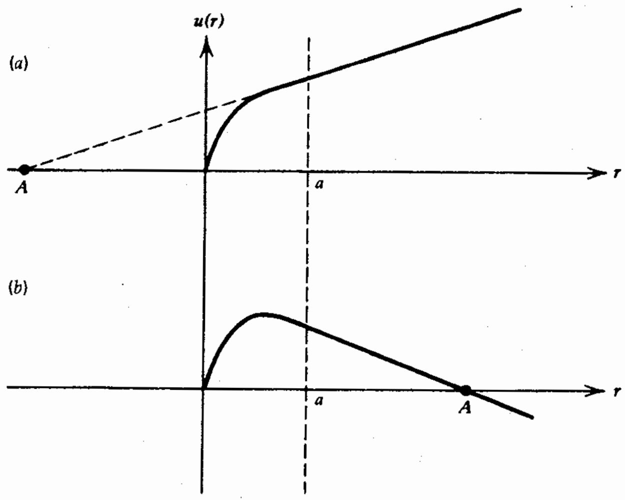

Figure 19-6 Sketch of the s-wave solution $u(r)$ near threshold. Outside the range radius $r = a$, the wave function has the form $C(r - A)$. [This is not in conflict with (19-73), which is an expansion of $\sin(kr + \delta)$. We could equally well have taken the form of $u(r)$ to be $(C/k)\sin(kr + \delta)$, since the normalization is arbitrary. It is, in fact, the interior wave function and the position of A that determine the slope of the line.] The sign of A depends on whether the interior wave function has or has not turned over cases (b) and (a), respectively. Since the wave function must turn over if there is a weakly bound state (so that it can match a slowly falling exponential) and since one does not expect the wave function inside the potential to be very sensitive to variations in E about zero, one expects that for a potential that has a bound state with E_B small, $A > 0$.

account, for example, that the effective mass of the proton in a molecule is different from that of a free proton, and that the molecules are not really at rest, but are moving with a distribution appropriate to the (low ~ 20 K) temperature. The large discrepancy between the two cross sections is not changed much by these corrections, and it can only be explained if A_s is indeed negative.

19-3 THE BORN APPROXIMATION

At higher energies many partial waves contribute to the scattering, and it is therefore preferable to avoid the angular momentum decomposition. A procedure that leads to a very useful approximation both when the potential is weak and when the energy is high is the Born approximation, in which we consider the scattering process as a transition, just like the transitions studied in Chapter 15. The difference is that here we consider the transitions

$$\text{continuum} \rightarrow \text{continuum}$$

If we work in the center-of-mass system, we have effectively a one-particle problem, and this particle makes a transition from an initial state, described by the eigenfunction

$$\psi_i(\mathbf{r}) = \frac{1}{\sqrt{V}} e^{i\mathbf{p}_i \cdot \mathbf{r}/\hbar} \tag{19-74}$$

to the final state, described by

$$\psi_f(\mathbf{r}) = \frac{1}{\sqrt{V}} e^{i\mathbf{p}_f \cdot \mathbf{r}/\hbar} \tag{19-75}$$

where $\mathbf{p}_i$ and $\mathbf{p}_f$ are the initial and final momenta, respectively. The transition rate, following the Golden Rule (15-20) is given by

$$R_{i\to f} = \frac{2\pi}{\hbar}\int \frac{V\,d^3\mathbf{p}_f}{(2\pi\hbar)^3}\,|M_{fi}|^2\,\delta\left(\frac{p_f^2}{2m} - \frac{p_i^2}{2m}\right) \tag{19-76}$$

The delta function expresses energy conservation. If the particles that emerge have a different mass from those that enter, or if the target is excited, that delta function takes a somewhat different form. It will, however, always be of the form $\delta[(p_f^2/2m) - E]$ where E is the energy available for kinetic energy of the final particle. The matrix element M_{fi} is given by

$$\begin{aligned} M_{fi} = \langle\psi_f|V|\psi_i\rangle &= \int d^3\mathbf{r}\,\frac{e^{-i\mathbf{p}_f\cdot\mathbf{r}/\hbar}}{\sqrt{V}}\,V(\mathbf{r})\,\frac{e^{i\mathbf{p}_i\cdot\mathbf{r}/\hbar}}{\sqrt{V}} \\ &= \frac{1}{V}\int d^3\mathbf{r}\,e^{-i\mathbf{\Delta}\cdot\mathbf{r}}\,V(\mathbf{r}) \end{aligned} \tag{19-77}$$

where $\mathbf{\Delta} = \frac{1}{\hbar}(\mathbf{p}_f - \mathbf{p}_i)$. We write the matrix element as

$$M_{fi} = \frac{1}{V}\,\tilde{V}(\Delta) \tag{19-78}$$

The integral in (19-76) can be rewritten in the form

$$\begin{aligned} R_{i\to f} &= \frac{2\pi}{\hbar}\int d\Omega\,\frac{Vp_f^2\,dp_f}{(2\pi\hbar)^3}\,\frac{1}{V^2}\,|\tilde{V}(\Delta)|^2\,\delta\left(\frac{p_f^2}{2m} - E\right) \\ &= \frac{2\pi}{\hbar}\,\frac{1}{(2\pi\hbar)^3}\,\frac{1}{V}\int d\Omega\,p_f m\,\frac{p_f\,dp_f}{m}\,\delta\left(\frac{p_f^2}{2m} - E\right)|\tilde{V}(\Delta)|^2 \\ &= \frac{1}{4\pi^2\hbar^4}\,\frac{1}{V}\int d\Omega\,p_f m|\tilde{V}(\Delta)|^2 \end{aligned} \tag{19-79}$$

To get the last line, we noted that $p_f\,dp_f/m = d(p_f^2/2m)$ and carried out the delta function integration. Thus, p_f must be evaluated at $p_f = (2mE)^{1/2}$, and we must not forget that m here is the reduced mass in the final state.

This expression has an undesirable dependence on the volume of the quantization box, but this is not really surprising. Our wave functions were normalized to one particle in the box V, so that the number of transitions should certainly go down as V increases. This difficulty arises because we are asking a question that does not correspond to an experiment. What one does is send a flux of incident particles at each other (in the center-of-mass frame; in the laboratory, one particle is stationary, of course). If we want a flux of one particle per m^2 per second, we must multiply the preceding by V divided by the volume of a cylinder with 1-m^2 base, and the relative velocity of the particles in the center-of-mass frame in the initial state. The number of transitions for unit flux is just the cross section. We therefore have

$$d\sigma = \frac{1}{4\pi^2\hbar^4}\,\frac{1}{|v_{\text{rel}}|}\,d\Omega\,p_f m|\tilde{V}(\Delta)|^2 \tag{19-80}$$

Since in the center-of-mass frame the two incident particles are moving toward each other with equal and opposite momenta of magnitude p_i, their relative velocity is

$$|v_{\text{rel}}| = \frac{p_i}{m_1} + \frac{p_i}{m_2} = p_i\left(\frac{1}{m_1} + \frac{1}{m_2}\right) = \frac{p_i}{m_{\text{red}}^{(i)}} \tag{19-81}$$

if m_1 and m_2 are their masses. Thus, if the initial and final reduced masses and momenta are not the same, we have

$$\frac{d\sigma}{d\Omega} = \frac{1}{4\pi^2}\frac{p_f}{p_i} m_{\text{red}}^{(f)} m_{\text{red}}^{(i)} \left|\frac{1}{\hbar^2}\tilde{V}(\Delta)\right|^2 \tag{19-82}$$

When the initial and final particles are the same,

$$\frac{d\sigma}{d\Omega} = \frac{m_{\text{red}}^2}{4\pi^2}\left|\frac{1}{\hbar^2}\tilde{V}(\Delta)\right|^2 \tag{19-83}$$

When one particle is a great deal more massive than the other, $m_{\text{red}} \to m$, the mass of the lighter particle. When we compare the above with (19-15) we see that

$$f(\theta, \phi) = -\frac{m_{\text{red}}}{2\pi\hbar^2}\tilde{V}(\Delta) \tag{19-84}$$

Actually, to determine the sign, one must go through a more detailed comparison with the partial wave expansion. We will not bother to do this here.

As an illustration of the application of the Born approximation, we will calculate the cross section for the scattering of a particle of mass m and charge Z_1 by a Coulomb potential of charge Z_2. The source of the Coulomb field is taken to be infinitely massive, so that the mass in (19-83) is the mass of the incident particle. For generality (and, as we will see, for technical reasons) we take the Coulomb field to be screened, so that

$$V(\mathbf{r}) = \frac{Z_1 Z_2 e^2}{4\pi\varepsilon_0}\frac{e^{-r/a}}{r} \tag{19-85}$$

where a is the screening radius. We thus need to evaluate

$$\tilde{V}(\Delta) = \frac{Z_1 Z_2 e^2}{4\pi\varepsilon_0}\int d^3 r e^{i\Delta\cdot\mathbf{r}}\frac{e^{-r/a}}{r} \tag{19-86}$$

We choose the direction of $\mathbf{\Delta}$ as z-axis, and then get

$$\begin{aligned}\int d^3\mathbf{r}\, e^{-i\Delta\cdot\mathbf{r}}\frac{e^{-r/a}}{r} &= \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\, d\theta \int_0^{\infty} r^2\, dr\, e^{-i\Delta r\cos\theta}\frac{e^{-r/a}}{r} \\ &= 2\pi \int_0^{\infty} r dr\, e^{-r/a}\int_{-1}^{1} d(\cos\theta)\, e^{-i\Delta r\cos\theta} \\ &= \frac{2\pi}{i\Delta}\int_0^{\infty} dr\, e^{-r/a}\left(e^{i\Delta r} - e^{-i\Delta r}\right) \\ &= \frac{2\pi}{i\Delta}\left(\frac{1}{(1/a) - i\Delta} - \frac{1}{(1/a) + i\Delta}\right) = \frac{4\pi}{(1/a^2) + \Delta^2}\end{aligned} \tag{19-87}$$

Now

$$\Delta^2 = \frac{1}{\hbar^2}(\mathbf{p}_f - \mathbf{p}_i)^2 = \frac{1}{\hbar^2}(2p^2 - 2\mathbf{p}_f\cdot\mathbf{p}_i) = \frac{2p^2}{\hbar^2}(1 - \cos\theta) \tag{19-88}$$

so that the cross section becomes

$$\frac{d\sigma}{d\Omega} = \frac{m^2}{4\pi^2\hbar^4}\left(\frac{Z_1Z_2e^2}{4\pi\varepsilon_0}\right)^2 \frac{16\pi^2}{[(2p^2/\hbar^2)(1-\cos\theta)+(1/a^2)]^2}$$

$$= \left(\frac{2m\dfrac{Z_1Z_2e^2}{4\pi\varepsilon_0}}{4p^2\sin^2(\theta/2)+(\hbar^2/a^2)}\right)^2 \qquad (19\text{-}89)$$

$$= \left(\frac{\dfrac{Z_1Z_2e^2}{4\pi\varepsilon_0}}{4E\sin^2(\theta/2)+(\hbar^2/2ma^2)}\right)^2$$

In the last line we replaced $p^2/2m$ by E, and we used $\frac{1}{2}(1 - \cos\theta) = \sin^2(\theta/2)$. The angle θ defined in (19-88) is the center-of-mass scattering angle. In the absence of screening ($a \to \infty$) this reduces to the well-known Rutherford formula. There is no $\hbar$ in it, and it is the same as the classical formula. Had we left out the screening factor in (19-86) we would have had an ill-defined integral. One often evaluates ambiguous integrals with the aid of such convergence factors.

The Born approximation has its limitations. For example, we found that $\tilde{V}(\mathbf{\Delta})$ was purely real so that $f(\theta)$ is also real in this approximation. This implies, by the optical theorem, that the cross section is zero. In fact, the Born approximation is only good when either (a) the potential is weak, so that the cross section is of second order in a small parameter; this would make the use of it consistent with the optical theorem, or (b) at high energies for potentials such that the cross section goes to zero. This is true for most smooth potentials. It is not true for real particles; there it seems that the cross sections stay constant at very high energies, and one cannot expect the Born approximation to serve as more than a guide of the behavior of the scattering amplitude.

As a final comment, we observe that if the potential V has a spin dependence, then (19-77) is trivially modified by the requirement that the initial and final states be described by their spin wave functions, in addition to the spatial wave functions. Thus, for example, if the neutron–proton potential has the form

$$V(r) = V_1(r) + \boldsymbol{\sigma}_P \cdot \boldsymbol{\sigma}_N V_2(r)$$

the Born approximation reads

$$M_{fi} = \frac{1}{V}\int d^3r\, e^{-i\Delta\cdot r}\, \xi_f^+\, V(r)\, \xi_i$$

where ξ_i and ξ_f represent the initial and final spin states of the neutron–proton system.

19-4 SCATTERING OF IDENTICAL PARTICLES

When two identical particles scatter, there is no way of distinguishing a deflection of a particle through an angle θ and a deflection of $\pi - \theta$ in the center-of-mass frame, since momentum conservation demands that if one of the particles scatters through θ, the other goes in the direction $\pi - \theta$ (Fig. 19-7). Classically, too, the cross section for scattering is affected by the identity of the particles, since the number of counts at a certain counter will be the sum of the counts due to the two particles. Thus

$$\sigma_{cl}(\theta) = \sigma(\theta) + \sigma(\pi - \theta) \qquad (19\text{-}90)$$