MASSACHUSETTS INSTITUTE OF TECHNOLOGY
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Lecture Notes 4

MORE DYNAMICS OF NEWTONIAN COSMOLOGY

THE AGE OF A FLAT UNIVERSE:

We showed last time that the scale factor for a flat, matter-dominated universe behaves as

$$a(t) = bt^{2/3} , \tag{4.1}$$

for some constant of proportionality b. This model universe is "flat" in the sense that $k = 0$ (or $\Omega = 1$), and it is "matter-dominated" in the sense that the mass density is dominated by the rest mass of nonrelativistic particles. (The assumption of matter-domination entered the derivation when we assumed that the total mass $M(r_i)$ contained within a given comoving radius r_i does not change as the system evolves. Photons or highly relativistic particles, by contrast, would redshift as the universe expands, and hence they would lose energy. In Lecture Notes 6 we will discuss the evolution of a universe dominated by relativistic particles, and in Lecture Notes 7 we will describe the effects of a cosmological constant, or vacuum energy. Since the energy loss of an expanding gas is proportional to its pressure, the "matter-dominated" case can also be described as the case in which the pressure is negligible.) Using Eq. (4.1), it is easy to calculate the age of such a universe in terms of the Hubble expansion rate. In Lecture Notes 2 it was shown that the Hubble expansion rate is given by

$$H(t) = \dot{a}/a , \tag{4.2}$$

where the over-dot has been used to denote a derivative with respect to time t. Thus,

$$H(t) = \frac{\frac{2}{3}bt^{-1/3}}{bt^{2/3}} = \frac{2}{3t} . \tag{4.3}$$

Recall that we have defined the origin of time so that the scale factor $a(t)$ vanishes at $t = 0$, so $t = 0$ is the earliest time that exists within the mathematical model. We therefore refer to t as the age of the universe, which can then be expressed in terms of the Hubble expansion rate by

$$\boxed{t = \frac{2}{3}H^{-1} .} \tag{4.4}$$

We should keep in mind, however, that we would be foolish to pretend that we actually understand the origin of the universe, so the phrase "age of the universe" is being used loosely. It is almost certain that our universe underwent an extremely hot dense phase at $t \approx 0$, and we can call this phase the big bang. The variable t represents the age of the universe since the big bang, but we can only speculate about whether the big bang actually represents the beginning of time. As we will see near the end of the course, the current understanding of inflationary cosmology suggests that the big bang was very likely not the beginning of time.

Consistency requires that the age of the universe be older than the age of the oldest stars, and this requirement has turned out to be a strong constraint. It was a very serious problem in the time shortly after Edwin Hubble's first measurement, when Hubble's bad estimate for the Hubble expansion rate led to age estimates of only several billion years. More recently, as quoted in Lecture Notes 2, the Planck satellite team, using their own data combined with other data, estimated that $H_0 = 67.66 \pm 0.42$ km-s^{-1}-Mpc^{-1}.* Using Eq. (4.4) and the relationship

$$\frac{1}{10^{10}\ \text{yr}} = 97.8\ \text{km-s}^{-1}\text{-Mpc}^{-1}\ ,$$

we find that a flat matter-dominated universe with a Hubble expansion rate in this range (i.e., 67.24 to 68.08 km-s^{-1}-Mpc^{-1}) must have an age between 9.58 and 9.70 billion years.

Today the oldest stars are believed to be those in globular clusters, which are tightly bound, nearly spherical distributions of stars that are found in the halos of galaxies. A careful study of the globular cluster M4 by Hansen et al.†, using data from 123 orbits of the Hubble Space Telescope, determined an age of 12.7 ± 0.7 billion years. Krauss and Chaboyer‡ argue that Hansen et al. did not take into account all the uncertainties, and claim that the globular clusters in the Milky Way (including M4) have an age of $12.6^{+3.4}_{-2.2}$ billion years. (Both sets of authors are quoting 95% confidence limit errors, also called 2σ errors, meaning that the probability of the true value lying in the quoted range is estimated at 95%.) Krauss and Chaboyer estimate that the stars must take at least 0.8 billion years to form, implying that the universe must be at least 11.2 billion years old.

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* N. Aghanim et al. (Planck Collaboration), "Planck 2018 results, VI: Cosmological parameters," Table 2, Column 6, arXiv:1807.06209.

† B.M.S. Hansen et al., "The White Dwarf Cooling Sequence of the Globular Cluster Messier 4," *Astrophysical Journal Letters*, vol. 574, p. L155 (2002), http://arXiv.org/abs/astro-ph/0205087.

‡ L.M. Krauss and B. Chaboyer, "Age Estimates of Globular Clusters in the Milky Way: Constraints on Cosmology," *Science,* vol. 299, pp. 65-70 (2003), available with MIT certificates at http://www.sciencemag.org.libproxy.mit.edu/content/299/5603/65.abstract or for purchase at http://www.sciencemag.org/content/299/5603/65.abstract

(That is, the stars are at least 12.6 - 2.2 = 10.4 billion years old, and could not have formed until the universe was already 0.8 billion years old.)

With either of these estimates for the age of the oldest stars, however, the age of the universe is inconsistent with the measured value of the Hubble expansion rate and the assumption of a flat, matter-dominated universe. We will see later in these lecture notes that the age calculation gives a larger answer if we assume an open universe ($\Omega \equiv \rho/\rho_c < 1$), but inflationary cosmology predicts that Ω should be very close to 1, and starting with the first BOOMERANG long duration balloon experiment of 1997,* measurements of the fluctuations in the CMB have indicated a nearly flat universe. The CMB measurements now imply that $\Omega = 1$ to high accuracy. Combining their own data with results of other experiments, the Planck team† concluded that $\Omega = 0.9993 \pm 0.0037$. We will see in Lecture Notes 7, however, that the age discrepancy problem is completely resolved by the inclusion of dark energy. With about 70% of the mass density in the form of dark energy, $H_0 = 67.7$ km-s^{-1}-Mpc^{-1} is consistent with an age of 13.79 ± 0.02 billion years, which is our current best estimate‡ for the age of the universe. Since the age estimates are not consistent if we assume a matter-dominated flat universe, the age calculations help to support the proposition that our universe is dominated by dark energy, which we will discuss in detail in Lecture Notes 7.

THE BIG BANG SINGULARITY:

This mathematical model of the universe starts from a configuration with $a(t) = 0$, which corresponds to infinite density. From Eq. (4.3) we see that the initial value of the Hubble expansion rate is also infinite. We will see shortly that these infinities are not peculiarities of the flat universe model, but occur also in the models for either a closed or open universe. This instant of infinite density is called a singularity.

One should realize, however, that there is no reason to believe that the equations which we have used are valid in the vicinity of this singularity. Thus, although our mathematical models of the universe certainly begin with a singularity, it is an open question whether the universe actually began with a singularity. If we use our equations to follow the history of the universe further and further into the past, the universe becomes denser and denser without limit. At some point one encounters densities that are so far beyond our experience that the equations are no longer to be trusted. We will discuss later in the course where this point may occur, but for now I just want to make it clear that the singularity should not be considered a reliable consequence of the theory.

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* P. D. Mauskopf et al., "Measurement of a peak in the cosmic microwave background power spectrum from the North American test flight of BOOMERANG," *Astrophysical Journal Letters,* vol. 536, pp. L59–L62 (2000), http://arXiv.org/abs/astro-ph/9911444.

† N. Aghanim et al. (Planck Collaboration), cited above, Table 4, Column 4.

‡ N. Aghanim et al. (Planck Collaboration), cited above, Table 2, Column 6.

THE HORIZON DISTANCE:

Since the age of the universe in the big bang model is finite, it follows that there is a theoretical upper limit to how far we can see. Since light travels at a finite speed, there will be particles in the universe that are so far away that light emitted from these particles will not yet have reached us. The present distance of the furthest particles from which light has had time to reach us is called the horizon distance. If the universe were static and had age t, then the horizon distance would be simply ct. In the real universe, however, everything is constantly in motion, and the value of the horizon distance has to be calculated.

The calculation of the horizon distance can be done most easily by using comoving coordinates. The speed of light rays in a comoving coordinate system was given in Eq. (2.8) as

$$\frac{dx}{dt} = \frac{c}{a(t)} \ . \tag{4.5}$$

(Recall that this formula is based on the statement that the speed of light in meters per second is constant, but the scale factor $a(t)$ is needed to convert from meters per second to notches per second.) One can then calculate the coordinate horizon distance (i.e., the horizon distance in "notches") by calculating the distance which light rays could travel between time zero and some arbitrary final time t. By integrating Eq. (4.5), one sees that

$$\ell_{c,\mathrm{horizon}}(t) = \int_0^t \frac{c}{a(t')} dt' \ . \tag{4.6}$$

The physical horizon distance at time t is then given by

$$\ell_{p,\mathrm{horizon}}(t) = a(t) \int_0^t \frac{c}{a(t')} dt' \ . \tag{4.7}$$

For the special case of a flat, matter-dominated universe, one can use Eq. (4.1) to obtain

$$\ell_{p,\mathrm{horizon}}(t) = bt^{2/3} \int_0^t \frac{c}{bt'^{2/3}} dt' \ . \tag{4.8}$$

Carrying out the integration, the physical horizon distance for a flat matter-dominated universe is found to be

$$\boxed{\ell_{p,\mathrm{horizon}}(t) = 3ct \ .} \tag{4.9}$$

The horizon distance can also be expressed in terms of the Hubble expansion rate, by using Eq. (4.4):

$$\boxed{\ell_{p,\mathrm{horizon}}(t) = 2cH^{-1} \ .} \tag{4.10}$$

Taking $H_0 = 67.7$ km-s^{-1}-Mpc^{-1}, the horizon distance today, under the assumption of a flat matter-dominated universe, is found to be about 29 billion light-years.

The factor of 3 on the right-hand side of Eq. (4.9) implies that the horizon distance is three times larger than it would be in a static universe, and hence it is significantly larger than most of us would probably guess. The reason, of course, is that the horizon distance refers to the ***present*** distance of the most distant matter that can in principle be seen. However, the light that we receive from distant sources was emitted long ago, and the present distance is large because the matter has been moving away from us ever since. In the limiting case of matter exactly at the horizon, the light that we receive today left the object at $t = 0$, at the instant of the big bang.

You might wonder why light from every object did not reach us immediately, since the scale factor $a(t)$ was zero at $t = 0$, so the initial distance between any two objects was zero. To be honest, you are pretty safe in believing whatever you like about what happens at the initial singularity. The classical description certainly breaks down as the mass density approaches infinity, and there does not yet exist a satisfactory quantum description. So, if you want to believe that everything could communicate with everything else at the instant of the singularity, nobody whom I know could prove that you are wrong. But nobody whom I know could prove that you are right, either. If one ignores the possibility of communication at the singularity, however, it is then a well-defined question to ask how far light signals can travel once the classical description becomes valid. As $t \to 0$ the scale factor $a(t)$ approaches zero, but its time derivative $\dot{a}(t)$ approaches infinity. If we think about some object at coordinate distance ℓ_c from us, at early times its physical distance $a(t)\ell_c$ was arbitrarily small, but its velocity of recession $\dot{a}(t)\ell_c$ was arbitrarily large. If such an object emitted a light pulse in our direction at some very early time $t = \epsilon > 0$, even though the light pulse would have traveled toward us at the speed of light, the expansion of the universe was so fast that the distance between us and the light pulse would have initially increased with time. The coordinate distance that the light pulse could travel between then and now is at most equal to $\ell_{c,\text{horizon}}$, as given Eq. (4.6), so the pulse could reach us by now only if the present distance to the object is less than the horizon distance as given by Eq. (4.7).

EVOLUTION OF A CLOSED UNIVERSE:

The time evolution equations are easiest to solve for the case of a flat ($k = 0$) universe, but the equations for a closed or open universe are also soluble. We will first consider the closed universe, for which $k > 0$, $E < 0$, and $\Omega > 1$.

From Lecture Notes 3, we write the Friedmann equation as

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3} G\rho - \frac{kc^2}{a^2} \ , \tag{4.11}$$

where the mass density ρ can be written as

$$\rho(t) = \left[\frac{a(t_i)}{a(t)}\right]^3 \rho(t_i) \ . \tag{4.12}$$

(The above equation is a slight generalization of Eq. (3.23), $\rho(t) = \rho_i/a^3(t)$. Eq. (3.23) was derived under the assumption that $a(t_i) = 1$, while Eq. (4.12) makes no such requirement.) Here we will use Eq. (4.12) to conclude that the quantity $\rho(t)a^3(t)$ is independent of time.

To obtain the desired solution in an economical way, it is useful to identify from the beginning the quantities of physical interest. Note that a is measured in units of, for example, meters per notch, while k is measured in units of notch^{-2}. Thus the quantity $a/\sqrt{k}$ has units of physical length (meters, for example), and is therefore independent of the definition of the notch. We will therefore choose

$$\tilde{a}(t) \equiv \frac{a(t)}{\sqrt{k}} \tag{4.13}$$

as the variable to use in our solution of the differential equation. Similarly we will use

$$\tilde{t} \equiv ct \ , \tag{4.14}$$

rather than t, so that all the spacetime variables have units of length.

Multiplying the Friedmann equation (4.11) by $a^2/(kc^2)$, it can be rewritten as

$$\begin{aligned} \frac{1}{kc^2}\left(\frac{da}{dt}\right)^2 &= \frac{8\pi}{3}\frac{G\rho a^2}{kc^2} - 1 \\ &= \frac{8\pi}{3}\frac{G\rho a^3}{k^{3/2}c^2}\frac{\sqrt{k}}{a} - 1 \ . \end{aligned} \tag{4.15}$$

In the second line the factors have been arranged so that we can rewrite it as

$$\left(\frac{d\tilde{a}}{d\tilde{t}}\right)^2 = \frac{2\alpha}{\tilde{a}} - 1 \ , \tag{4.16}$$

where

$$\alpha \equiv \frac{4\pi}{3}\frac{G\rho\tilde{a}^3}{c^2} \ . \tag{4.17}$$

Note that the parameter α also has the units of length. While the above expression for α contains the product $\rho(t)\tilde{a}^3(t)$, Eq. (4.12) guarantees that this quantity is independent of time, so α is a constant. Eq. (4.16) can be solved formally by rewriting it as

$$d\tilde{t} = \frac{d\tilde{a}}{\sqrt{\frac{2\alpha}{\tilde{a}} - 1}} = \frac{\tilde{a}\,d\tilde{a}}{\sqrt{2\alpha\tilde{a} - \tilde{a}^2}} \tag{4.18}$$

and then integrating both sides. One could use indefinite integrals, as we did for the $k = 0$ case in Eq. (3.37). Using indefinite integrals, the arbitrary constant of integration would become an arbitrary constant in the solution to the equation. We found, however, that the arbitrary constant could be eliminated by choosing the zero of time so that $a(t) = 0$ at $t = 0$. Here, mainly for the purpose of demonstrating an alternative method, I will use definite integrals. In this method the arbitrary constant in the solution will be fixed by the specification of the limits of integration. Following the standard convention, I will again choose the zero of time to be the time at which the scale factor is equal to zero. Using a subscript f to denote an arbitrary final time, one has

$$\tilde{t}_f = \int_0^{\tilde{t}_f} d\tilde{t} = \int_0^{\tilde{a}_f} \frac{\tilde{a}\, d\tilde{a}}{\sqrt{2\alpha\tilde{a} - \tilde{a}^2}} \ , \tag{4.19}$$

where $\tilde{a}_f \equiv \tilde{a}(\tilde{t}_f)$. The subscripts f will be dropped when the problem is finished, but for now it is convenient to use them to distinguish the limits of integration from the variables of integration.

The only remaining step is to carry out the integration shown in Eq. (4.19). Completing the square in the denominator, and then replacing the variable of integration by

$$x \equiv \tilde{a} - \alpha \ , \tag{4.20}$$

one finds

$$\begin{aligned} \tilde{t}_f &= \int_0^{\tilde{a}_f} \frac{\tilde{a}\, d\tilde{a}}{\sqrt{\alpha^2 - (\tilde{a} - \alpha)^2}} \\ &= \int_{-\alpha}^{\tilde{a}_f - \alpha} \frac{(x + \alpha)\, dx}{\sqrt{\alpha^2 - x^2}} \ . \end{aligned} \tag{4.21}$$

The integral can now be simplified by the trigonometric substitution

$$x = -\alpha \cos\theta \ , \tag{4.22}$$

which leads to

$$\tilde{t}_f = \alpha \int_0^{\theta_f} (1 - \cos\theta) d\theta = \alpha(\theta_f - \sin\theta_f) \ . \tag{4.23}$$

Since θ_f denotes the final value of θ, we can combine Eqs. (4.20) and (4.22) to find $x_f = \tilde{a}_f - \alpha = -\alpha\cos\theta_f$, so

$$\tilde{a}_f = \alpha(1 - \cos\theta_f) \ . \tag{4.24}$$

Dropping the subscript f and recalling the definitions (4.13) and (4.14), the two equations above provide a solution to our problem:

$$ct = \alpha(\theta - \sin\theta) \; , \tag{4.25a}$$

$$\frac{a}{\sqrt{k}} = \alpha(1 - \cos\theta) \; . \tag{4.25b}$$

These two equations provide a ***parametric*** description of the function $a(t)$. That is, Eq. (4.25a) determines in principle the function $\theta(t)$, and then Eq. (4.25b) defines the function $a\big(\theta(t)\big)$. (There is, however, no explicit expression for $\theta(t)$, so the function $a(t)$ cannot be constructed explicitly.) Some of you may recognize these equations as the equations for a cycloid. The curve which they trace can be generated by imagining a graph of $a/\sqrt{k}$ vs. ct, with a disk of radius α that rolls along the t-axis, as shown below. As the disk rolls, the point P traces the graph of $a/\sqrt{k}$ vs. ct:

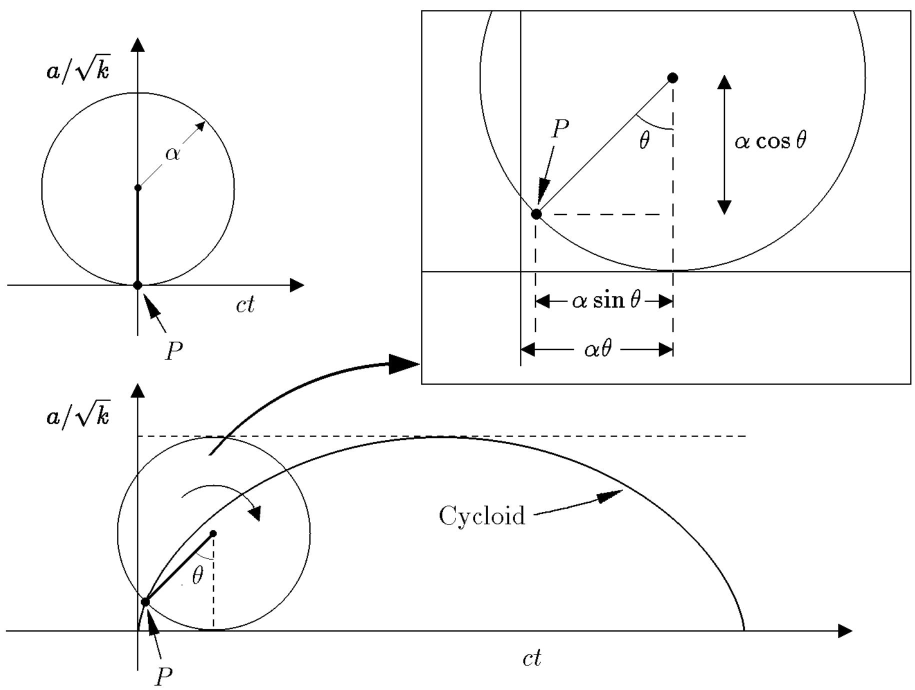

Figure 4.1: Evolution of a closed universe. The figure shows a graph of $a/\sqrt{k}$ vs. ct for a closed matter-dominated universe. If one imagines a circle rolling on the ct axis, a point on the circle traces out a cycloid, which is exactly the equation for $a/\sqrt{k}$ for a closed universe. The insert at the upper right includes labels for various distances, showing the connection with Eq. (4.25).

The relation between Eqs. (4.25) and the rolling disk can be seen in the insert at the top right: after the disk has rolled through an angle θ, the horizontal component of P is given by $\alpha\theta - \alpha\sin\theta$, and the vertical component is given by $\alpha - \alpha\cos\theta$.

The model for the closed universe reaches a maximum scale factor when $\theta = \pi$, which corresponds to a time $ct = \pi\alpha$. The corresponding value of a is given by

$$\frac{a_{\max}}{\sqrt{k}} = 2\alpha \ . \tag{4.26}$$

By that point the pull of gravity has overcome the inertia of the expansion, and the universe begins to contract. The scale factor during the contracting phase reverses the behavior it had during the expansion phase, and the universe ends in a "big crunch". The total lifetime of this universe is then

$$t_{\text{total}} = \frac{2\pi\alpha}{c} = \frac{\pi a_{\max}}{c\sqrt{k}} \ . \tag{4.27}$$

The angle θ is sometimes called the "development angle," because it describes the stage of development of the universe. The universe begins at $\theta = 0$, reaches its maximum expansion at $\theta = \pi$, and then is terminated by a big crunch at $\theta = 2\pi$.

Eqs. (4.25) contain the two parameters α and k, which might lead one to believe that there is a two-parameter family of closed universes. We must remember, however, that these equations still allow us the freedom to define the notch, so the numerical value of k is physically irrelevant. Many books, in fact, ***define*** k to always have the value $+1$ for a closed universe. The parameter α, on the other hand, is physically meaningful, and is related to the total lifetime of the closed universe. (When general relativity effects are described in Lecture Notes 5, we will learn that a closed universe actually has a finite size, and that the maximum size is determined by α.) Thus there is really only a one-parameter class of solutions.

THE AGE OF A CLOSED UNIVERSE:

The formula for the age of a closed universe can be obtained from the formulas in the previous section, but we have to do a little work. Eqs. (4.25) tell how to express the age in terms of α and θ, but this is not the result we want. We would prefer to relate α and θ to other quantities that are in principle measurable. Since we need to determine two variables, α and θ, we will have to imagine measuring two physical quantities. These two measurable quantities can be taken to be the Hubble expansion rate H and the mass density parameter $\Omega \equiv \rho/\rho_c$.

Our goal, then, is to express all the quantities related to the closed universe model in terms of H and Ω. To start, the mass density ρ can be rewritten as $\Omega\rho_c$, where $\rho_c = 3H^2/8\pi G$ (see Eq. (3.33)). So

$$\rho = \frac{3H^2\Omega}{8\pi G} \ , \tag{4.28}$$

or equivalently

$$\frac{8\pi}{3} G\rho = H^2\Omega \ . \tag{4.29}$$

Recalling that $H = \dot{a}/a$ (see Eq. (2.7)), the above formula can be substituted into the Friedmann equation (4.11), yielding

$$H^2 = H^2\Omega - \frac{kc^2}{a^2} \ , \tag{4.30}$$

which can then be solved for a^2 to give

$$\tilde{a}^2 = \frac{a^2}{k} = \frac{c^2}{H^2(\Omega - 1)} \ . \tag{4.31}$$

If we want to get our signs right for the entire evolution of the closed universe, we need to be careful. Note that Eq. (4.31) does not determine the sign of $\tilde{a}$, since it only specifies the value of $\tilde{a}^2$. The scale factor is by definition positive, however, and for the case under consideration $k > 0$. We adopt the standard convention that the square root of a positive number is positive, so $\sqrt{k} > 0$. Thus, the definition $\tilde{a} \equiv a/\sqrt{k}$ implies that $\tilde{a} > 0$, so Eq. (4.31) implies that

$$\tilde{a} = \frac{a}{\sqrt{k}} = \frac{c}{|H|\sqrt{\Omega - 1}} \ . \tag{4.32}$$

We write Eq. (4.32) with absolute value indicators around H, because during the contracting phase H is negative, while we know that only the positive square root of Eq. (4.31) is physically relevant.

We are now ready to evaluate α, using the definition (4.17). Using Eqs. (4.28) and (4.32) to replace ρ and $\tilde{a}$, we find

$$\alpha = \frac{c}{2|H|} \frac{\Omega}{(\Omega - 1)^{3/2}} \ . \tag{4.33}$$

Recall that α has direct physical meaning — if our universe is closed, then the total lifetime of the universe (from big bang to big crunch) would be given by $2\pi\alpha/c$.

The value of θ can now be found from Eq. (4.25b), using Eq. (4.32) to replace $a/\sqrt{k}$ on the left-hand side, and Eq. (4.33) to replace α on the right-hand side. After these substitutions, Eq. (4.25b) becomes

$$\frac{c}{|H|\sqrt{\Omega - 1}} = \frac{c}{2|H|} \frac{\Omega}{(\Omega - 1)^{3/2}} (1 - \cos\theta) \ , \tag{4.34}$$

which can be solved for either $\cos\theta$ or for Ω:

$$\cos\theta = \frac{2 - \Omega}{\Omega} \ , \tag{4.35}$$

$$\Omega = \frac{2}{1+\cos\theta} \ . \tag{4.36}$$

Using

$$\sin\theta = \pm\sqrt{1-\cos^2\theta} = \pm\frac{2\sqrt{\Omega-1}}{\Omega} \ , \tag{4.37}$$

Eq. (4.25a) can now be rewritten to obtain the desired formula for the age of the universe:

$$t = \frac{\Omega}{2|H|(\Omega-1)^{3/2}}\left\{\arcsin\left(\pm\frac{2\sqrt{\Omega-1}}{\Omega}\right) \mp \frac{2\sqrt{\Omega-1}}{\Omega}\right\} \quad \text{(for a closed universe)}, \tag{4.38}$$

where the sign choices correspond to Eq. (4.37) (i.e., if the upper sign is used in Eq. (4.37), then both upper signs should be used in Eq. (4.38).) Both the square root and the inverse sine function are multibranched functions, so the evaluation of Eq. (4.38) requires some additional information. It is easy to see which branch to use, however, if one remembers that Eq. (4.38) is a rewriting of Eq. (4.25a), and that in Eq. (4.25a) the variable θ runs monotonically from 0 to 2π over the lifespan of the closed universe.

To describe the branches in detail, it is necessary to divide the full cycle, with θ varying from 0 to 2π, into quadrants. The first quadrant is from $\theta = 0$ to $\theta = \pi/2$, where we see from Eq. (4.36) that $\theta = \pi/2$ corresponds to $\Omega = 2$. Thus, the first quadrant corresponds to the beginning of the expanding phase, with $1 \le \Omega \le 2$. For this quadrant $\sin\theta > 0$, so we use the upper signs in Eq. (4.38), and the inverse sine is evaluated in the range 0 to $\pi/2$. The other quadrants are understood in the same way, producing the following table of rules:

Quadrant	Phase	Ω	Sign Choice	$\sin^{-1}()$
1	Expanding	1 to 2	Upper	0 to $\frac{\pi}{2}$
2	Expanding	2 to ∞	Upper	$\frac{\pi}{2}$ to π
3	Contracting	∞ to 2	Lower	π to $\frac{3\pi}{2}$
4	Contracting	2 to 1	Lower	$\frac{3\pi}{2}$ to 2π

Our universe is not currently matter-dominated, but it could be just barely closed. If we considered the hypothesis that our universe was matter-dominated, it would certainly be in the expanding phase, with $\Omega < 2$, and so it would be in the first quadrant. That means that the age would be given by Eq. (4.38), using the upper signs, and evaluating the inverse sine function in the range of 0 to $\pi/2$.

EVOLUTION OF AN OPEN UNIVERSE:

The evolution of an open universe can be calculated in a very similar way, except that one must use hyperbolic trigonometric substitutions in order to carry out the crucial integral. Fortunately, none of the complications with multibranched functions will occur in this case. For the open universe $k < 0$, $E > 0$, and $\Omega < 1$. We will define $\kappa \equiv -k$ to avoid the inconvenience of working with a negative quantity, and we will define

$$\tilde{a}(t) \equiv \frac{a(t)}{\sqrt{\kappa}} \tag{4.39}$$

instead of $\tilde{a} = a(t)/\sqrt{k}$. Eq. (4.16) is then replaced by

$$\left(\frac{d\tilde{a}}{d\tilde{t}}\right)^2 = \frac{2\alpha}{\tilde{a}} + 1 \ , \tag{4.40}$$

while Eq. (4.17),

$$\alpha \equiv \frac{4\pi}{3}\frac{G\rho\tilde{a}^3}{c^2} \ ,$$

continues to be valid, although the meaning of $\tilde{a}$ has changed. Eqs. (4.19) and (4.21) are then replaced by

$$\begin{aligned} \tilde{t}_f &= \int_0^{\tilde{a}_f} \frac{\tilde{a}\,d\tilde{a}}{\sqrt{2\alpha\tilde{a}+\tilde{a}^2}} \\ &= \int_0^{\tilde{a}_f} \frac{\tilde{a}\,d\tilde{a}}{\sqrt{(\tilde{a}+\alpha)^2-\alpha^2}} \\ &= \int_\alpha^{\tilde{a}_f+\alpha} \frac{(x-\alpha)\,dx}{\sqrt{x^2-\alpha^2}} \ . \end{aligned} \tag{4.41}$$

In this case one uses the substitution

$$x = \alpha\cosh\theta \ , \tag{4.42}$$

and one obtains the result

$$ct = \alpha(\sinh\theta - \theta) \ , \tag{4.43a}$$

$$\frac{a}{\sqrt{\kappa}} = \alpha(\cosh\theta - 1) \ . \tag{4.43b}$$

THE AGE OF AN OPEN UNIVERSE:

The methods are again identical to the ones used previously, so I will just state the results. Eqs. (4.35), (4.36) and (4.37) are replaced by

$$\Omega = \frac{2}{1+\cosh\theta}\ , \quad \cosh\theta = \frac{2-\Omega}{\Omega} \tag{4.44}$$

and

$$\sinh\theta = \sqrt{\cosh^2\theta - 1} = \frac{2\sqrt{1-\Omega}}{\Omega}\ , \tag{4.45}$$

and the final result for the age of the universe becomes

$$t = \frac{\Omega}{2H(1-\Omega)^{3/2}}\left\{\frac{2\sqrt{1-\Omega}}{\Omega} - \sinh^{-1}\left(\frac{2\sqrt{1-\Omega}}{\Omega}\right)\right\} \quad \text{(for an open universe).} \tag{4.46}$$

Below is a graph of Ht versus Ω, using Eqs. (4.38) and (4.46). The graph shows that the curve is actually continuous at $\Omega = 1$, even though the expressions (4.38) and (4.46) look rather different. In fact, these two expressions are really not so different. Although it is not obvious, the two expressions are different ways of writing the same analytic function. You can verify this by using the usual techniques to evaluate square roots of negative numbers, and the trigonometric and hypertrigonometric functions of imaginary arguments.

To summarize, the age the universe can be expressed as a function of H and Ω as

$$|H|\,t = \begin{cases} \dfrac{\Omega}{2(1-\Omega)^{3/2}}\left[\dfrac{2\sqrt{1-\Omega}}{\Omega} - \sinh^{-1}\left(\dfrac{2\sqrt{1-\Omega}}{\Omega}\right)\right] & \text{if } \Omega < 1 \\ 2/3 & \text{if } \Omega = 1 \\ \dfrac{\Omega}{2(\Omega-1)^{3/2}}\left[\sin^{-1}\left(\pm\dfrac{2\sqrt{\Omega-1}}{\Omega}\right) \mp \dfrac{2\sqrt{\Omega-1}}{\Omega}\right] & \text{if } \Omega > 1 \end{cases} \tag{4.47}$$

Graphically,

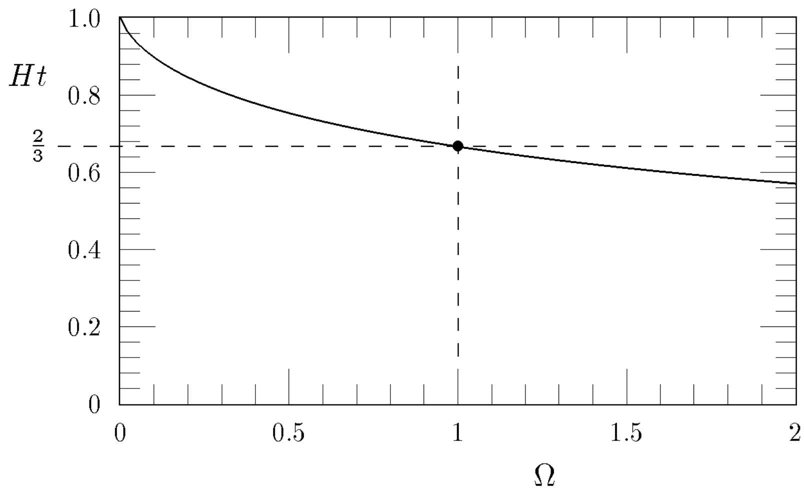

Figure 4.2: The age of a matter-dominated universe, expressed as Ht (where t is the age and H is the Hubble expansion rate), as a function of Ω. The curve describes all three cases of an open ($\Omega < 1$), flat ($\Omega = 1$), and closed ($\Omega > 1$) universe.

THE EVOLUTION OF A MATTER-DOMINATED UNIVERSE:

The following graph shows the evolution of the scale factor for all three cases: open, closed, or flat. The graphs were constructed using Eqs. (4.1) for the flat case, (4.43) for the open case, and (4.25) for the closed case. The parametric equations are plotted by choosing a finely spaced grid of values for θ, using the formulas to determine t and a for each value of θ. Since the only parameter, α, appears merely as an overall factor, a graph that is valid for all values of α can be obtained by plotting the dimensionless ratios $a/(\alpha\sqrt{|k|})$ and ct/α:

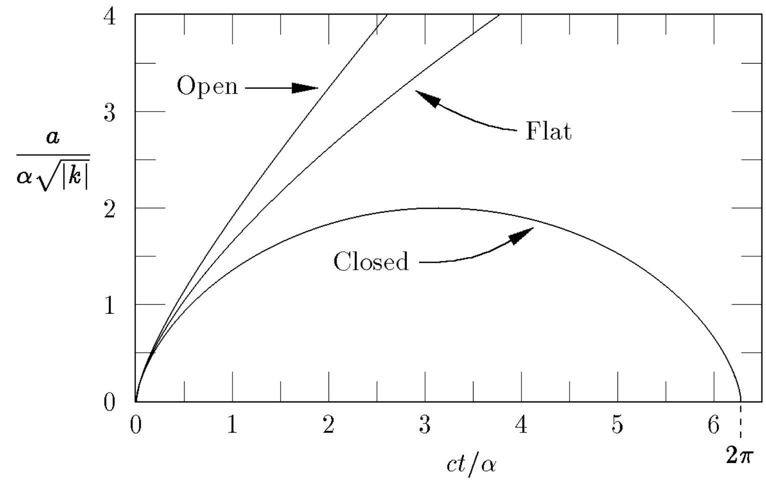

Figure 4.3: The evolution of a matter-dominated universe. Closed and open universes can be characterized by a single parameter α, defined by Eq. (4.17). With the scalings shown on the axis labels, the evolution of a matter-dominated universe is described in all cases by the curves shown in this graph.

Although the graph shows all three cases, it must be remembered that it is still restricted by the assumption that the universe is matter-dominated — that is, the mass density is dominated by nonrelativistic matter, for which pressure forces are negligible.