Massachusetts Institute of Technology
Physics Department

Physics 8.322 — **Spring 2007**
Quantum Theory II — **February 12, 2007**

Assignment 2

DUE FEBRUARY 23, 2007, AT THE END OF THE DAY

Announcements

- Problem Sets can be downloaded from http://web.mit.edu/8.322/

Reading topics for this period

- Relativistic quantum mechanics and the Dirac equation
- Perturbation theory

Reading Recommendations 2

- Relativistic quantum mechanics: Bjorken and Drell *Relativistic Quantum Mechanics* §1, and §2 scanned material posted on the 8.322 website.
- The MIT Bag Model: Phys. Rev. article posted on the 8.322 website.
- Perturbation theory: Sakurai, §5.1, 5.2; Gottfried and Yan, §3.7 a and b; Shankar, §17.

Problem Set 2

Topics covered in the problems

- The Dirac Equation and applications.

Important background for the problems

There is some background information that will make these problems considerably easier. The topics should have been covered in prerequisities for 8.322, but in case not, here is a lightning quick summary. Please, if you're not familiar with this material, derive it for yourself

- **Properties of the Levi-Civita Symbol**

  The three-index Levi-Civita symbol, ϵ_{ijk}, where i, j, k range over 1,2, and 3, is defined by $\epsilon_{123} = 1$ and ϵ changes sign whenever any two indices are interchanged. The aim of this problem is to familiarize yourself with properties of ϵ.

  – Show that:

$$\begin{aligned} \epsilon_{ijk}\epsilon_{ilm} &= \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl} \\ \epsilon_{ijk}\epsilon_{ijm} &= 2\delta_{km} \\ \epsilon_{ijk}\epsilon_{ijk} &= 6 \\ \epsilon_{ijk}\epsilon_{lmn} &= \begin{vmatrix} \delta_{il} & \delta_{im} & \delta_{in} \\ \delta_{jl} & \delta_{jm} & \delta_{jn} \\ \delta_{kl} & \delta_{km} & \delta_{kn} \end{vmatrix} \end{aligned} \tag{1}$$

  In all these formulas any repeated index is summed over values 1,2, and 3.

  – Use the result above to prove the following formula for the determinant of a 3×3 matrix,

$$\epsilon_{ijk} \det M = \epsilon_{lmn} M_{il} M_{jm} M_{kn}$$

- **Some results from the tensor formulation of special relativity**

  In the tensor formulation of special relativity with metric,

$$g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

  with $v_\mu = g_{\mu\nu} v^\nu$ (common indices summed 0,1,2,3), *etc....*

  If $x^\mu = (ct, \vec{x})$, then $x_\mu = (ct, -\vec{x})$ and $\partial_\mu = (\frac{1}{c}\frac{\partial}{\partial t}, +\vec{\nabla})$ and $\partial^\mu = (\frac{1}{c}\frac{\partial}{\partial t}, -\vec{\nabla})$. The "contraction" of a contravariant and a covariant vector, $v^\mu w_\mu = v^0 w^0 - \vec{v} \cdot \vec{w}$, is Lorentz invariant. For example, the 4-divergence, $\partial_\mu v^\mu = \frac{1}{c}\dot{v} + \vec{\nabla} \cdot \vec{v}$ is Lorentz invariant. Note the essential point: the *covariant* form of the derivative has all plus signs. See Jackson's book on *Electrodynamics* chapter on special relativity for a discussion of covariant (lower) and contravariant (upper) indices.

  The electromagnetic field strength tensor is defined by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

Using the definition of the vector potential, $A^\mu = (\Phi, \vec{A})$, and the standard definitions, $\vec{B} = \vec{\nabla} \times \vec{A}$ and $\vec{E} = -\vec{\nabla}\Phi - \frac{1}{c}\frac{\partial}{\partial t}\vec{A}$, then

$$F_{\mu\nu} = \begin{pmatrix} 0 & E^1 & E^2 & E^3 \\ -E^1 & 0 & -B^3 & B^2 \\ -E^2 & B^3 & 0 & -B^1 \\ -E^3 & -B^2 & B^1 & 0 \end{pmatrix} \qquad F^{\mu\nu} = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{pmatrix}$$

where the physical 3-vector electric (magnetic) field, $\vec{E}$ ($\vec{B}$), is identified with the contravariant indices, $\vec{E} = (E^1, E^2, E^3)$ ($\vec{B} = (B^1, B^2, B^3)$).

Problems

1. **The Dirac Equation with a Central Potential**

   Consider the Dirac Equation with a central potential that is Lorentz invariant (therefore it enters the Dirac equation the same way as the mass)

   $$(\vec{\alpha} \cdot \vec{p} + \beta(m + V(r)))\psi = E\psi \tag{2}$$

   where $\hbar = c = 1$ and $\vec{p} = -i\vec{\nabla}$. From the form of this equation we can identify the Dirac *Hamiltonian* as $H_D \equiv \vec{\alpha} \cdot \vec{p} + \beta(m + V(r))$.

   (a) Show that the parity operator, $\Pi_D \equiv \beta\Pi$, commutes with H_D, where Π is the operator that reflects $\vec{r}$ to $-\vec{r}$:

   $$\Pi\, \vec{r}\, \Pi = -\vec{r}.$$

   (b) Show that if $\psi_E(\vec{r})$ is a solution with positive energy, E, then $\gamma_5 \Pi \psi_E(\vec{r})$ is a solution with negative energy, $-E$. Thus the spectrum is symmetric in energy. Show that this is not true if the potential is proportional to the time component of a four vector, $\Phi(r)$, in which case the Dirac equation reads,

   $$(\vec{\alpha} \cdot \vec{p} + \beta m + \Phi(r))\psi = E\psi \tag{3}$$

   [Note that this is the situation for an electrostatic potential as in the Dirac version of the hydrogen atom. When the negative energy solutions are re-interpreted as antiparticle solutions, this situation makes sense: The positive and negative energy solutions correspond to opposite charges and experience opposite electrostatic forces.]

   (c) Show that the orbital angular momentum $\vec{L} = \vec{r} \times \vec{p}$ *does not* commute with H_D even for $V(r) = 0$, but that the *total angular momentum*,

   $$\vec{J} = \vec{r} \times \vec{p} + \frac{1}{2}\vec{\sigma}$$

   does commute with H_D, where $\vec{\sigma}$ is the vector of Pauli matrices, $\{\sigma_1, \sigma_2, \sigma_3\}$.

(d) So the energy eigenstates of the Dirac equation with a central potential can be labeled by $\{j, m_j, \pi\}$, the eigenvalues of $\vec{J}^2$ $(j(j+1))$, J_3 (m_j), and Π_D (π). The labels of a *non-relativistic* spin-1/2 particle in a central potential are usually taken to be either $\{\ell, m_l, m_s\}$ or $\{\ell, j, m_j\}$. As we go from the non-relativistic to the relativistic case, we don't expect states to suddenly disappear, so there should be a one-to-one correspondence between the two sets of states. What is it?

2. **Relativistic Corrections in a Central Potential.**

Once again consider a Dirac particle in a central potential,

$$(\vec{\alpha} \cdot \vec{p} + \beta(m + V(r)))\psi = E\psi \tag{4}$$

It is convenient to use Bjorken and Drell's choice of Dirac matrices. Write ψ as

$$\psi = \begin{pmatrix} \varphi \\ \chi \end{pmatrix}$$

where φ and χ are two component spinors. When the potential is weak ($|V(r)| \ll m$), the energy, E, is close to m. In this case in lecture I showed that φ obeys the Schrödinger equation. Here I want to go a little beyond this...

(a) Show that to lowest order in v/c,

$$\chi = \frac{\vec{\sigma} \cdot \vec{p}}{2m}\varphi$$

provided the potential satisfies $|V(r)| \ll m$.

(b) Compute the matrix element of the magnetic moment operator,

$$\vec{\mu} = \frac{1}{2}\vec{r} \times \vec{j} = \frac{e}{2}\vec{r} \times \vec{\alpha}$$

and show that it has the expected form,

$$\langle \vec{\mu} \rangle \equiv \int d^3r \psi^\dagger(\vec{r})\vec{\mu}\psi(\vec{r}) = \frac{e(\hbar)}{2m(c)} \int d^3r \varphi^\dagger(\vec{r}) \left(\vec{L} + \vec{\sigma}\right) \varphi(\vec{r})$$

The factors of $\hbar$, and c, in parentheses are there to remind you that they were set to unity before. Note that the magnetic moment operator couples upper and lower components of the Dirac wavefunction and is therefore an $\mathcal{O}(v/c)$ effect.

(c) One of the operators that mediates β-decay is the "axial charge", defined by

$$\vec{A} = \frac{1}{2}\gamma^0\vec{\gamma}\gamma_5$$

[This type of β decay is known as Gamow-Teller decay.] Show that in the non-relativistic limit, the axial charge is just equal to the spin. More specifically, let $\varphi(\vec{r}) = f(\vec{r})u$, where u is a two component constant spinor. Then

$$\int d^3r\, \psi^\dagger(\vec{r})\vec{A}\psi(\vec{r}) = \frac{1}{2}u^\dagger \sigma u$$

Next compute the first, non-trivial relativistic correction to the axial charge in the case where the space wavefunction is a function only of $r = |\vec{r}|$. Note that it is $\mathcal{O}(v^2/c^2)$.

3. **Anomalous Magnetic Moment**

Consider the Dirac equation for a particle in an external electromagnetic field with a non-minimal coupling of the form,

$$\left(i\gamma^\mu \partial_\mu - e\gamma^\mu A_\mu - \frac{\kappa e}{4m}\sigma^{\mu\nu}F_{\mu\nu} - m\right)\psi = 0$$

where $\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field strength tensor. Note that the "non-minimal" term proportional to κ is gauge invariant.

(a) Develop an interpretation of the new term by studying the non-relativistic limit of this equation when $F_{\mu\nu}$ corresponds to a uniform, time-independent magnetic field. Specifically, can this term be absorbed into a redefinition of one of the properties of the particle that has already been defined?

(b) What is the form that this new term contributes to the Dirac Hamiltonian?

(c) Under the transformation $\vec{x} \to -\vec{x}$ the electric and magnetic fields transform according to: $\vec{E}(\vec{x}) \to -\vec{E}(-\vec{x})$ and $\vec{B}(\vec{x}) \to +\vec{B}(-\vec{x})$. [$\vec{E}$ is a "normal vector" and $\vec{B}$ is a "pseudo- (or axial-)vector".] Prove that the new term in the Dirac Hamiltonian is invariant under the parity transformation defined in Problem 1 when $\vec{E}$ and $\vec{B}$ are functions only of r ($r \equiv |\vec{x}|$).

(d) Suppose that the new term was multiplied by γ_5,

$$\frac{\kappa e}{4m}\sigma^{\mu\nu}F_{\mu\nu} \to \frac{\tilde{\kappa} e}{4m}\sigma^{\mu\nu}\gamma_5 F_{\mu\nu}$$

Repeat parts (a) and (b) and show that this term is not invariant under parity.

4. **MIT Bag Model**

A very simple model of a proton is three massless quarks confined to a potential hole $r < R$. [The up and down quarks that make up a proton are so light that their masses can be ignored compared to the energy associated with their confinement in a small domain.]

The first order of business is to find a boundary condition at $r = R$ that will keep the quarks inside. Because the Dirac equation is *first order* in derivatives, the boundary condition $\psi(R) = 0$ has only the trivial solution $\psi(\vec{r}) = 0$ for all $\vec{r}$. A better choice is

$$\text{NOTE THE SIGN} \quad -i\hat{r}\cdot\vec{\gamma}\psi = \psi|_{r=R}$$

So the simplest model of a massless quark confined to a proton is

$$\begin{aligned} -i\vec{\alpha}\cdot\vec{\nabla}\psi &= E\psi \quad \text{for} \quad r < R && (5) \\ \text{NOTE THE SIGN} \quad -i\hat{r}\cdot\vec{\gamma}\psi &= \psi \quad \text{at} \quad r = R && (6) \end{aligned}$$

(a) Show that the current $j_\mu = \bar{\psi}\gamma_\mu\psi$ is conserved. This requires both $\partial_\mu j^\mu = 0$ for $r < R$ and $\hat{r}\cdot\vec{j} = 0$ at $r = R$. Therefore the charge $Q = \int_{r\leq R} d^3r\psi^\dagger\psi$ is time independent.

(b) Show that the Hamiltonian in eq. (5), $H = -i\vec{\alpha}\cdot\vec{\nabla}$, is Hermitian on the space of states obeying the bag boundary condition, eq. (6).

(c) Find the Dirac wavefunction and energy eigenvalue of the lowest positive energy eigenstate in a bag. [Hint: The upper component (using $Bj\&D$ γ-matrices) of the wavefunction of the lowest energy eigenstate is spherically symmetric.]

The other ingredient in the bag model is a uniform "vacuum pressure", B, that resists the growth of the bag. It contributes an energy $BV = 4\pi BR^3/3$ to the energy of a proton. The total energy of the proton is therefore

$$E(R) = 3\times\frac{2.0428...}{R} + \frac{4\pi BR^3}{3}$$

where the number 2.0428... appeared in the solution to part (c). Note that $\hbar$ and c have been set equal to unity in this expression. The radius is determined dynamically to minimize the energy. At this radius the zero point pressure of the quarks just exactly balances the confining pressure of the vacuum.

(d) Find R_0, the equilibrium radius of a bag in terms of B. Then choose B so that the energy of the proton is 939 MeV (the proton's rest mass). Restore more familiar units and express the pressure in atmospheres.

5. **Helicity and Chirality**

The eigenstates of the Dirac matrix γ_5 are called eigenstates of "chirality",

$$\gamma_5 u_R = u_R \quad \text{and} \quad \gamma_5 u_L = -u_L$$

where the subscripts refer to "right" and "left" chirality. Note that parity changes $u_{R,L}$ into $u_{L,R}$, which at least in part justifies the names.

(a) Consider the Dirac Hamiltonian in momentum space. Show that the "helicity", defined by $h \equiv \vec{\sigma} \cdot \hat{p}$ commutes with H, so eigenstates of momentum can be further labeled by their helicity eigenvalues. What are the eigenvalues of h?

(b) Write the helicity eigenstates with momentum $\vec{p}$ and energy E as a linear superposition of eigenstates of chirality. Explore what happens to these superpositions in the limits $E \to m$ and $E \to \infty$.

Conclusions: Helicity is a very useful label for relativistic particles with spin. Chirality becomes the label of choice when the particles' masses are negligible compared to their energies.

6. **Duffin-Kemmer Formulation of Maxwell's Equations**

Maxwell's equations for $\vec{E}$ and $\vec{B}$ in empty space are

- Lorentz covariant
- First order in space and time derivatives

So it is probably not surprising that they can be recast in a form very similar to the Dirac equation, *i.e.* as a Hamiltonian wave mechanics describing the time evolution of a wavefunction for a massless, relativistic "particle" with spin one. This formulation was first worked out by R. J. Duffin (Phys. Rev. **54**, 1114 (1938)) and N. Kemmer (Proc. Roy. Soc. London **A173**, 91 (1939)) and is therefore known as the "Duffin-Kemmer" formulation of electrodynamics.

Start with Maxwell's equations in empty space:

$$\begin{aligned} \frac{1}{c}\frac{\partial \vec{B}}{\partial t} &= -\vec{\nabla} \times \vec{E} \\ \frac{1}{c}\frac{\partial \vec{E}}{\partial t} &= \vec{\nabla} \times \vec{B} \end{aligned} \tag{7}$$

Note that it is not necessary to view $\vec{\nabla} \cdot \vec{E} = \vec{\nabla} \cdot \vec{B} = 0$ as independent equations because they follow from eqs. (7) plus the initial condition that the divergences vanish at some initial time.

Define a pair of complex wavefunctions,

$$\vec{v}_{\pm}(\vec{x}, t) \equiv \vec{E} \pm i\vec{B}$$

(a) Show that Maxwell's equations, eqs. (7), can be written in the form

$$i\hbar\frac{\partial}{\partial t}\vec{v}_\pm = \pm c\left(\vec{\Sigma}\cdot\vec{p}\right)\vec{v}_\pm \tag{8}$$

where $\vec{p} = -i\hbar\vec{\nabla}$ as usual, and $\vec{\Sigma}$ are the three dimensional matrix representation of the generators of rotations, $[\Sigma^j]_{ik} = i\epsilon_{ijk}$, or, for example,

$$\Sigma^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

and similarly for Σ^2 and Σ^3.

Note that eqs. (8) is *exactly* analagous to the Weyl equation(s) for two component massless spinors. Note that in both cases the appearance of quantum mechanics is a cheat: the $\hbar$ cancels on both sides of the equation.

(b) Eq. (8) must be Lorentz invariant because they came from Maxwell's equations which were. However they are not separately invariant under parity. Show that the parity operation, $\vec{x} \to -\vec{x}$ transforms the equation for $\vec{v}_+$ into the equation for $\vec{v}_-$.

The two Weyl equations (with $\pm$ signs) can be combined into a single Dirac equation that conserves parity. The same can be done in this case.

(c) Show that eqs. (8) can be combined in the form

$$i\hbar\frac{\partial}{\partial t}f = -i\hbar c\, \vec{\boldsymbol{\Sigma}}\cdot\vec{\nabla}f \tag{9}$$

where f is a six component wavefunction and $\vec{\boldsymbol{\Sigma}}$ is a set of three 6×6 matrices. What is f in terms of $\vec{v}_\pm$? What is $\vec{\boldsymbol{\Sigma}}$ in terms of $\vec{\Sigma}$?

(d) In lecture we showed that the "square" of Dirac's equation yielded the Klein Gordon equation. Show that the square of the Duffin-Kemmer equation yields the wave equation *for the transverse electric and magnetic fields*, $\vec{E}_t$ and $\vec{B}_t$, which satisfy $\vec{\nabla}\cdot\vec{E}_t = \vec{\nabla}\cdot\vec{B}_t = 0$. Show that eqs. (8) require $\partial(\vec{\nabla}\cdot\vec{E})/\partial t = 0$ and $\partial(\vec{\nabla}\cdot\vec{B})/\partial t = 0$. All these results are expected from Maxwell's equations.

(e) Seek plane wave solutions to eqs. (8) of the form

$$\vec{v}_\pm = \vec{\varepsilon}_\pm(p)\exp(iEt/\hbar - i\vec{p}\cdot\vec{x}/\hbar)$$

Show that these solutions correspond to left and right circularly polarized plane waves.