Massachusetts Institute of Technology
Physics Department

Physics 8.322 **Spring 2007**
Quantum Theory II **April 30, 2007**

Assignment 11

Due May 11, 2007, at the end of the day

Announcements

- There will be one more problem set in 8.322, due on May 16. I'll post it as soon as possibe

Reading topics for this period

- Formal scattering theory; Born Approximation; Separable Potential

Reading Recommendations 11

- Sakurai, Formal development and Born Approximation: §7.1 — 7.3, and Method of Partial Waves: 7.5 — 7.6.
- Gottfried, Partial waves, §8.1; Formal development, 8.2(a); Born approximation, 8.3(a),(b).
- Schiff has a lengthy and pretty good section on scattering theory. §5 for basics and §9 for approximation methods.

Problem Set 11

Topics covered in the problems

- The phase shift and the radial wave function.
- Scattering from atoms in the first Born approximation.
- The Born approximation for a δ-shell
- Spin dependent scattering

Problems

1. **A formula for the phase shift (postponed from last week.)**

   In our studying of scattering theory we derived a general formula for elastic scattering amplitude from a potential $V(\vec{x})$,

   $$f(\vec{k}', \vec{k}) = \sum_{\ell} (2\ell+1) f_\ell(E) P_\ell(\cos\theta_{\vec{k}'\vec{k}}) = -\frac{2m}{4\pi}(2\pi)^3 \langle \vec{k}'|V|\psi_{\vec{k}}^{(+)}\rangle$$

   Here, $E = k^2/2m$, $f_\ell(E) = e^{i\delta_\ell(E)}\sin\delta_\ell(E)/k$, the state $|\vec{k}\rangle$ is a plane wave, $\langle\vec{x}|\vec{k}\rangle = e^{i\vec{k}\cdot\vec{x}}$, and the state $|\psi_{\vec{k}}^{(+)}\rangle$ is a solution regular at the origin and obeying outgoing wave boundary conditions. In lecture we showed that for a spherically symmetric potential,

   $$\langle\vec{x}|\psi_{\vec{k}}^{(+)}\rangle = \sum_{\ell}(2\ell+1) i^\ell e^{i\delta_\ell(k)} P_\ell(\cos\theta_{\vec{k}\vec{x}}) R_\ell^E(r),$$

   where R_ℓ is normalized so

   $$\lim_{r\to\infty} R_\ell(k,r) = \frac{1}{kr}\sin(kr - \ell\pi/2 + \delta_\ell(k)).$$

   (a) From this starting point, derive an exact expression for the phase shift in the ℓ^{th} partial wave,

   $$\sin\delta_\ell(k) = -2mk\int_0^\infty dr r^2 V(r) j_\ell(kr) R_\ell(k,r)$$

   Note that three different angles appear in this problem: $\cos\theta_{\vec{k}\vec{x}} = \hat{k}\cdot\hat{x}$, $\cos\theta_{\vec{k}'\vec{x}} = \hat{k}'\cdot\hat{x}$, and $\cos\theta_{\vec{k}'\vec{k}} = \hat{k}'\cdot\hat{k}$. You will have to find identities involving Legendre polynomials and spherical harmonics in order to evaluate the angular integrals that occur here.

   (b) Find an expression for $\sin\delta_\ell(k)$ to *first order* in V. This is the "first Born approximation", $\delta_\ell^{\text{1BA}}(k)$.

   (c) Is $|\sin\delta_\ell^{\text{1BA}}(k)|$ less than unity? If not, why?

2. **Scattering from a atom or a nucleus in the first Born approximation**

   When low-energy electrons are scattered off atoms, the motion is non-relativistic and the atomic recoil can be neglected. If the atom is left in its ground state, where it began, the scattering is called *elastic*. Neglect spin. The cross section for scattering from incoming momentum $\vec{k}$ to outgoing momentum $\vec{k}'$ can equally well be regarded as function of the angles θ and ϕ defined relative to the $\hat{k}$ axis.

   (a) What is the relation between the momentum transfer $q \equiv |\vec{q}| = |\vec{k} - \vec{k}'|$ and the angles θ and ϕ?

(b) Show that *in the first Born approximation*, the differential cross section is given by

$$\frac{d\sigma}{d\phi d\cos\theta} = \left(\frac{2}{a_0 q^2}\right)^2 (Z - F(\vec{q}))^2$$

where $a_0 = \hbar^2/me^2$. $F(\vec{q})$ is known as the *form factor* and contains all the information, measurable in elastic scattering, about the atomic electron charge distribution, $\rho(\vec{x})$.

$$F(\vec{q}) = \int d^3x e^{i\vec{q}\cdot\vec{x}} \rho(\vec{x})$$

(c) Assuming that the charge density is spherically symmetric, $\rho(\vec{x}) = \rho(r)$, note $F(0) = Z$, and show that

$$\left.\frac{dF}{dq^2}\right|_{q=0} = -\frac{1}{6}\langle r^2\rangle,$$

where $\langle r^2\rangle$ is the measn square charge radius of the atom. What is the *forward* ($\theta \to 0$) limit of the differential cross section?

(d) For hydrogen $\rho(r) = \frac{1}{\pi a_0^3} e^{-2r/a_0}$. Find $F(q)$ and $\langle r^2\rangle$.

Now consider scattering from a nucleus in the same approximation: non-relativistic, no spin, no recoil.

(e) Nuclei have rather sharp surfaces. Model a nucleus with charge Z and mass number A as a uniform spherical charge distribution $\rho(r, Z, A) = \rho_0(Z, A)$ for $r \le R(A)$ and $\rho = 0$ for $r > R(A)$. A good estimate for $R(A)$ is $R(A) = R_0 A^{1/3}$, with $R_0 = 1.2$fm. Compute the cross section for scattering in the first Born approximation as a function of Z and A.

(f) Plot the results of the previous section as a function of θ at fixed electron energy. Choose a few values of electron energy that show the characteristic structure of the cross section.

(g) Real nuclei do not have sharp surfaces. Can you describe *qualitatively* what happens to the cross section you computed in the previous part if the surface is smeared out over $\delta R \ll R(A)$?

3. **Born Approximation for a δ-shell**

Consider scattering from a δ-shell potential,

$$2mV(r) = -\lambda\delta(r - a).$$

(a) Find the first Born approximation to the scattering amplitude, $f(k, \theta)$.

(b) Sketch $f(k, \theta)$ as a function of θ at fixed k.

(c) Extract from $f(k, \theta)$ the first Born approximation to the s-wave phase shift, $\delta_0(k)$.

(d) Compute the s-wave phase shift directly by matching interior and exterior solutions. How good is the Born approximation?

4. **Spin dependent scattering**

A particle with spin-1/2 (mass m) scatters from a spin dependent potential. Suppose the spin "up" state is subject to a potential $V(\vec{x})$, while the spin "down" state is subject to no potential. In addition, suppose there is a "spin flip" potential $W(\vec{x})$ which connects spin up and down, so

$$V = \begin{pmatrix} V(\vec{x}) & W(\vec{x}) \\ W(\vec{x}) & 0 \end{pmatrix}$$

in matrix notation. Assume V to be real, and assume both $V(\vec{x})$ and $W(\vec{x})$ to be short range.

(a) Write down the Schrödinger equation (in terms of a two-component spinor wave-function) for motion in this potential.

(b) Introduce a spin index into the notation for scattering states:

$$\begin{aligned} |\phi_{\vec{k}}\rangle &\longrightarrow |\phi_{\vec{k},s}\rangle \qquad s = \uparrow \text{ or } \downarrow \\ |\psi^{(+)}_{\vec{k}}\rangle &\longrightarrow |\psi^{(+)}_{\vec{k},s}\rangle \end{aligned}$$

and write the Lippmann–Schwinger integral equation for $|\psi^{(+)}_{\vec{k},s}\rangle$

(c) Calculate the scattering amplitude

$$f_{s,s'}\left(\vec{k},\vec{k}'\right) = -\frac{2m}{4\pi}\langle \vec{k}'\, s'|V|\psi^{(+)}_{\vec{k},s}\rangle$$

to *lowest non-trivial order* in the Born expansion for all four values of (s, s'). Express your answer in terms of the Fourier transforms of $V(\vec{x})$ and $W(\vec{x})$. Note that the expansion for $f_{\downarrow\downarrow}$ begins with a term quadratic in V. Simplify these results in the case that V and W depend only on $r \equiv |\vec{x}|$.