(b) Show that Equation 10.43 does *not* work, in this case. Use Equation 10.62 to determine c_- (in Equation 10.54). Confirm that the last term in Equation 10.55 is second order in ω (don't forget the $\epsilon = \omega/\omega_1$ out front). Show that $\gamma_+(t)$ (Equation 10.64) does satisfy the *corrected* version of Equation 10.43, Equation 10.56.

*****Problem 10.8** Work out the analog to Equation 10.71 for a particle of spin 1. *Answer*: $-\Omega$ (for spin s the result is $-s\Omega/2$).

10.2.4 The Aharonov-Bohm Effect

In classical electrodynamics the potentials (φ and $\mathbf{A}$)[13] are not directly measurable—the physical quantities are the electric and magnetic fields:

$$\mathbf{E} = -\nabla\varphi - \frac{\partial \mathbf{A}}{\partial t}, \quad \mathbf{B} = \nabla \times \mathbf{A}. \qquad [10.72]$$

The fundamental laws of the theory (Maxwell's equations and the Lorentz force law) make no reference to potentials, which are (from a logical point of view) no more than convenient but dispensible scaffolding for getting a better purchase on the real structure (the fields). Indeed, you're perfectly free to *change* the potentials:

$$\varphi \to \varphi' = \varphi - \frac{\partial \Lambda}{\partial t}, \quad \mathbf{A} \to \mathbf{A}' = \mathbf{A} + \nabla\Lambda, \qquad [10.73]$$

where Λ is an arbitrary function of position and time; this is called a gauge transformation, and it has no effect at all on the fields.

In quantum mechanics the potentials play a more significant role, for the Hamiltonian (Equation 4.201) is expressed in terms of φ and $\mathbf{A}$, not $\mathbf{E}$ and $\mathbf{B}$:

$$H = \frac{1}{2m}\left(\frac{\hbar}{i}\nabla - q\mathbf{A}\right)^2 + q\varphi. \qquad [10.74]$$

Nevertheless, the theory is still invariant under gauge transformations (see Problem 4.53), and it was taken for granted until quite recently that there could be no electromagnetic influences in regions where $\mathbf{E}$ and $\mathbf{B}$ are zero—any more than there can be in the classical theory. But in 1959 Aharonov and Bohm[14] showed that the vector potential *can* affect the quantum behavior of a charged particle that never encounters an electromagnetic field. I'll work out a simple example first, then discuss the

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[13] I'm sorry, but we have reached a notational impasse: It is customary in quantum mechanics to use the letter V for *potential energy*, but in electrodynamics the same letter is reserved for the scalar potential. To avoid confusion I'll use φ for the scalar potential. See Problems 4.51, 4.52, and 4.53 for background on this material.

[14] Y. Aharonov and D. Bohm, *Phys. Rev.* **115**, 485 (1959). For a significant precursor, see W. Ehrenberg and R. E. Siday, *Proc. Phys. Soc. London* **B62**, 8 (1949).

Aharanov-Bohm effect itself, and finally indicate how it can be thought of as an example of Berry's phase.

Imagine a particle constrained to move in a circle of radius b (a bead on a wire ring, if you like). Along the axis runs a solenoid of radius $a < b$, carrying a magnetic field $\mathbf{B}$ (see Figure 10.12). If the solenoid is extremely long, the field inside is uniform, and the field outside is zero. But the vector potential outside the solenoid is *not* zero; in fact (adopting the convenient gauge condition $\nabla \cdot \mathbf{A} = 0$),[15]

$$\mathbf{A} = \frac{\Phi}{2\pi r}\hat{\phi}, \quad (r > a), \tag{10.75}$$

where $\Phi = \pi a^2 B$ is the **magnetic flux** through the solenoid. Meanwhile, the solenoid is uncharged, so the scalar potential φ is zero. In this case the Hamiltonian (Equation 10.74) becomes

$$H = \frac{1}{2m}\left[-\hbar^2\nabla^2 + q^2A^2 + 2i\hbar q\mathbf{A}\cdot\nabla\right]. \tag{10.76}$$

But the wave function depends only on the azimuthal angle ϕ, ($\theta = \pi/2$ and $r = b$) so $\nabla \to (\hat{\phi}/b)(d/d\phi)$, and the Schrödinger equation reads

$$\frac{1}{2m}\left[-\frac{\hbar^2}{b^2}\frac{d^2}{d\phi^2} + \left(\frac{q\Phi}{2\pi b}\right)^2 + i\frac{\hbar q\Phi}{\pi b^2}\frac{d}{d\phi}\right]\psi(\phi) = E\psi(\phi). \tag{10.77}$$

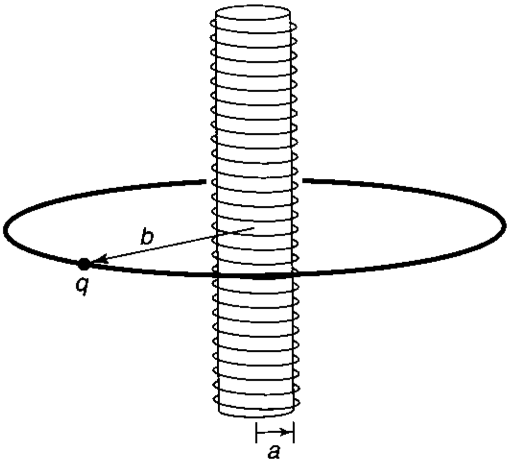

Figure 10.12: Charged bead on a circular ring through which a long solenoid passes.

This is a linear differential equation with constant coefficients:

$$\frac{d^2\psi}{d\phi^2} - 2i\beta\frac{d\psi}{d\phi} + \epsilon\psi = 0, \tag{10.78}$$

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[15] See, for instance, D. J. Griffiths, *Introduction to Electrodynamics*, 2nd ed. (Englewood Cliffs, NJ: Prentice Hall, 1989), Equation 5.65.

where

$$\beta \equiv \frac{q\Phi}{2\pi\hbar} \quad \text{and} \quad \epsilon \equiv \frac{2mb^2E}{\hbar^2} - \beta^2. \tag{10.79}$$

Solutions are of the form

$$\psi = Ae^{i\lambda\phi}, \tag{10.80}$$

with

$$\lambda = \beta \pm \sqrt{\beta^2 + \epsilon} = \beta \pm \frac{b}{\hbar}\sqrt{2mE}. \tag{10.81}$$

Continuity of $\psi(\phi)$, at $\phi = 2\pi$, requires that λ be an *integer*:

$$\beta \pm \frac{b}{\hbar}\sqrt{2mE} = n, \tag{10.82}$$

and it follows that

$$E_n = \frac{\hbar^2}{2mb^2}\left(n - \frac{q\Phi}{2\pi\hbar}\right)^2, \quad (n = 0, \pm 1, \pm 2, \ldots). \tag{10.83}$$

The solenoid lifts the twofold degeneracy of the bead on a ring (Problem 2.43): Positive n, representing a particle traveling in the *same* direction as the current in the solenoid, has a somewhat *lower* energy (assuming q is positive) than negative n, describing a particle traveling in the *opposite* direction. And, more important, the allowed energies clearly depend on the field inside the solenoid, *even though the field at the location of the particle is zero.*[16]

More generally, suppose a particle is moving through a region where **B** is zero (so $\nabla \times \mathbf{A} = 0$), but **A** itself is *not*. (I'll assume that **A** is static, although the method can be generalized to time-dependent potentials.) The (time-dependent) Schrödinger equation,

$$\left[\frac{1}{2m}\left(\frac{\hbar}{i}\nabla - q\mathbf{A}\right)^2 + V\right]\Psi = i\hbar\frac{\partial\Psi}{\partial t}, \tag{10.84}$$

with potential energy V—which may or may not include an electrical contribution $q\varphi$—can be simplified by writing

$$\Psi = e^{ig}\Psi', \tag{10.85}$$

where

$$g(\mathbf{r}) \equiv \frac{q}{\hbar}\int_{\mathcal{O}}^{\mathbf{r}} \mathbf{A}(\mathbf{r}') \cdot d\mathbf{r}', \tag{10.86}$$

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[16] It is a peculiar property of **superconducting** rings that the enclosed flux is *quantized*: $\Phi = (2\pi\hbar/q)n'$, where n' is an integer. In this case the effect is undetectable, since $E_n = (\hbar^2/2mb^2)(n + n')^2$, and $(n + n')$ is just another integer. (Incidentally, the charge q here turns out to be *twice* the charge of an electron; the superconducting electrons are locked together in pairs.) However, **flux quantization** is enforced by the *superconductor* (which induces circulating currents to make up the difference), not by the solenoid or the electromagnetic field, and it does not occur in the example considered here.

and $\mathcal{O}$ is some (arbitrarily chosen) reference point. Note that this definition makes sense *only* when $\nabla \times \mathbf{A} = 0$ throughout the region in question—otherwise the line integral depends entirely on the *path* taken from $\mathcal{O}$ to $\mathbf{r}$, and hence does not define a function of $\mathbf{r}$. In terms of Ψ', the gradient of Ψ is

$$\nabla\Psi = e^{ig}(i\nabla g)\Psi' + e^{ig}(\nabla\Psi');$$

but $\nabla g = (q/\hbar)\mathbf{A}$, so

$$\left(\frac{\hbar}{i}\nabla - q\mathbf{A}\right)\Psi = \frac{\hbar}{i}e^{ig}\nabla\Psi', \qquad [10.87]$$

and it follows that

$$\left(\frac{\hbar}{i}\nabla - q\mathbf{A}\right)^2\Psi = -\hbar^2 e^{ig}\nabla^2\Psi'. \qquad [10.88]$$

Putting this into Equation 10.84, and canceling the common factor of e^{ig}, we are left with

$$-\frac{\hbar^2}{2m}\nabla^2\Psi' + V\Psi' = i\hbar\frac{\partial\Psi'}{\partial t}. \qquad [10.89]$$

Evidently Ψ' satisfies the Schrödinger equation *without* $\mathbf{A}$. If we can solve Equation 10.89, correcting for the presence of a (curl-free) vector potential is trivial: You just tack on the phase factor e^{ig}.

Aharonov and Bohm proposed an experiment in which a beam of electrons is split in two, and passed either side of a long solenoid, before being recombined (Figure 10.13). The beams are kept well away from the solenoid itself, so they encounter only regions where $\mathbf{B} = 0$. But $\mathbf{A}$, which is given by Equation 10.75, is *not* zero, and (assuming V is the same on both sides), the two beams arrive with *different phases*:

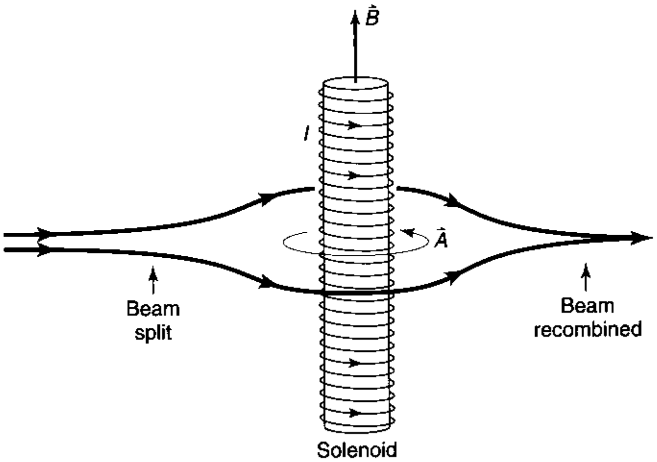

Figure 10.13: The Aharonov-Bohm effect: electron beam splits, with half passing either side of a long solenoid.

$$g = \frac{q}{\hbar}\int \mathbf{A}\cdot d\mathbf{r} = \frac{q\Phi}{2\pi\hbar}\int\left(\frac{1}{r}\hat{\phi}\right)\cdot(r\hat{\phi}\,d\phi) = \pm\frac{q\Phi}{2\hbar}. \qquad [10.90]$$

The plus sign applies to the electrons traveling in the same direction as **A**—which is to say, in the same direction as the current in the solenoid. The beams arrive *out of phase* by an amount proportional to the magnetic flux their paths encircle:

$$\text{phase difference} = \frac{q\Phi}{\hbar}. \qquad [10.91]$$

This phase shift leads to measurable interference (as in Equation 10.53), and has been confirmed experimentally by Chambers and others.[17]

The Aharonov-Bohm effect can be regarded as an example of geometric phase, as Berry himself noted in his first paper. Suppose the charged particle is confined to a box (which is centered at point **R** outside the solenoid) by a potential $V(\mathbf{r}-\mathbf{R})$—see Figure 10.14. (In a moment we're going to transport the box around the solenoid, so **R** will become a function of time, but for now it is just some fixed vector.) The eigenfunctions of the Hamiltonian are determined by

$$\left\{\frac{1}{2m}\left[\frac{\hbar}{i}\nabla - q\mathbf{A}(\mathbf{r})\right]^2 + V(\mathbf{r}-\mathbf{R})\right\}\psi_n = E_n\psi_n. \qquad [10.92]$$

We have already learned how to solve equations of this form:

$$\psi_n = e^{ig}\psi_n', \qquad [10.93]$$

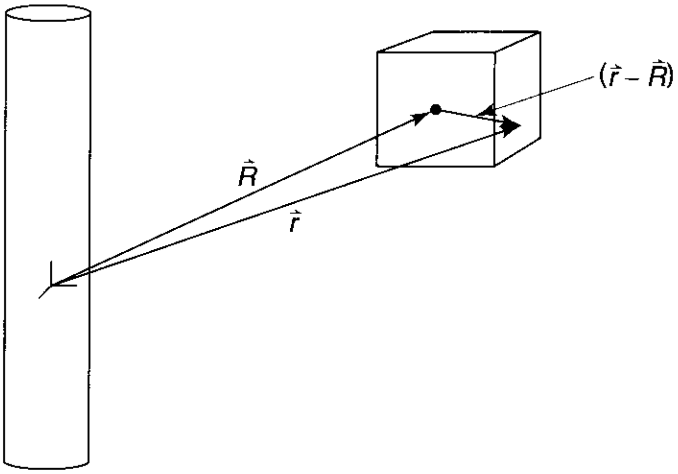

Figure 10.14: Particle confined to a box, by a potential $V(\mathbf{r}-\mathbf{R})$.

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[17] R. G. Chambers, *Phys. Rev. Lett.* **5**, 3 (1960).

where[18]

$$g \equiv \frac{q}{\hbar}\int_{\mathbf{R}}^{\mathbf{r}} \mathbf{A}(\mathbf{r}')\cdot d\mathbf{r}' \qquad [10.94]$$

and ψ' satisfies the same eigenvalue equation, only with $\mathbf{A} \to 0$:

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{r}-\mathbf{R})\right]\psi_n' = E_n\psi_n'. \qquad [10.95]$$

Notice that ψ_n' is a function only of the combination $(\mathbf{r}-\mathbf{R})$, not (like ψ_n) of $\mathbf{r}$ and $\mathbf{R}$ separately.

Now let's carry the box around the solenoid (in this case the process doesn't even have to be adiabatic). To determine Berry's phase, we must first evaluate the quantity $\langle\psi_n|\nabla_R\psi_n\rangle$. Noting that

$$\nabla_R\psi_n = \nabla_R\left[e^{ig}\psi_n'(\mathbf{r}-\mathbf{R})\right] = -i\frac{q}{\hbar}\mathbf{A}(\mathbf{R})e^{ig}\psi_n'(\mathbf{r}-\mathbf{R}) + e^{ig}\nabla_R\psi_n'(\mathbf{r}-\mathbf{R}),$$

we find

$$\begin{aligned}&\langle\psi_n|\nabla_R\psi_n\rangle\\ &= \int e^{-ig}[\psi_n'(\mathbf{r}-\mathbf{R})]^* e^{ig}\left[-i\frac{q}{\hbar}\mathbf{A}(\mathbf{R})\psi_n'(\mathbf{r}-\mathbf{R}) + \nabla_R\psi_n'(\mathbf{r}-\mathbf{R})\right]d^3\mathbf{r}\\ &= -i\frac{q}{\hbar}\mathbf{A}(\mathbf{R}) - \int[\psi_n'(\mathbf{r}-\mathbf{R})]^*\nabla\psi_n'(\mathbf{r}-\mathbf{R})\,d^3\mathbf{r}.\end{aligned} \qquad [10.96]$$

The ∇ with no subscript denotes the gradient with respect to $\mathbf{r}$, and I used the fact that $\nabla_R = -\nabla$, when acting on a function of $(\mathbf{r}-\mathbf{R})$. But the last integral is $i/\hbar$ times the expectation value of momentum, in an eigenstate of the Hamiltonian $-(\hbar^2/2m)\nabla^2 + V$, which we know from Section 2.1 is *zero*. So

$$\langle\psi_n|\nabla_R\psi_n\rangle = -i\frac{q}{\hbar}\mathbf{A}(\mathbf{R}). \qquad [10.97]$$

Putting this into Berry's formula (Equation 10.49), we conclude that

$$\gamma_n(T) = \frac{q}{\hbar}\oint\mathbf{A}(\mathbf{R})\cdot d\mathbf{R} = \frac{q}{\hbar}\int(\nabla\times\mathbf{A})\cdot d\mathbf{a} = \frac{q\Phi}{\hbar}, \qquad [10.98]$$

which neatly confirms the Aharonov-Bohm result (Equation 10.91), and reveals that the Aharonov-Bohm effect is a particular instance of geometric phase.[19]

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[18] It is convenient to set the reference point $\mathcal{O}$ at the center of the box, for this guarantees that we recover the original phase convention for ψ_n when we complete the journey around the solenoid. If you use a fixed point in space, for example, you'll have to readjust the phase "by hand", at the far end; this leads to exactly the same answer, but it's a crude way to do it. In general, when choosing the phase convention for the eigenfunctions in Equation 10.39, you want to make sure that $\psi_n(x, T) = \psi_n(x, 0)$ so that no spurious phase changes are introduced.

[19] Incidentally, in this case the analogy between Berry's phase and magnetic flux (Equation 10.59) becomes *almost* an identity: "$\mathbf{B}$" = $(q/\hbar)\mathbf{B}$.

What are we to make of the Aharonov-Bohm effect? Evidently our classical preconceptions are simply *mistaken*: There *can* be electromagnetic effects in regions where the fields are zero. Note, however, that this does not make **A** itself measurable—only the enclosed *flux* comes into the final answer, and the theory remains gauge invariant.

Problem 10.9

(a) Derive Equation 10.76, from Equation 10.74.

(b) Derive Equation 10.88, starting with Equation 10.87.

FURTHER PROBLEMS FOR CHAPTER 10

* * ***Problem 10.10** Suppose the one-dimensional harmonic oscillator (mass m, frequency ω) is subjected to a driving force of the form $F(t) = m\omega^2 f(t)$, where $f(t)$ is some specified function [I have factored out $m\omega^2$ for notational convenience; notice that $f(t)$ has the dimensions of *length*]. The Hamiltonian is

$$H(t) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}m\omega^2 x^2 - m\omega^2 x f(t). \qquad [10.99]$$

Assume that the force was first turned on at time $t = 0$: $f(t) = 0$ for $t \leq 0$. This system can be solved exactly, both in classical mechanics and in quantum mechanics.[20]

(a) Determine the *classical* position of the oscillator, assuming it started out at rest at the origin $[x_c(0) = \dot{x}_c(0) = 0]$. *Answer*:

$$x_c(t) = \omega \int_0^t f(t') \sin[\omega(t - t')]\, dt'. \qquad [10.100]$$

(b) Show that the solution to the (time-dependent) Schrödinger equation for this oscillator, assuming it started out in the nth state of the *undriven* oscillator $[\Psi(x, 0) = \psi_n(x)$, where $\psi_n(x)$ is given by Equation 2.50], can be written as

$$\Psi(x, t) = \psi_n(x - x_c)e^{\frac{i}{\hbar}\left[-(n+\frac{1}{2})\hbar\omega t + m\dot{x}_c(x - \frac{x_c}{2}) + \frac{m\omega^2}{2}\int_0^t f(t')x_c(t')dt'\right]}. \qquad [10.101]$$

(c) Show that the eigenfunctions and eigenvalues of $H(t)$ are

$$\psi_n(x, t) = \psi_n(x - f); \quad E_n(t) = \left(n + \frac{1}{2}\right)\hbar\omega - \frac{1}{2}m\omega^2 f^2. \qquad [10.102]$$

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[20] See Y. Nogami, *Am. J. Phys.* **59**, 64 (1991) and references therein.

(d) Show that in the adiabatic approximation the classical position (Equation 10.100) reduces to $x_c(t) \cong f(t)$. *Hint*: Use the integration-by-parts trick of Section 10.1.2. State the precise criterion—analogous to Equation 10.15—for adiabaticity.

(e) Confirm the adiabatic theorem for this example, by using the results in (c) and (d) to show that

$$\Psi(x,t) \cong \psi_n(x,t)e^{i\theta_n(t)}e^{i\gamma_n(t)}. \qquad [10.103]$$

Check that the dynamic phase has the correct form (Equation 10.41). Is the geometric phase what you would expect?

*****Problem 10.11** In time-dependent perturbation theory, we used the completeness of the unperturbed eigenfunctions (of H_0) to expand $\Psi(x,t)$ (see Equation 9.81). But we could as well use the instantaneous eigenfunctions of $H(t)$ (Equation 10.39), since they, too, are complete:

$$\Psi(x,t) = \sum_n c_n(t)\psi_n(x,t)e^{i\theta_n}, \qquad [10.104]$$

where $\theta_n(t)$ is given by Equation 10.41. We can use this expansion to develop an **adiabatic series**, whose leading term is the adiabatic approximation itself and whose successive terms represent the *departure* from perfect adiabaticity.

(a) Insert Equation 10.104 into the (time-dependent) Schrödinger equation, and obtain the following formula for the coefficients:

$$\dot{c}_m = -\sum_n \langle\psi_m|\frac{\partial\psi_n}{\partial t}\rangle c_n e^{i(\theta_n-\theta_m)}. \qquad [10.105]$$

(b) Suppose the system starts out in the N^{th} state; in the adiabatic approximation, it *remains* in the N^{th} state, picking up (at most) a time-dependent geometric phase (compare Equations 10.40 and 10.104):

$$c_n(t) = \delta_{nN}e^{i\gamma_N(t)}. \qquad [10.106]$$

Substitute this into the right side of Equation 10.105, and obtain the "first correction" to adiabaticity:

$$c_m(t) = -\int_0^t \langle\psi_m|\frac{\partial\psi_N}{\partial t}\rangle e^{i\gamma_N}e^{i(\theta_N-\theta_m)}\,dt'. \qquad [10.107]$$

This enables us to calculate transition probabilities in the *nearly* adiabatic regime. To develop the "second correction," we would insert Equation 10.107 on the right side of Equation 10.105, and so on.

(c) As an example, apply Equation 10.107 to the driven-oscillator (Problem 10.10). Show that (in the near-adiabatic approximation) transitions are possible only to the two immediately adjacent levels, for which

$$c_{N+1}(t) = i\sqrt{\frac{m\omega}{2\hbar}}\sqrt{N+1}\int_0^t \dot{f}(t')e^{i\omega t'}\,dt',$$

$$c_{N-1}(t) = i\sqrt{\frac{m\omega}{2\hbar}}\sqrt{N}\int_0^t \dot{f}(t')e^{-i\omega t'}\,dt'.$$

(The transition *probabilities* are the absolute squares of these, of course.)