MASSACHUSETTS INSTITUTE OF TECHNOLOGY
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Lecture Notes 6
BLACK-BODY RADIATION AND THE EARLY HISTORY OF THE UNIVERSE

INTRODUCTION:

In Lecture Notes 3 and 4 we discussed the dynamics of Newtonian cosmology under the assumption that mass is conserved as the universe expands. In that case, since the physical volume is proportional to $a^3(t)$, the mass density $\rho(t)$ is proportional to $1/a^3(t)$. In these lecture notes we will extend our understanding to include the dynamical effects of electromagnetic and other forms of radiation. Electromagnetic radiation is intrinsically relativistic ($v \equiv c$!), so we need to begin by discussing the concepts of mass and energy in the context of relativity.

According to special relativity, mass and energy are equivalent, with the conversion of units given by the famous formula,

$$E = mc^2 \ . \tag{6.1}$$

When one says that mass and energy are equivalent, one is saying that they are just two different ways of expressing precisely the same thing. The total energy of any system is equal to the total mass of the system — sometimes called the relativistic mass — times c^2, the square of the speed of light.

Although c^2 is a large number in conventional units, one can still think of it conceptually as being merely a unit conversion factor. For example, one can imagine measuring the mass/energy of an object in either grams or ergs, with

$$1 \text{ gram} = 8.9876 \times 10^{20} \text{ erg} \ , \tag{6.2}$$

where $c^2 = 8.9876 \times 10^{20}\ \text{cm}^2/\text{s}^2$. So one gram is a ***huge*** number of ergs. For SI units,

$$1 \text{ kg} = 8.9876 \times 10^{16} \text{ joule} = 2.497 \times 10^{10} \text{ kw-hr.} \tag{6.3}$$

To put this number in perspective, we might compare it to the world power supply, which is about 1.8×10^{10} kilowatts, according to the International Energy Agency.* Thus, if we

* *Key World Energy Statistics, 2017*, http://www.iea.org/publications/freepublications/publication/KeyWorld2017.pdf. The 2015 annual "Total Primary Energy Supply" is given as 13,647 million tonnes of oil equivalent (Mtoe), which in more familiar units is 1.587×10^{14} kW-hr. If this energy production were uniformly spread over the year, the average power would be 1.811×10^{10} kW. With a 2015 world population of 7.35 billion people, this corresponds to 2.46 kW per person.

could build a machine that would convert 1 kg per hour entirely into energy, its power output would be about 1.5 times the world's total power supply. A 15 gallon tank of gasoline, if it could be converted entirely to energy, would power the world for two and a half days. Unfortunately, however, it is not so easy: when a uranium-235 nucleus undergoes fission, for example, only about 0.09% of its mass is converted to energy.

Since c is conceptually a unit conversion factor, many physicists (especially nuclear and particle physicists) work in unit systems for which $c \equiv 1$. A common choice is to use the MeV (10^6 eV) or GeV (10^9 eV) as the unit of energy, where

$$1\,\text{eV} = 1 \text{ electron volt} = 1.6022 \times 10^{-19} \text{ J}, \tag{6.4}$$

and then

$$1\,\text{GeV} = 1.7827 \times 10^{-27} \text{ kg}. \tag{6.5}$$

The mass of a proton is 0.938 GeV.

It will be useful to know some basic properties of the energy-momentum four-vector, so I will summarize them here. The energy-momentum four-vector is defined by starting with the momentum three-vector $(p^1, p^2, p^3) \equiv (p^x, p^y, p^z)$, and appending a fourth component

$$p^0 = \frac{E}{c} \ , \tag{6.6}$$

so the four-vector can be written as

$$p^\mu = \left(\frac{E}{c}, \vec{p}\right) \ . \tag{6.7}$$

As with the three-vector momentum, the energy-momentum four-vector can be defined for a system of particles as the sum of the vectors for the individual particles. The motivation for putting the four components together is that the four-vector obeys a simple transformation law that describes how to calculate the components measured by an inertial observer in terms of the components measured by another inertial observer who is moving relative to the first. The transformation law is identical to one that describes the transformation of the spacetime coordinate vector, $x^\mu = (ct, \vec{x})$, known as the Lorentz transformation. The mass of a particle in its own rest frame is called its rest mass, which we denote by m_0. If the particle moves with velocity $\vec{v}$, then the relativistic expressions for its momentum and energy are given by

$$\begin{aligned} \vec{p} &= \gamma m_0 \vec{v} \ , \\ E &= \gamma m_0 c^2 = \sqrt{(m_0 c^2)^2 + |\vec{p}|^2 c^2} \ , \end{aligned} \tag{6.8}$$

where as usual γ is defined by

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} . \tag{6.9}$$

Like the Lorentz-invariant interval that we discussed with Eq. (5.30), the energy-momentum four-vector has a Lorentz-invariant square:

$$p^2 \equiv |\vec{p}|^2 - \left(p^0\right)^2 = |\vec{p}|^2 - \frac{E^2}{c^2} = -\left(m_0 c\right)^2 . \tag{6.10}$$

For a particle at rest, Eq. (6.8) implies that the energy E_0 is given by

$$E_0 = m_0 c^2 , \tag{6.11}$$

since $\vec{p} = 0$. To see the implications of this equation, we can imagine a hydrogen atom, which is composed of a proton and an electron. If the two particles are started an infinite distance apart, then the initial total energy is given by $E_{\rm tot} = (m_p + m_e)c^2$, where we are defining the zero of potential energy so that it vanishes at infinite separation. As the particles come together they attract each other, and therefore accelerate. They gain kinetic energy, and the potential energy becomes negative. If the particles combine to form a hydrogen atom in its ground state (i.e., its lowest energy state), then an energy ΔE is given off. This energy is called the binding energy of the hydrogen, and has a value of 13.6 eV. The energy is most commonly given off in the form of photons. (There is also some kinetic energy associated with the recoil of the hydrogen atom, but the recoil energy is very small when the rest energy of the recoiling object is large compared to the energy given off. Here we will ignore the recoil.) The mass m_H of the resulting hydrogen atom is then given by

$$\boxed{m_H = m_p + m_e - \Delta E/c^2 ,} \tag{6.12}$$

where m_p is the mass of the proton, and m_e is the mass of the electron. The rest mass of the system is reduced by the energy given off, divided by c^2. Thus, a small part of the rest mass of the proton and electron has been converted into other forms of energy.

For a particle in motion, one can define a relativistic mass $m_{\rm rel}$ by

$$m_{\rm rel} = \frac{E}{c^2} . \tag{6.13}$$

Many authors prefer to never introduce the concept of relativistic mass, and it is certainly not necessary. Since it is defined solely in terms of the energy, anything that can be thought or said in terms of the relativistic mass of a particle can equally well be expressed in terms of its energy. However, when one discusses the gravitational field of a system

including relativistic particles, then the concept of relativistic mass can be useful. The gravitational field of a single moving particle, according to general relativity, is anisotropic and rather complicated, but fortunately we will not have to deal with this. However, if one has a gas of relativistic particles with no net momentum in the frame of interest, then the gravitational field can be computed as if the particles were at rest, but using the relativistic mass, as defined by Eq. (6.13). If one adopts the concept of relativistic mass, then the famous equation $E = mc^2$ can be described by saying that energy and mass are equivalent, related in all cases by a factor of c^2. The concept of relativistic mass is also useful when discussing the gravitational force that acts on a body. If a gas of relativistic particles were sealed inside a box, and the box were placed on a scale, then the scale would register the relativistic mass of the particles in the gas.

THE MASS OF RADIATION:

We are perhaps not used to thinking of electromagnetic radiation as having mass, but it is well-known that radiation has an energy density. If the energy density is denoted by u, then the electromagnetic radiation has a relativistic mass density ρ given by

$$\boxed{\rho = u/c^2 .} \tag{6.14}$$

That is, the formula above describes the amount of relativistic mass ($m_{\rm rel}$) per unit volume. According to general relativity, such a mass density contributes to the gravitational field just like any other mass density.*

To my knowledge nobody has ever actually "weighed" electromagnetic radiation in any way, but the theoretical evidence in favor of Eq. (6.14) is overwhelming — light does have mass. Nonetheless, the photon has zero rest mass, meaning that it cannot be brought to rest. The general relation for the square of the four-momentum reads $p^2 = -(m_0 c)^2$, as in Eq. (6.10), so for the photon this becomes $p^2 = 0$. Writing out the square of the four-momentum leads to the following relation for photons:

$$\boxed{|\vec{p}|^2 - \frac{E^2}{c^2} = 0 , \quad \text{or} \quad E = c|\vec{p}| .} \tag{6.15}$$

In this set of notes we will examine the role which the mass of electromagnetic radiation plays in the early stages of the universe.

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* Authors who avoid the concept of relativistic mass would reach the same conclusion, but would describe it by saying that the energy density u creates the same gravitational field as a mass density u/c^2.

RADIATION IN AN EXPANDING UNIVERSE:

If we ignore the interactions of photons, then as the universe expands the photons travel on geodesics, and their number is conserved. We will learn later that even when we take into account the emission and absorption of photons by the matter in the universe, their number is still very accurately conserved during the long period after inflation (to be discussed later) and before the formation of the earliest stars. As long as the number is conserved, the number density n_γ of photons varies as $1/a^3(t)$ as the universe expands, just like the number density of nonrelativistic particles:

$$n_\gamma \propto \frac{1}{a^3(t)} \ . \tag{6.16}$$

Note that the Greek letter γ ("gamma") is often used to denote the photon, even when the energy of the photon is far from the range of 10^4–10^7 eV that normally characterizes what are called gamma rays.

Unlike nonrelativistic particles, however, the frequency of each photon is redshifted as the universe expands, as we learned in Lecture Notes 2. The ratio of the period Δt at the time t_2 to the period at the time t_1 is given by the redshift factor

$$\frac{\Delta t(t_2)}{\Delta t(t_1)} \equiv 1 + z = \frac{a(t_2)}{a(t_1)} \ . \tag{6.17}$$

Since the frequency ν (Greek letter "nu") of each photon is related to the period by $\nu = 1/\Delta t$, the frequency of each photon decreases as $1/a(t)$ as the universe expands. According to elementary quantum mechanics, the energy of the photon is related to the frequency by

$$E = h\nu \ , \tag{6.18}$$

where h is Planck's constant ($h = 4.136 \times 10^{-15}$ eV-s). Thus the energy of the photon decreases as $1/a(t)$ as the universe expands. The energy density u_γ of the radiation is given by

$$u_\gamma = n_\gamma \, E_\gamma \ , \tag{6.19}$$

where E_γ is the mean energy per photon, so

$$n_\gamma \propto \frac{1}{a^3(t)} \ \ , \ \ E_\gamma \propto \frac{1}{a(t)} \quad \Longrightarrow \quad \boxed{\rho_\gamma = \frac{u_\gamma}{c^2} \propto \frac{1}{a^4(t)} \ .} \tag{6.20}$$

(Although I have justified this relation with quantum mechanical arguments, it can also be derived from classical electromagnetic theory. However, in this case the quantum argument is simpler.)

THE RADIATION–DOMINATED ERA:

Today the energy density u_r in the cosmic background radiation is given approximately by

$$u_r = 7.01 \times 10^{-14} \ \text{J/m}^3 \ . \tag{6.21}$$

(Here I have used the subscript "r" for radiation, rather than "γ" for photons, because I have included both the energy density of photons and the expected density of neutrinos, which we will talk about later.) To find the corresponding mass density, use

$$\begin{aligned}\rho_r = \frac{u}{c^2} &= \frac{7.01 \times 10^{-14} \ \left(\text{kg-m}^2\text{-s}^{-2}\right) \text{m}^{-3}}{\left(3 \times 10^8 \ \text{m-s}^{-1}\right)^2} \\ &= 7.80 \times 10^{-31} \ \text{kg/m}^3 = 7.80 \times 10^{-34} \ \text{g/cm}^3 \ .\end{aligned} \tag{6.22}$$

This can be compared with the critical mass density ρ_c, which was calculated in Eq. (3.34):

$$\rho_c = 1.88 \, h_0^2 \times 10^{-29} \ \text{g/cm}^3 \ , \tag{3.34}$$

where

$$H_0 = 100 \, h_0 \, \text{km-s}^{-1}\text{-Mpc}^{-1} \ .$$

One finds that the fraction Ω_r of closure density in radiation is given by

$$\Omega_r \equiv \frac{\rho_r}{\rho_c} = \frac{7.80 \times 10^{-34} \ \text{g-cm}^{-3}}{1.88 h_0^2 \times 10^{-29} \ \text{g-cm}^{-3}} = 4.15 \times 10^{-5} \, h_0^{-2} \ , \tag{6.23}$$

For $h_0 = 0.67$, one finds $\Omega_r = 9.2 \times 10^{-5}$. This is only a very small fraction, but Ω_r was larger in the past. Since $\rho_r \propto 1/a^4$, while the mass density ρ_m of nonrelativistic matter behaves as $1/a^3$, it follows that

$$\rho_r/\rho_m \propto 1/a(t) \ . \tag{6.24}$$

Then density of nonrelativistic matter in our universe (visible and dark matter combined) gives $\Omega_m \approx 0.30$, so today $\rho_r/\rho_m \approx 9.2 \times 10^{-5}/0.30 \approx 3.1 \times 10^{-4}$. The constant of proportionality in Eq. (6.24) is then determined, giving

$$\frac{\rho_r(t)}{\rho_m(t)} = \left[a(t_0)\frac{\rho_r(t_0)}{\rho_m(t_0)}\right]\frac{1}{a(t)} = \frac{a(t_0)}{a(t)} \times 3.1 \times 10^{-4} \ . \tag{6.25}$$

Since $a(t) \to 0$ as $t \to 0$, the right-hand-side approaches infinity in this limit. Thus there was a time at which the value of the right-hand-side went through one, and this time is denoted by t_{eq}, the time of radiation-matter equality. We will assume that the universe is flat, and that for $t > t_{\rm eq}$ we can make the crude approximation that the universe can be treated as if it were dominated by nonrelativistic matter. This approximation ignores the effect of radiation for times shortly after $t_{\rm eq}$, and it also ignores the effect of dark energy (and the consequent acceleration) during the past 5 billion years or so. As discussed in Lecture Notes 3, during the matter-dominated era the scale factor behaves as $a(t) \propto t^{2/3}$. Thus, writing Eq. (6.25) for $t = t_{\rm eq}$ gives

$$\frac{\rho_r(t_{\rm eq})}{\rho_m(t_{\rm eq})} \equiv 1 = \frac{a(t_0)}{a(t_{\rm eq})} \times 3.1 \times 10^{-4} \ . \tag{6.26}$$

Remembering that $a(t_0)/a(t_{\rm eq}) = 1 + z_{\rm eq}$ (see Eq. (2.15)), the redshift $z_{\rm eq}$ of matter-radiation equality is given by

$$z_{\rm eq} = \frac{1}{3.1 \times 10^{-4}} - 1 \approx 3200 \ . \tag{6.27}$$

If we ignore for now the acceleration that our universe has undergone during the last 5 billion years or so, we can approximate it as a flat matter-dominated universe, with $a(t) \propto t^{2/3}$. This gives $t_{\rm eq} = 5.5 \times 10^{-6}\, t_0$, so for $t_0 = 13.8$ Gyr, $t_{\rm eq} \approx 75{,}000$ years. Our approximations have been crude, but Barbara Ryden quotes a more precise numerical calculation (on p. 97), where she finds $t_{\rm eq} \approx 47{,}000$ years.

DYNAMICS OF THE RADIATION–DOMINATED ERA:

When we studied the dynamics of a matter-dominated universe (i.e., a universe whose mass density is dominated by nonrelativistic matter) in Lecture Notes 3, we learned that the evolution of such a universe can be described by the two Friedmann equations:

$$\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi}{3} G\rho - \frac{kc^2}{a^2} \tag{6.28a}$$

$$\ddot a = -\frac{4\pi}{3} G\rho a \ , \qquad \begin{pmatrix}\text{matter-dominated}\\ \text{universe}\end{pmatrix} \tag{6.28b}$$

where $a(t)$ is the scale factor, $\rho(t)$ is the mass density, and an overdot represents differentiation with respect to time t. In such a matter-dominated universe we found that the mass density behaves as

$$\rho(t) \propto \frac{1}{a^3(t)} \qquad \text{(matter-dominated).} \tag{6.29}$$

The three equations above are not independent, but in fact any two of them can be used to derive the third. For example we can derive Eq. (6.28b) by multiplying Eq. (6.28a) by

a^2 and then differentiating it with respect to time. The resulting equation will contain a term proportional to $\dot\rho$. Eq. (6.28b) can then be obtained by replacing $\dot\rho$ by

$$\dot\rho = -3\frac{\dot a}{a}\rho \qquad \text{(matter-dominated),} \tag{6.30}$$

which can be derived from Eq. (6.29).

For a universe dominated by radiation, we have already learned (see Eq. (6.20)) that

$$\rho(t) \propto \frac{1}{a^4(t)} \qquad \text{(radiation-dominated),} \tag{6.31}$$

in contrast to Eq. (6.29). This implies that Eqs. (6.28a) and (6.28b) will no longer be consistent with each other, since the derivation of Eq. (6.28b) described in the previous paragraph will give a different result. To correctly describe a radiation-dominated universe, we will have to reconcile this inconsistency.

While we have not yet used the word, Eq. (6.31) can be viewed as a statement about the *pressure* of radiation. Pressure is relevant, because it is the pressure of a gas that determines how much energy it looses if it expands. Consider, as a thought experiment, a volume of gas contained in a chamber with a movable piston, as shown below:

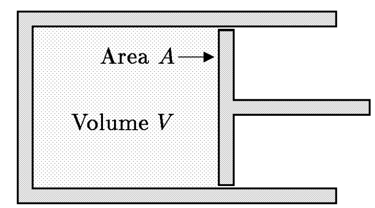

Figure 6.1: A piston chamber, used to discuss the effect of pressure on the rate of change of the energy density of an expanding gas.

We will assume that the piston chamber is small enough so that gravity plays no role in our thought experiment. Let U denote the total energy of the gas, and let p denote the pressure. Suppose that the piston is moved a distance dx to the right. (We suppose that the motion is slow, so that the gas particles have time to respond and to maintain a uniform pressure throughout the volume.) The gas exerts a force pA on the piston, so the gas does work $dW = pA\,dx$ as the piston is moved. The volume increases by an amount $dV = A\,dx$, so $dW = p\,dV$. The energy of the gas decreases by this amount, so

$$dU = -p\,dV \ . \tag{6.32}$$

It can be shown that this relation is valid whenever the volume of a gas is changed, regardless of the shape of the volume.

Now consider a homogeneous, isotropic, expanding universe, described by a scale factor $a(t)$. Let $u = \rho c^2$ denote the energy density of the gas that fills it. We will consider a fixed coordinate volume V_{coord}, so the physical volume will vary as

$$V_{\rm phys}(t) = a^3(t) V_{\rm coord} \ , \tag{6.33}$$

and the energy of the gas in this region is given by

$$U = V_{\rm phys} u \ . \tag{6.34}$$

Using these relations, you will show in Problem Set 6 that

$$\frac{d}{dt}\left(a^3 \rho c^2\right) = -p \frac{d}{dt}(a^3) \ , \tag{6.35}$$

and then that

$$\dot\rho = -3\frac{\dot a}{a}\left(\rho + \frac{p}{c^2}\right) \ . \tag{6.36}$$

By comparing this equation with the matter-dominated relation of Eq. (6.30), we see that nonrelativistic matter has zero pressure. This could have been expected, since nonrelativistic matter means a gas of approximately motionless particles, and we assumed starting in Lecture Notes 3 that there is no loss of energy when the universe filled with nonrelativistic matter expands — the energy spreads out as the volume increases, but otherwise it is not changed. By contrast, you will also show in Problem Set 6 that radiation, with a mass density that falls off as $1/a^4(t)$, has a pressure given by

$$p = \frac{1}{3}\, u = \frac{1}{3}\, \rho c^2 \ . \tag{6.37}$$

Thus, the new ingredient that is introduced by radiation, which is causing an inconsistency between Eqs. (6.28a) and (6.28b), is pressure.

The treatment of pressure in general relativity is unambiguous, and the implication for this situation is simple: the $\dot a$ equation (6.28a) is not modified, but the $\ddot a$ equation (6.28b) needs to be modified. By accepting Eq. (6.28a) and using Eq. (6.36) for $\dot\rho$, you will show in Problem Set 6 that Eq. (6.28b) must be modified to read

$$\boxed{\frac{d^2 a}{dt^2} = -\frac{4\pi}{3} G \left(\rho + \frac{3p}{c^2}\right) a \ .} \tag{6.38}$$

While general relativity might be needed to prove the above equation, Newtonian arguments are sufficient to at least make this result seem extremely plausible. We know that when the pressure is non-negligible, $\dot\rho$ is given by Eq. (6.36), and that then Eqs. (6.28a) and (6.28b) become incompatible. One or both of these equations, therefore, must be modified by the presence of pressure. The two equations are different from each other, however, in an obvious way. The $\ddot a$ equation is a force equation, as in $\vec F = m\vec a$, and in fact we derived it in our Newtonian model by applying $\vec F = m\vec a$ to each particle in the model universe. The $\dot a$ equation, on the other hand, was derived by finding a first integral of the $\ddot a$ equation, and therefore looks like a conservation of energy equation. In fact, we showed in Problem 3, Problem Set 3, that for the Newtonian model with a finite radius $R_{\rm max}$, the $\dot a$ equation is precisely equivalent to the statement that the total energy of the Newtonian model universe is fixed. Does it make sense to add a pressure term to a conservation of energy equation? No, it does not. As a toy problem, we can ask what would happen if the universe were filled with TNT, and at a certain pre-arranged time little gremlins throughout the universe ignited the TNT, so the pressure suddenly changed. The pressure change can in principle be very large and fast, but there is no mechanism to cause any of the other quantities in Eq. (6.28a) to change rapidly. We can consider a small region of space, in which the velocities associated with the Hubble expansion are all small, so we can expect that we can trust our Newtonian understanding of how matter should behave. In that case ρ describes an energy density that cannot change discontinuously, and a and $\dot a$ describe the positions and velocities of particles, which also cannot change discontinuously. So, our conclusion is that a term depending on the pressure cannot be added to Eq. (6.28a), and then Eq. (6.38) follows as a consequence.

Note that Eq. (6.38) is implying something that is perhaps very surprising: the pressure is contributing to the gravitational acceleration. That is, the pressure as well as the energy density can act as a source for the gravitational field. We will not make much use of Eq. (6.38) in the rest of this chapter, as Eq. (6.28a) will be sufficient for most of our conclusions. But we can keep in mind that Eq. (6.28a) would not be consistent with $\rho(t) \propto 1/a^4(t)$ if Eq. (6.38) were not true. We will learn later that the pressure term in Eq. (6.38) can have dramatically new consequences. In particular, we will learn that pressures, unlike mass densities, can sometimes be negative. Eq. (6.38) implies that a negative pressure can result in a gravitational repulsion. We believe that the current acceleration of the universe, which we mentioned briefly in Lecture Notes 3, can be attributed to the negative pressure of an unidentified material that is called *dark energy*. Many of us also believe that the early universe underwent a very brief period of incredibly rapid acceleration, called *inflation*, which was also driven by a negative pressure. We will return to both of these topics in later sets of lecture notes.

DYNAMICS OF A FLAT RADIATION-DOMINATED UNIVERSE:

As a simple (but important) special case, consider the evolution of a radiation-dominated universe with $k = 0$. From Eqs. (6.20) and (6.28a), one has

$$\frac{1}{a^2}\left(\frac{da}{dt}\right)^2 = \frac{\text{const}}{a^4} , \tag{6.39}$$

which leads to

$$\frac{da}{dt} = \frac{\sqrt{\text{const}}}{a} . \tag{6.40}$$

This equation can be solved by rewriting it as

$$a da = \sqrt{\text{const}}\, dt \tag{6.41}$$

and then integrating both sides to obtain

$$\frac{1}{2}a^2 = \sqrt{\text{const}}\, t + \text{const}' . \tag{6.42}$$

The convention is to choose the zero of time so that $a(t) = 0$ for $t = 0$, which implies that $\text{const}' = 0$. Thus, the final result can be written as

$$\boxed{a(t) \propto \sqrt{t} \qquad \text{(radiation-dominated)} .} \tag{6.43}$$

The Hubble expansion rate $H(t)$ is given by Eq. (2.8), which says that

$$H(t) = \dot{a}/a . \tag{6.44}$$

Combining this equation with Eq. (6.43), one has immediately that

$$\boxed{H(t) = \frac{1}{2t} \qquad \text{(radiation-dominated)} .} \tag{6.45}$$

The age of a radiation-dominated universe is therefore related to the Hubble constant by $t = \frac{1}{2}H^{-1}$. (Recall for comparison that for a matter-dominated flat universe with $a(t) \propto t^{2/3}$, the age is $\frac{2}{3}H^{-1}$.) The horizon distance is given by Eq. (4.7), and the result here is

$$\begin{aligned} \ell_{p,\text{horizon}}(t) &= a(t)\int_0^t \frac{c}{a(t')}dt' \\ &= \boxed{2ct \qquad \text{(radiation-dominated)} .} \end{aligned} \tag{6.46}$$

(Recall that this answer is to be compared with $3ct$ for the matter-dominated universe.) If one inserts Eq. (6.45) into Eq. (6.28a) (with $k = 0$, still), one obtains a relation for the mass density as a function of time:

$$\rho = \frac{3}{32\pi G t^2} . \tag{6.47}$$

Note that the $1/t^2$ behavior in the above equation is consistent with what we already know: $\rho \propto 1/a^4(t)$, and $a(t) \propto \sqrt{t}$.

BLACK–BODY RADIATION:

If a cavity is carved out of any material, and the walls of the cavity are kept at a uniform temperature T, then the cavity will fill with radiation. Assuming that the walls are thick enough so that no radiation can get through them, then the energy density (and also the entire spectrum of the radiation) is determined solely by the temperature T — the composition of the material is entirely irrelevant. The material is serving solely to keep the radiation at a uniform temperature. Radiation of this type is generally called either thermal radiation or black-body radiation.

The motivation for the name "black-body radiation" stems from the fact that a "black" body in empty space can be shown to emit radiation of exactly this intensity and spectrum. Here the word "black" is used to describe an object that absorbs all light that hits it, so there is no reflected light, although there is emission due to thermal effects. Emission is distinguished from reflection by the fact that reflection is an immediate response to the radiation that is currently hitting the material. To understand the radiation emitted by a black body, imagine a block of such material inside the cavity described in the previous paragraph. Since thermal equilibrium has been established, one concludes that the block at temperature T must emit radiation which precisely matches the radiation that it is absorbing — otherwise it would either heat up or cool down, and that would violate the assumption of thermal equilibrium. In fact, not only must the energy densities match, but the entire spectrum must match — otherwise one could imagine introducing a frequency-selecting filter that would cause the black body to heat or cool. That is, if there were any frequency band for which the radiation emitted by the block did not match the radiation hitting the block, then we could surround the block by a filter that transmits only in that frequency band, and we would see the block heat up or cool down. Since objects will never heat up or cool down once thermal equilibrium is reached, the emitted and absorbed radiation must match in every frequency band. Since the block is assumed to be black, none of the emitted radiation is reflection, so all of it is thermal emission that will continue to be emitted even if the block is removed from the cavity. Thus, a black body will emit radiation with an intensity and a spectrum that depends only on the temperature, and not on any property of the material other than the fact that it is black.

The energy density and other properties of the radiation can be derived using the standard principles of statistical mechanics, but the derivation will not be included in this course. However, I will make a few comments about the underlying physics, and then I will state the results. The rule of thumb for classical statistical mechanics is the "equipartition theorem," which says that under certain circumstances (which I will not specify), each degree of freedom of a system at temperature T acquires a mean thermal energy of $\frac{1}{2}kT$. For example, in a gas of point particles each particle acquires a mean thermal energy of $\frac{3}{2}kT$, since motion in the x, y and z directions constitutes three degrees of freedom. For the system of radiation inside a cavity, each possible standing wave pattern corresponds to one degree of freedom. In a rectangular cavity, for example, a standing wave can be described in terms of a polarization, which has two linearly independent values, and a wave vector $\vec{k}$, with the wave amplitude proportional to $\mathrm{Re}\{e^{i\vec{k}\cdot\vec{x}}\}$. For the standing wave to exist, each component of $\vec{k}$ must satisfy the condition that the wave amplitude must vary either an integral or half-integral number of cycles from one side of the cavity to the other. Thus a standing wave pattern exists only for a discrete set of frequencies. The discrete set of frequencies is, however, infinite, since there is no upper limit to the frequency of a standing wave. The number of degrees of freedom is therefore infinite, and the equipartition theorem cannot be applied. This problem is known as the "Jeans catastrophe," and represents an important failure of classical physics. The implications can be stated as follows: if classical physics were correct, then a region of space containing an electromagnetic field could never come into thermal equilibrium — instead it would continue indefinitely to absorb energy from its surroundings, and the energy absorbed would be used to excite higher and higher frequency standing waves of the field. The electromagnetic field would be an infinite heat sink, draining away all thermal energy.

Of course the electromagnetic field does not drain away all thermal energy, and the reason comes from quantum theory. Classically it would be possible to excite a standing wave by an arbitrary amount, but quantum theory requires that the excitations occur only by the addition of discrete photons, each with an energy $h\nu$, where ν is the frequency of the standing wave. For cases in which $h\nu \ll kT$, the classical answer is not changed — such standing waves acquire a mean energy of $\frac{1}{2}kT$ for each polarization. However, for those standing waves with $h\nu \gg kT$, the minimum excitation is much larger than the energy which is classically expected. These modes are then only rarely excited, and the total energy is convergent.

When the calculation is done quantum mechanically, one finds that black-body electromagnetic radiation has an energy density given by

$$\boxed{u = g\frac{\pi^2}{30}\,\frac{(kT)^4}{(\hbar c)^3}\ ,} \tag{6.48}$$

where

$$\begin{aligned} k = \text{Boltzmann's constant} &= 1.381 \times 10^{-16}\ \text{erg/K} \\ &= 8.617 \times 10^{-5}\ \text{eV/K}\ , \end{aligned} \tag{6.49}$$

$$\begin{aligned} \hbar = \frac{h}{2\pi} &= 1.055 \times 10^{-27}\ \text{erg-sec} \\ &= 6.582 \times 10^{-16}\ \text{eV-sec}\ , \end{aligned}$$

and

$$\boxed{g = 2 \quad (\text{for photons})\ .} \tag{6.50}$$

The factor of g is introduced to prepare for the discussion below of black body radiation of particles other than photons. g is taken as 2 for photons because the photon has two possible polarization states. The polarization states can be described as linearly polarized, or as circularly polarized, depending on one's choice of basis. In either case, however, there are two polarizations. A photon traveling along the z-axis can be linearly polarized in either the x or y directions, or it can have a circular polarization of left or right. The polarization is related to the intrinsic angular momentum, or spin, of the photon: right circular polarization corresponds to the spin being aligned with the momentum, while left circular polarization is the opposite. Thus one could say that g is taken as 2 because the photon has two spin states.

One also finds that the radiation has a pressure, given by

$$\boxed{p = \frac{1}{3}u\ .} \tag{6.51}$$

The number density of photons is found to be

$$\boxed{n = g^*\frac{\zeta(3)}{\pi^2}\,\frac{(kT)^3}{(\hbar c)^3}\ ,} \tag{6.52}$$

where

$$\zeta(3) = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots \approx 1.202 \tag{6.53}$$

is the Riemann zeta function evaluated at 3, and

$$\boxed{g^* = 2 \quad \text{(for photons)}\ .} \tag{6.54}$$

Finally, the radiation has an entropy density s (entropy per unit volume) given by

$$\boxed{s = g\frac{2\pi^2}{45}\,\frac{k^4T^3}{(\hbar c)^3}\ .} \tag{6.55}$$

We will not need to know the precise meaning of entropy, but it will suffice to say that the entropy is a measure of the degree of disorder (or uncertainty) in the statistical system. Entropy is conserved if the system remains in thermal equilibrium, and this assumption appears to be quite accurate for most processes in the early universe. (The inflationary process, to be discussed later, is a colossal exception.) When departures from thermal equilibrium occur, the entropy is monotonically increasing, a principle known as the second law of thermodynamics.

In the laboratory the only kind of thermal radiation that can be achieved is that of photons. The radiation in the early universe, on the other hand, is believed to have also contained neutrinos. During the 20th century these neutrinos were thought to have zero rest mass, like the photon, but that is no longer the case. We now believe that neutrinos have a very small but nonzero mass. Nonetheless, as long as $m_0c^2 \ll kT$, which is certainly the case throughout the history of the universe, the neutrinos contribute to the thermal radiation as if they were massless particles.

Besides having a nonzero rest mass, neutrinos differ from photons in another property which has an important effect on their thermal radiation. The photon belongs to a class of particles called bosons, and these particles have the property that there is no limit to the number of particles that can exist simultaneously in a given quantum state. It is precisely because of this property that the photon can give rise to a classical electromagnetic field. The field behaves classically because it is composed of huge numbers of photons. The neutrino, on the other hand, belongs to a class of particles called fermions. For these particles it is impossible to have more than one particle in a given quantum state at one time. An electron is also a fermion, and the principle of one electron per quantum state is sometimes called the "Pauli Exclusion principle."

In relativistic quantum field theory it is possible to prove the *spin-statistics theorem,* which says that the boson/fermion property of a particle is connected to its intrinsic angular momentum, also called the particle's spin. If the spin is an integer (in units of $\hbar$), then the particle must be a boson. The only other possibility is that the spin is half-integer (more precisely, half-odd-integer, again in units of $\hbar$), in which case the particle

is a fermion. The proof requires relativistic invariance, so there is no analogous theorem in nonrelativistic quantum mechanics.

Since fermions obey the Pauli exclusion principle, which is a restriction on the states that they can occupy, the fact that a particle is a fermion leads to a reduction in the number of particles that will be present in black-body radiation. The equations that describe the black-body radiation of fermions have the same form as the equations for bosons, so the energy density u, the pressure p, the number density n, and the entropy density s are again described by Eqs. (6.48), (6.51), (6.52), and (6.55) above. The Pauli exclusion principle, however, causes the factor g to be multiplied by 7/8 if the particle is a fermion, and the factor g^* to be multiplied by 3/4.

To find the values of g and g^* for neutrinos, we must count how many types of neutrinos exist. While there is only one kind of photon, we believe that there are three different species, or *flavors*, of neutrinos: the electron neutrino ν_e, the muon neutrino ν_μ, and the tau neutrino ν_τ. The existence of the three species causes g and g^* to be multiplied by 3. In addition, neutrinos exist as particles and antiparticles, in contrast to the photon which is its own antiparticle. The particle/antiparticle option leads to a factor of 2 for both g and g^*. While the photon has two spin states, the neutrino has only 1: neutrinos are *left-handed*, which means that their spin points in the opposite direction from their momentum, while antineutrinos are *right-handed.* Thus the values of g and g^* for neutrinos are given by

$$g_\nu = \underbrace{\frac{7}{8}}_{\text{Fermion factor}} \times \underbrace{3}_{\substack{\text{3 species}\\ \nu_e,\nu_\mu,\nu_\tau}} \times \underbrace{2}_{\text{Particle/antiparticle}} \times \underbrace{1}_{\text{Spin states}} = \frac{21}{4}\ . \tag{6.56}$$

$$g^*_\nu = \underbrace{\frac{3}{4}}_{\text{Fermion factor}} \times \underbrace{3}_{\substack{\text{3 species}\\ \nu_e,\nu_\mu,\nu_\tau}} \times \underbrace{2}_{\text{Particle/antiparticle}} \times \underbrace{1}_{\text{Spin states}} = \frac{9}{2}\ . \tag{6.57}$$

[One might wonder why neutrinos are not produced when a piece of metal is heated until it glows. The answer is that neutrinos interact very weakly at these low energies, and their production rate is totally negligible. Thermal equilibrium neutrino radiation can in principle be seen at any temperature, but it is very difficult to produce. The radiation would reach thermal equilibrium only if it were confined to a box opaque to neutrinos, which means that the walls of the box would have to be much thicker than the diameter of the earth. In the early universe, however, the temperatures were much higher. Neutrino interaction rates increase with energy, so in the early universe they interacted rapidly with the other particles, and were quickly brought to thermal equilibrium.]

As the temperature is increased, more and more types of particles contribute to the thermal radiation. Any particle with $mc^2 \ll kT$ will contribute in essentially the same way as a massless particle. In particular, when kT is much larger than the value of mc^2 for an electron (0.511 MeV), then electron-positron pairs contribute to the thermal radiation. Electrons and positrons each have two spin states, and they are antiparticles of each other. They are again fermions, so

$$\boxed{\begin{aligned} g_{e^+e^-} &= \underbrace{\frac{7}{8}}_{\text{Fermion factor}} \times \underbrace{1}_{\text{Species}} \times \underbrace{2}_{\text{Particle/antiparticle}} \times \underbrace{2}_{\text{Spin states}} = \frac{7}{2} . && (6.58) \\ g^*_{e^+e^-} &= \underbrace{\frac{3}{4}}_{\text{Fermion factor}} \times \underbrace{1}_{\text{Species}} \times \underbrace{2}_{\text{Particle/antiparticle}} \times \underbrace{2}_{\text{Spin states}} = 3 . && (6.59) \end{aligned}}$$

Including photons, three species of neutrinos, and the electron-positron pairs, the total value of g is given by

$$g_{\rm tot} = 2 + \frac{21}{4} + \frac{7}{2} = 10\frac{3}{4} . \tag{6.60}$$

This value is appropriate for values of kT which are larger than 0.511 MeV, but smaller than 106 MeV (where muons begin to be produced).

THE ENERGY DENSITY OF RADIATION

In Eq. (6.21) we stated an estimate for the energy density in radiation of the current universe, which we are now prepared to justify. The value can be calculated in terms of the current temperature T_γ of the cosmic microwave background. The best single measurement of T_γ to date was done by the FIRAS (Far InfraRed Absolute Spectrophotometer) instrument on the COBE (Cosmic Background Explorer) satellite, which released its final analysis in 1999,* reporting a value of $T_\gamma = 2.725 \pm 0.002$ K. In 2009 Fixsen† combined the results of all experiments to date to obtain a value 2.7255 ± 0.0006 K.

The radiation that exists in the universe today consists of photons and neutrinos. The energy density is therefore given by Eq. (6.48), using $g = 2$ for the photon contribution,

* J.C. Mather, D.J. Fixsen, R.A. Shafer, C. Mosier, and D.T. Wilkinson, "Calibrator Design for the COBE Far-Infrared Absolute Spectrophotometer (FIRAS)," *Astrophysical Journal*, vol. 512, pp. 511–520 (1999), http://arxiv.org/abs/astro-ph/9810373.

† D.J. Fixsen, "The Temperature of the Cosmic Microwave Background," *Astrophysical Journal*, vol. 707, pp. 916–920 (2009), http://arxiv.org/abs/arXiv:0911.1955.

and $g = 21/4$ for the neutrino contribution, as given by Eq. (6.56). There is a further complication, which you explore in Problem Set 7: the temperature T_ν of the neutrinos is not the same as the temperature T_γ of the photons, but instead

$$T_\nu = \left(\frac{4}{11}\right)^{1/3} T_\gamma , \tag{6.61}$$

This temperature differential is established as the e^+e^- pairs disappear from the thermal equilibrium mix, as kT falls below the electron rest energy of 0.511 MeV. The asymmetry results from the fact that the neutrinos interact too weakly to absorb any significant amount of the energy from the e^+e^- pairs, so all the energy goes into heating the photons relative to the neutrinos. Combining the two contributions to the energy density,

$$\begin{aligned} u_{\rm rad,0} &= \left[2 + \frac{21}{4}\left(\frac{4}{11}\right)^{4/3}\right] \frac{\pi^2}{30}\frac{(kT_\gamma)^4}{(\hbar c)^3} \\ &= 7.01 \times 10^{-14}\ \mathrm{J/m^3} , \end{aligned} \tag{6.62}$$

in agreement with Eq. (6.21).

NEUTRINO MASSES:

The fact that neutrinos have mass has become known only relatively recently, and we still do not know what the masses are. The status of particle data is tallied by the Particle Data Group at Lawrence Berkeley Laboratory, which can be found on the web at http://pdg.lbl.gov/. In 1996 the Particle Data Group reported that there is "no direct, unconstested evidence for massive neutrinos," while in 1998 it added that suggestive evidence had been found. In 2000 the evidence was "rather convincing," and by 2002 the evidence had become "compelling."

The evidence remains indirect, however. The mass of a neutrino has never been measured, but instead the existence of a nonzero mass is inferred from the fact that we see neutrinos "oscillate" from one species to another. For many years it was a mystery why we do not detect as many neutrinos from the Sun as is expected, but we are now convinced that the deficit is caused by the fact that the electron neutrinos produced in the Sun can oscillate to become muon or tau neutrinos, which are much harder to detect. The muon and tau neutrinos can now be detected by the Sudbury Neutrino Observatory buried 2100 m underground in a mine near Sudbury, Ontario, and by SuperKamiokande, buried 1000 m in a mine at Hida-city, Gifu prefecture, Japan. In addition, starting in 1998, experiments at SuperKamiokande and other locations have found that muon neutrinos produced by cosmic ray collisions in the upper atmosphere can undergo oscillations into other species before reaching the ground. The 2015 Nobel Prize in Physics was awarded to Takaaki Kajita and Arthur McDonald "for the discovery of neutrino oscillations, which shows that neutrinos have mass."

Such oscillations would not be possible if the neutrinos were massless, essentially because a massless particle experiences an infinite time dilation, so time effectively stops. A massless particle in vacuum cannot do anything except travel at the speed of light. The measurements of the oscillations do not allow a determination of the mass, but instead allow one to infer the differences between the squares of the masses. As of 2016, the Particle Data Group reports

$$\Delta m_{21}^2\, c^4 = (7.53 \pm 0.18) \times 10^{-5}\ \text{eV}^2 \ ,$$
$$\Delta m_{32}^2\, c^4 = (2.44 \pm 0.06) \times 10^{-3}\,\text{eV}^2 \ ,$$

or

$$\Delta m_{32}^2\, c^4 = (2.51 \pm 0.06) \times 10^{-3}\,\text{eV}^2 \ , \tag{6.63}$$

where the two options for Δm_{32}^2 depend on assumptions about the ordering of the masses. The masses are labeled 1, 2, and 3, which are related to the better-known flavor labels ν_e, ν_μ, and ν_τ in a complicated way. The PDG also reports that the rest energy of each type of neutrino is known to be less than 2 eV. The flavor labels ν_e, ν_μ, and ν_τ indicate how the neutrinos are produced, but in the peculiar context of quantum theory these states do not have a well-defined mass. Instead each state of definite mass is a superposition of different flavor states, and vice versa. Although these issues are fascinating, we will not have cause to pursue them any further. If you have not studied quantum theory you will probably have no idea what the last few sentences mean, and that is okay as far as this course is concerned.

Nonetheless, the presence of any mass for the neutrino, no matter how small, raises an important question about the counting of spin states, which is important in our formulas for the black-body radiation of neutrinos. The bottom line will be that the mass makes no difference, but the reasoning is not simple.

We said above that the neutrino has one spin state, because neutrinos are always left-handed: their spin points in the opposite direction from their momentum. If the neutrino were massless, this statement could be precisely true. It can be shown that for massless particles, if the statement is true for one observer, then the spin and the momentum measured by any other observer would align in the same way. Thus, if the neutrino were massless, its left-handedness would be a relativistically invariant property. While it is difficult to prove this invariance, it is easy to see that the invariance fails if the mass of the neutrino is not zero. For definiteness, consider a left-handed neutrino moving along the z axis in the positive direction, so its spin points in the negative z direction. If it has a nonzero mass then it moves slower than the speed of light, so we can always imagine an observer who moves faster, also along the z axis in the positive direction.

To the moving observer the neutrino will be moving in the negative z direction, but the spin will still point along the negative z direction. Hence, the moving observer will see a right-handed particle. But what is this mysterious right-handed particle? Is this a new spin state that must be counted in our calculations of black-body radiation?

We do not yet have a unique theory of neutrino masses, but there are two possibilities. The neutrino might have a *Majorana* mass, in which case the mysterious right-handed particle in the above thought experiment would be an ordinary antineutrino. Since the antineutrino has already been included in the black-body formulas, they will not be changed. The other possibility is that the neutrino can have a *Dirac* mass, which would be the same type of mass that an electron has. In that case, the mysterious right-handed particle in the thought experiment would be a new spin state of the neutrino. The statement that neutrinos are always left-handed would be blatantly false. Nonetheless, our theories would allow us to calculate the strength of the interactions of these right-handed neutrinos, and they would be incredibly weak. They would be so weak that they would essentially never be produced in the early inverse, so again our black-body formulas would not require modification.

THERMAL HISTORY OF THE UNIVERSE:

We now have all the ingredients necessary to calculate the temperature of the universe as a function of time. Eq. (6.47) gives the mass density as a function of time, and Eq. (6.48) relates the energy density to the temperature. Recalling that $u = \rho c^2$, one can combine these relations and solve for the temperature as a function of time:

$$kT = \left(\frac{45\hbar^3 c^5}{16\pi^3 g G}\right)^{1/4} \frac{1}{\sqrt{t}} \ . \tag{6.64}$$

To find the temperature at 1 sec after the big bang, we now need only plug in numbers:

$$\begin{aligned} kT &= \left[\frac{45\left(1.055\times10^{-34}\ \text{J-s}\right)^3 \left(2.998\times10^{8}\ \text{m-s}^{-1}\right)^5}{16\pi^3(10.75)\left(6.673\times10^{-11}\ \text{m}^3\text{-kg}^{-1}\text{s}^{-2}\right)}\right]^{1/4} \\ &\quad \times \frac{1}{(1\ \text{s})^{1/2}} \times \left(\frac{1\ \text{J}}{\text{kg-m}^2\text{-s}^{-2}}\right)^{1/4} \\ &= 1.378\times10^{-13}\ \text{J} \ , \end{aligned}$$

where the factor $(1\ \text{erg/gm-cm}^2\text{-sec}^{-2})^{1/4}$ is equal to 1, and has been inserted to convert the units to the desired form. Using $1\,\text{eV} = 1.602\times10^{-12}\,\text{erg}$, one can convert this result if one wishes to

$$kT = 0.860\ \text{MeV} \ .$$

Since one knows that $T \propto t^{-1/2}$, one can write down a general expression for the time-temperature relation, for $0.511\ \text{MeV} \ll kT \ll 106\ \text{MeV}$, as

$$\boxed{kT = \frac{0.860\ \text{MeV}}{\sqrt{t\ (\text{in sec})}}\ ,} \tag{6.65a}$$

or equivalently

$$\boxed{T = \frac{9.98 \times 10^9\ \text{K}}{\sqrt{t\ (\text{in sec})}}\ .} \tag{6.65b}$$

As an example one can use Eq. (6.65b) to calculate the temperature of the universe at the end of the first seven days. (Here we are making a minor error, since the value $g_{\text{tot}} = 10\frac{3}{4}$ is not appropriate when kT falls below 0.5 MeV.) One finds $T \approx 1.3 \times 10^7$ K, which is roughly the temperature which is believed to exist in the core of a bright star.

RELATIONSHIP BETWEEN a AND T:

When a gas of black-body radiation expands in thermal equilibrium, there is a simple relationship between the scale factor a and the temperature T. We have already seen that the energy density $\rho \propto 1/a^4$, and that $\rho \propto T^4$. It follows that the product aT remains constant as the universe expands. The constancy of aT is actually a direct consequence of statistical mechanics, and has nothing to do with the dynamics of the expanding universe. As long as the expansion of the universe is slow enough so that the radiation stays in thermal equilibrium, which it is, then the entropy of the expanding gas remains constant. According to Eq. (6.55) the entropy density is proportional to gT^3, so the total entropy S contained in a fixed region in the comoving coordinate system obeys the relation

$$S = sV_{\text{phys}} = sa^3(t)V_{\text{coord}} \propto ga^3T^3\ , \tag{6.66}$$

where V_{coord} is the coordinate volume of the region. As long as g does not change, then the conservation of entropy implies that aT remains constant. Eq. (6.66) allows us to also understand what happens when g does change, which happens when there is a change in the kinds of particles that contribute to the black-body radiation. For example, when kT falls below 0.5 MeV and the electron-positron pairs disappear from the thermal equilibrium mix, the entropy that had been contained in the electron-positron component of the gas must be given to the other components. However, at this point the neutrinos have decoupled, which means that they are no longer undergoing significant interactions with the rest of the gas. The entropy from the electron-positron pairs is therefore given entirely to the photons, and essentially none is given to the neutrinos. The photons are heated relative to the neutrinos, and they continue to be hotter than the neutrinos into the present era. On Problem Set 7 you will show that this transfer of entropy from the electron-positron pairs to the photons increases the quantity aT_γ, where T_γ is the photon temperature, by a factor of $(11/4)^{1/3} = 1.40$.

RECOMBINATION AND DECOUPLING:

The observed baryonic matter in the universe — the matter made of protons, neutrons, and electrons — is about 80% hydrogen by mass. Most of the rest is helium, with an almost negligible amount of heavier elements. One can use statistical mechanics to understand the behavior of this hydrogen under the conditions prevalent in the early universe, but I will not attempt such a calculation in this course. As one might guess, hydrogen will ionize (*i.e.* break up into separate protons and electrons) if the temperature is high enough. The temperature necessary to cause ionization depends on the density, but for the history of our universe one can say that the hydrogen is ionized when T is greater than about 4,000 K.

Thus, when the temperature falls below 4,000 K, the ionized hydrogen coalesces into neutral atoms. The process is usually called "recombination," although I am at a loss to explain the significance of the prefix "re-". When recombination occurs, the universe becomes essentially transparent to photons. The photons cease to interact with the other particles, and this process is called "decoupling". Decoupling occurs slightly later than recombination, at a temperature of about 3,000 K, since even a small residual density of free electrons is enough to keep the photons coupled to the other particles. The photons which we observe today in the cosmic background radiation are photons which for the most part have last scattered at the time of decoupling.

We can estimate the time of decoupling by using the constancy of aT. Here T indicates the temperature of the photons, since the neutrinos have decoupled and are not relevant to the current discussion. It is very accurate to assume that aT has remained constant from the time of decoupling to the present, since the photons are not interacting significantly with anything else, so the conservation of photon entropy implies that $a^3 s_\gamma \propto a^3T^3$ is constant. Using the subscript d to denote quantities evaluated at the time of decoupling, and subscript 0 to denote quantities evaluated at the present time, one has

$$a_d T_d = a_0 T_0\ , \tag{6.67}$$

from which one has immediately that

$$\frac{a_d}{a_0} = \frac{T_0}{T_d}\ . \tag{6.68}$$

Assuming that the universe is flat, and making the crude approximation that it can be treated as matter-dominated from t_d to the present, one has $a(t) \propto t^{2/3}$ and

$$\left(\frac{t_d}{t_0}\right)^{2/3} = \frac{T_0}{T_d}\ . \tag{6.69}$$

Solving, one has

$$\begin{aligned} t_d &= \left(\frac{T_0}{T_d}\right)^{3/2} t_0 \\ &\approx \left(\frac{2.7\,\mathrm{K}}{3000\,\mathrm{K}}\right)^{3/2} \times \left(13.7\times 10^9\ \mathrm{yr}\right) \approx 370{,}000\ \mathrm{yr}\ . \end{aligned} \tag{6.70}$$

On p. 159, Ryden quotes a more accurate numerical calculation, giving $t_d \approx 350{,}000$ yr.

THE SPECTRUM OF THE COSMIC BACKGROUND RADIATION:

The cosmic background radiation was discovered by Penzias and Wilson in 1965. They measured at one frequency only, but found that the radiation appeared to be coming uniformly from all directions in space. This radiation was quickly identified by Dicke, Peebles, Roll, and Wilkinson as the remnant radiation from the big bang. Since then the measurement of the cosmic background radiation has become a minor industry, and much data has been obtained about the spectrum of the radiation and about its angular distribution in the sky.

The prediction from big bang cosmology is that the spectrum should be thermal, corresponding to black-body radiation that has been redshifted from its initially very high temperature. It is a peculiar feature of the black-body spectrum that it maintains its thermal equilibrium form under uniform redshift, even though the photons in the radiation are noninteracting. That is, if each photon in the black-body probability distribution is redshifted by the same factor, the net effect is to produce a new probability distribution which is again of the black-body form, except that the temperature is modified by a factor of the redshift. Thus, the redshift reduces the temperature, but does not lead to departures from the thermal equilibrium spectrum.

The ideal Planck spectrum for such radiation has an energy density $\rho_\nu(\nu)d\nu$, for radiation in the wavelength interval between ν and $\nu + d\nu$, given by

$$\rho_\nu(\nu)d\nu = \frac{16\pi^2\hbar\nu^3}{c^3}\frac{1}{e^{2\pi\hbar\nu/kT}-1}d\nu\ . \tag{6.71}$$

The subscript ν on ρ_ν indicates that it is the energy density per frequency interval, while one could alternatively speak of the energy density per wavelength interval, ρ_λ. (As with the other statistical mechanics results in this set of Lecture Notes, we will use Eq. (6.71) without derivation.) Observers usually do not directly measure the energy density, however, but instead measure the intensity of the radiation. It can be shown

that the power hitting a detector per frequency interval per area of aperature per solid angle of aperture is given by

$$I_\nu(\nu) = \frac{c}{4\pi}\rho_\nu(\nu) = \frac{4\pi\hbar\nu^3}{c^2}\frac{1}{e^{2\pi\hbar\nu/kT}-1}\ . \tag{6.72}$$

The data on the spectrum available in 1975 is summarized on the two graphs on the following page. The graphs show measurements of the energy density in the cosmic background radiation at different frequencies (or wavelengths). The lower horizontal axis shows the frequency in gigahertz (10^9 cycles per second), and the upper horizontal axis shows the corresponding wavelength. The solid line is the expected blackbody distribution, shown for the best current determination of the temperature, 2.726 K. Part (a) shows the low frequency measurements, including those of Penzias & Wilson and Roll & Wilkinson (which was published about 6 months after the Penzias & Wilson result). Part (b) includes the full range of interesting frequencies. The circles show the results of each measurement, and the bars indicate the range of the estimated uncertainty. The measurements with small uncertainties are shown with dark shading. A high-frequency broad-band measurement is shown on part (b), labeled "1974 Balloon" — the measured energy density is shown as a solid line, and the estimated uncertainty is indicated by gray shading. The 1971 balloon measurements were taken by the MIT team of Dirk Muehlner and Rainer Weiss. (The energy density on both graphs is measured in electron volts per cubic meter per gigahertz.)

The earth's atmosphere poses a serious problem for measuring the high frequency side of the curve, so the best measurements must be done from balloons, rockets, or satellites. In 1987 a rocket probe was launched by a collaboration between the University of California at Berkeley and Nagoya University in Japan. The resulting paper* included a graph of the remarkable data shown in Figure 6.3.

Note that the points labeled 2 and 3 are much higher than the black body spectrum predicts. Using each of these points individually to determine a temperature, the authors find:

Point 2: $T = 2.955 \pm 0.017\,\mathrm{K}$
Point 3: $T = 3.175 \pm 0.027\,\mathrm{K}$

These numbers correspond to discrepancies of 12 and 16 standard deviations, respectively, from the temperature of $T = 2.74\,\mathrm{K}$ that fits the lower frequency points. In terms of energy, the excess intensity seen at high frequencies in this experiment amounts to about

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* T. Matsumoto, S. Hayakawa, H. Matsuo, H. Murakami, S. Sato, A.E. Lange, and P.L. Richards, "The Submillimeter Spectrum of the Cosmic Background Radiation," *Astrophysical Journal,* vol. 329, pp. 567–571 (1988), http://adsabs.harvard.edu/abs/1988ApJ...329..567M.

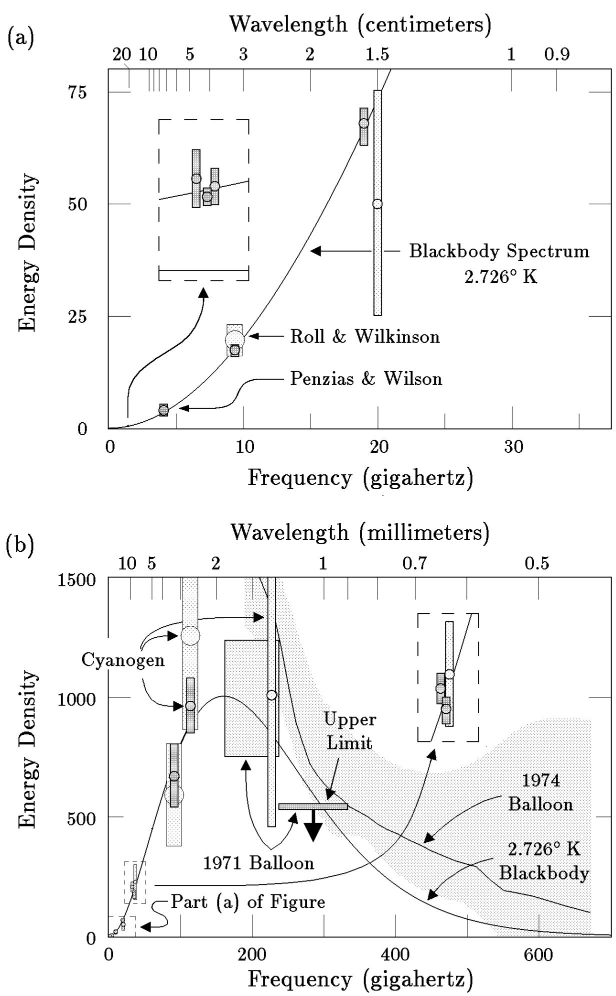


Figure 6.2: The spectrum of the cosmic microwave background as it was known in 1975. Each graph shows the energy density of the radiation, in electron volts per cubic meter per gigahertz, as a function of frequency. Part (a) shows the lowest frequencies, which include the original measurement of Penzias and Wilson, while part (b) includes the full range of interesting frequencies. The curve shows the black-body spectrum for 2.726 K.

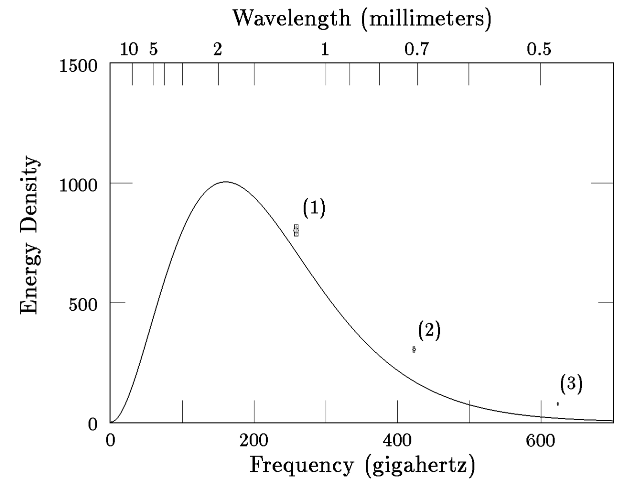


Figure 6.3: Three data points in the CMB spectrum measured by the Berkeley-Nagoya rocket experiment in 1987. Point (3) differs from the theoretically expected curve by 16 standard deviations. The lesson, apparently, is that one should not reject a previously successful theory until the evidence against it is reliably confirmed.

10% of the total energy in the cosmic background radiation. Cosmologists were stunned by the extremely significant disagreement with predictions. Some tried to develop theories to explain the radiation, without much success, while others banked on the theory that it would go away. The experiment looked like a very careful one, however, so it was difficult to dismiss. The most likely source of error in an experiment of this type is the possibility that the detectors were influenced by heat from the exhaust of the launch vehicle — but the experimenters very carefully tracked how the observed radiation varied with time as the detector moved away from the launch rocket, and it seemed clear that the rocket was not a factor.

The same group tried to check their results with a second flight a year later, but the rocket failed and no useful data was obtained.

In the fall of 1989 NASA launched the Cosmic Background Explorer, known as COBE (pronounced “koh-bee”). This marked the first time that a satellite was used to probe the background radiation. Within months, the COBE group announced their first results at a meeting of the American Astronomical Society in Washington, D.C., January 1990. The data was so spectacular that the audience rose to give the speaker, John Mather, a standing ovation. The detailed preprint, with a cover sheet showing a sketch of the satellite, was released the same day, and later published as an *Astrophysical Journal* letter.*

The data showed a perfect fit to the blackbody spectrum, with a temperature of $2.735 \pm 0.06\,\mathrm{K}$, with no evidence whatever for the "submillimeter excess" that had been seen by Matsumoto *et al.* The data was shown with estimated error bars of 1% of the peak intensity, which the group regarded as very conservative. The graph is reproduced here as Fig. 6.5.

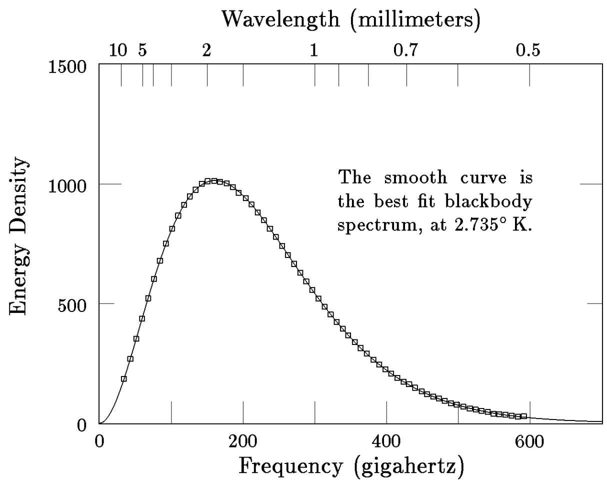


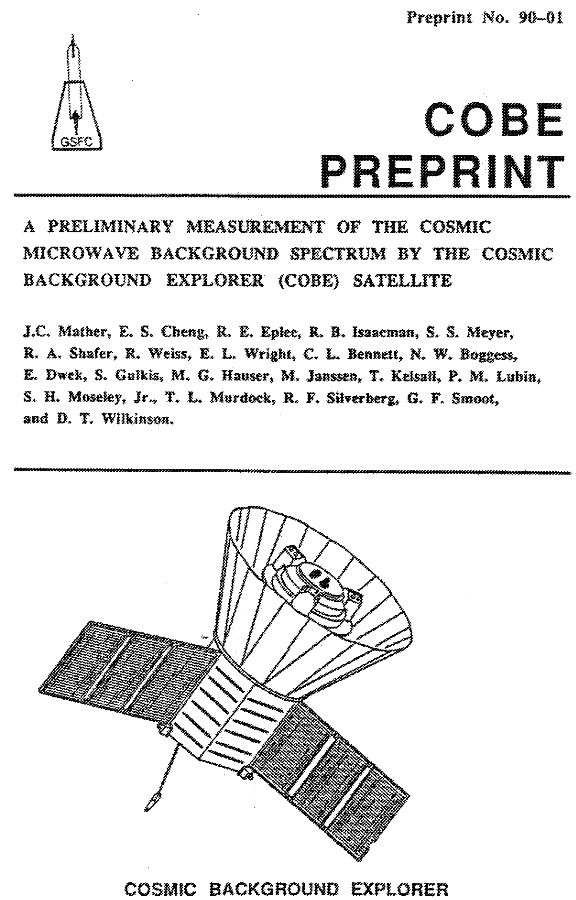
Figure 6.5: The original (1990) COBE measurement of the spectrum of the cosmic microwave background, based on only 9 minutes of data. The vertical axis shows the energy density in units of electron volts per cubic meter per gigahertz.

Once again, the vertical axis is calibrated in electron volts per cubic meter per gigahertz.

Since the COBE instrument is far more precise and has more internal consistency checks, there has been no doubt in the scientific community that the COBE result supercedes the previous one. Despite the 16σ discrepancy of 1988, the cosmic background radiation is now once again believed to have a nearly perfect black-body spectrum.

In January 1993, the COBE team released its final data on the cosmic background radiation spectrum. The first graph had come from just 9 minutes of data, but now the

* J.C. Mather *et al.*, "A preliminary measurement of the cosmic microwave background spectrum by the Cosmic Background Explorer (COBE) satellite," *Astrophysical Journal*, vol. 354, pp. L37–L40 (1990), http://adsabs.harvard.edu/abs/1990ApJ...354L..37M.


Preprint No. 90-01

GSFC

COBE PREPRINT

A PRELIMINARY MEASUREMENT OF THE COSMIC MICROWAVE BACKGROUND SPECTRUM BY THE COSMIC BACKGROUND EXPLORER (COBE) SATELLITE

J.C. Mather, E. S. Cheng, R. E. Eplee, R. B. Isaacman, S. S. Meyer, R. A. Shafer, R. Weiss, E. L. Wright, C. L. Bennett, N. W. Boggess, E. Dwek, S. Gulkis, M. G. Hauser, M. Janssen, T. Kelsall, P. M. Lubin, S. H. Moseley, Jr., T. L. Murdock, R. F. Silverberg, G. F. Smoot, and D. T. Wilkinson.

COSMIC BACKGROUND EXPLORER

Figure 6.4: The cover page of the original preprint of the COBE cosmic microwave background spectrum measurement.

team had analyzed the data from the entire mission. The error boxes were shrunk beyond visibility to only 0.03%, and the background spectrum was still perfectly blackbody, just as the big bang theory predicted. The new value for the temperature was just a little lower, 2.726 K, with an uncertainty of less than 0.01 K.

The perfection of the spectrum means that the big bang must have been very simple. The COBE team estimated that no more than 0.03% of the energy in the background radiation could have been released anytime after the first year of the life of the universe, since energy released after one year would not have had time to reach such a perfect state of thermal equilibrium. Theories that predict energy release from the decay of turbulent motions or exotic elementary particles, from a generation of exploding or massive stars preceding those already known, or from dozens of other interesting hypothetical objects, were all excluded at once.

Although a few advocates of the steady state universe have not yet given up, the COBE team announced that the theory is ruled out. A nearly perfect blackbody spectrum can be achieved in the steady state theory only by a thick fog of objects that could absorb and re-emit the microwave radiation, allowing the radiation to come to a uniform temperature. Steady state proponents have in the past suggested that interstellar space might be filled by a thin dust of iron whiskers that could create such a fog. However, a fog that is thick enough to explain the new data would be so opaque that distant sources would not be visible.

In this chapter we have discussed mainly the spectrum of the cosmic microwave background (CMB). Starting in 1992, however, with some preliminary results from the COBE satellite, astronomers have also been able to measure the anisotropies of the CMB. This is quite a tour de force, since the radiation is isotropic to an accuracy of about 1 part in 10^5. Since the photons of the CMB have been travelling essentially on straight lines since the time of decoupling, these anisotropies are interpreted as a direct measure of the degree of nonuniformity of the matter in the universe at the time of decoupling, about 380,000 years after the big bang. These non-uniformities are crucially important, because they give us clues about how the universe originated, and because they are believed to be the seeds which led to the formation of the complicated structure that the universe has today. We will return to discuss the physics of these nonuniformities near the end of the course.

The Nobel Prize in Physics 2006 was awarded jointly to John C. Mather and George F. Smoot "for their discovery of the blackbody form and anisotropy of the cosmic microwave background radiation."