum.

10. A Problem

The detailed experimental verification of the predictions of the nuclear model of the atom left little room for doubt concerning the validity of the model. However, the question of the *stability* of such an atom presented a serious problem.

If we assume that the electrons in the atom are stationary, it is easy to see that there exists no stable arrangement of the electrons surrounding the nucleus which would prevent the electrons from falling into the nucleus under the influence of its Coulomb attraction. We cannot allow the atom to collapse, because then its radius would be of the order of a nuclear radius, which is four orders of magnitude smaller than what we know the radius of an atom to be.

At first glance it appears that we can simply allow the electrons to circulate about the nucleus in orbits similar to the orbits of the planets circulating about the sun. Such a system can be stable mechanically because the centrifugal force can be made to balance the Coulomb attraction just as it balances the gravitational attraction in the planetary system. But a difficulty arises in trying to carry over this idea from the planetary system to the atomic system. The problem is that the charged electrons would be constantly accelerating in their motion around the nucleus and, according to the classical electromagnetic theory, all accelerating charged bodies radiate energy in the form of electromagnetic radiation. The energy would be emitted at the expense of the mechanical energy of the electron, and the electron would spiral into the nucleus. Again we have an atom which would rapidly collapse to nuclear dimensions. Furthermore, the continuous spectrum of the radiation which would be emitted is not in agreement with the discrete spectrum which is known to be emitted by atoms.

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EXERCISES

1. Use classical mechanics to prove the statement, made in section 4, that in a collision of a very massive body, initially moving at velocity v, with a free stationary body of small mass, the velocity of the light body after the collision can be no greater than $2v$. *Hint:* Make a Galilean transformation to the CM system, treat the collision, then transform back to the LAB system.

2. Derive equation (4–8) from Newton's law.

3. Adapt the calculation of section 7 to the case of a particle moving under an attractive inverse square force. Use the results to discuss the motion of a satellite.

4. Verify from equation (4–17) that $R \to D$ as $\phi \to \pi$, and that $R \to b$ as $\phi \to 0$. Give physical arguments explaining these results.

5. Adapt the calculation of section 7 to the case of an electron moving in a hydrogen atom, under the assumption that it radiates energy at the rate given by equation (2–7) because it is accelerated. If the radius of its orbit is initially 10^{-8} cm, how long would it take for it to spiral into the nucleus of radius 10^{-12} cm?