# Massachusetts Institute of Technology
## Physics Department

**Physics 8.322** **Spring 2007**
**Quantum Theory II** **February 19, 2007**

***Assignment 3***

DUE MARCH 2, 2007, AT THE END OF THE DAY

### Announcements

- Don't forget that there is no lecture on February 19 (President's Day), but that there will be a lecture on Tuesday, February 20 at the normal time and place.
- Problem Sets can be downloaded from http://web.mit.edu/8.322/

### Reading topics for this period

- Perturbation theory and applications to hydrogenic atoms. We will treat the fine structure of hydrogen and touch on the Zeeman effect in lecture, but only mention the Stark effect and hyperfine structure in passing.
- The variational principle. We will do much more on the variational principle than is usually included in texts. Take careful notes.

### Reading Recommendations 3

- Sakurai: Perturbation theory and applications to hydrogen: §5.1 – 5.3; Variational Principle: §5.4
- Gottfried and Yan, Perturbation theory: §3.7 a and b; Applications to fine structure and the Zeeman effect can be found in §5.3 and 5.4 (but Gottfried and Yan do much more on hyperfine structure, and the Stark effect than we will cover). Variational Principle: §3.7 e.

## Problem Set 3

## Topics covered in the problems

- Perturbation theory, both non-degenerate and degenerate. Hydrogen fine structure.

## Problems

1. **Anharmonic Oscillator**

   A particle of mass $m$ moves in one dimension under the influence of an harmonic oscillator potential of frequency $\omega$,

   $$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2$$

   The eigenstates are $|n\rangle = \frac{1}{\sqrt{n!}}[a^\dagger]^n|0\rangle$ with energies $(n + 1/2)\hbar\omega$.

   Some generic background introduces an additional smooth potential, $V(x)$ in the vicinity of the oscillator. $V(x)$ can be expanded in a Taylor series,

   $$V(x) = \sum_{k=1}^{\infty} \frac{1}{k!} v_k x^k$$

   where $v_k = \frac{d^k V(x)}{dx^k}|_{x=0}$. Compute the effect of this potential on the eigen *energies* of the oscillator with the following accuracy:

   (a) Compute the energy shift of all states *exactly* due to the terms through order $x^2$.

   (b) Compute the energy shift of the ground state to first order in $v_3$ and $v_4$.

   (c) Compute the energy shift of the ground state to second order in $v_3$.

2. **Quadrupole perturbation of the hydrogen $p$-states** [1]

   An electron in the $Np$-level of hydrogen is characterized by its principle quantum number, $N$, and azimuthal quantum number $m$. Remember in Bohr's notation $N = 2$ corresponds to the lowest $p$-state. So its state vector can be written $|N, \ell = 1, m\rangle$, where $m = -1, 0, +1$. The electron is subject to an external potential

   $$V(\vec{x}) = \lambda xy$$

   where $\lambda$ is a constant.

[1] Adapted from Sakurai, problem 5.9.

(a) Write $V(\vec{x})$ as a sum of spherical tensor operators, $T_q^{(k)}$. [For notation or review, see Sakurai §3.10.].

(b) Obtain the "correct" zeroth order energy eigenstates that diagonalize the perturbation. You can (and should) use the Wigner-Eckart Theorem to simplify this calculation. You may assume that the other angular momentum states, $d, f, g, \ldots$, have been split from the $p$-state by some other perturbation. Note, also, that you do not have to compute the matrix elements of the perturbation explicitly to solve this piece of the problem.

(c) What is the expectation value of the perturbation in these states for the case of $N = 2$ ($N = 2$ is the *lowest* energy $p$-state)?

3. **Form of perturbation expansions**

Consider a system with an unperturbed Hamiltonian, $H_0$, and a discrete unperturbed spectrum: $\{|n^0\rangle, n = 1, 2, \ldots\}$ with unperturbed energies $\{E_n^0, n = 1, 2, \ldots\}$. A perturbation, $V$, is introduced. Its matrix elements are denoted,

$$\langle n^0|V|m^0\rangle \equiv V_{nm}$$

(a) Assuming no degeneracy, write the expression for the shift in the energy of the $n^{\text{th}}$ eigenstate to *third* order in $V$ in Rayleigh-Schrödinger perturbation theory.

(b) Assuming no degeneracy, write the expression for the shift in the wavefunction of the $n^{\text{th}}$ eigenstate to *second* order in $V$ in Rayleigh-Schrödinger perturbation theory.

4. **Removing the degeneracy beyond leading order**

This problem is "extra credit". If you solve it, you will add to your grade an amount equal to 1/8 of the total problem set grade.

For the situation (and notation) described in Problem 3, suppose that the zeroth order spectrum contains degeneracies. Specifically, suppose $\mathcal{D}$ is a subspace of states in which the original Hamiltonian is degenerate:

$$H_0|n^0\rangle = \bar{E}^0|n^0\rangle \quad \text{for} \quad |n\rangle \in \mathcal{D}$$

All good students know that it is necessary to remove the degeneracy before attempting perturbation theory, so they diagonalize the perturbation within the subspace. However, this fails to entirely remove the degeneracy. To be specific, the states within a subspace $\mathcal{D}_1 \subset \mathcal{D}$ remain degenerate:

$$(H_0 + V)|n^0\rangle = \bar{E}^1|n^0\rangle \quad \text{for} \quad |n\rangle \in \mathcal{D}_1$$

(a) Show that the attempt to compute the perturbative energy shift of one of the states in $\mathcal{D}_1$ breaks down in *fourth* order of R-S perturbation theory.

(b) Show that this problem can be cured by choosing a basis for $\mathcal{D}_1$ that diagonalizes the operator $V^2$ in $\mathcal{D}_1$, where $V^2$ is defined by

$$V^2_{np} \equiv \langle n^0 | V^2 | p^0 \rangle = \sum_{k \notin \mathcal{D}_1} \frac{V_{nk} V_{kp}}{\bar{E}^1 - E^0_k}$$

See Sakurai, §5, Problem 12 for an example of such a Hamiltonian.

5. **Bounding an exotic particle**

Every once in a while, theorists invent a particle to solve some problem, only to find that precision measurements of atomic energy levels rule it out. Some years ago some theorists speculated the existence of a spinless particle, the $X$, which was supposed to have a mass of

$$m_X = 3.4\,\mathrm{MeV}/c^2 \approx 6.6 m_e$$

and was supposed to give rise to a potential between two charged particles of the form,

$$V_X(r) = \frac{m_1 e_1}{M} \frac{m_2 e_2}{M} \frac{e^{-r/\lambda}}{r}$$

where $m_i$ and $e_i$ is the mass and (electric) charge of the particle $i$. $\lambda$ is the Compton wavelength of the $X$, $\lambda = \hbar / m_X c$, and $M$ is some unknown, but very large mass. $M$ is expected to exceed 100 GeV ($\approx 2 \times 10^5\ m_e$).

(a) The measured $2p_{1/2} - 2s_{1/2}$ splitting in hydrogen agrees very well with the best theoretical calculation including all known effects.

- $\Delta E_{\mathrm{EXPT}} = 1057.862 \pm 0.020$ Megacycles
- $\Delta E_{\mathrm{TH}} = 1057.864 \pm 0.014$ Megacycles

What is the smallest value of $M$ allowed by the observed agreement between theory and experiment?

(b) The muon is an unstable particle with the charge and interactions of an electron but with a mass 207 times greater ($m_\mu \approx 207 m_e$). Muons form hydrogen-like atoms with protons and other nuclei, called *muonic atoms*. Muonic hydrogen has not been studied very well experimentally, but the muonic helium ion — a single muon bound to a helium nucleus — has. The $2p_{1/2} - 2s_{1/2}$ splitting in the muonic helium ion has been measured to be $\Delta E_{\mathrm{EXPT}} = 1.37 \pm 0.01$ eV. This result agrees with theory to about 1% (the major uncertainty comes from the uncertainty in the radius of the helium nucleus). What is the smallest value of $M$ allowed by agreement between theory and experiment in this case?

6. **The Stark effect in hydrogen**

   Consider an electron in hydrogen subject to a constant, external electric field,

$$H = \frac{\vec{p}^2}{2m} - \frac{e^2}{r} + \mathcal{E}ez$$

   where, of course, $z = r\cos\theta$.

   (a) What is the first order correction to the ground state energy?

   (b) Obtain a *lower* estimate for the second order correction to the ground state energy by the following method,

$$E^{(2)} = [\mathcal{E}e]^2 \sum_{n\neq 0} \frac{|\langle n|z|0\rangle|^2}{E_0 - E_n} \geq [\mathcal{E}e]^2 \sum_{n\neq 0} \frac{|\langle n|z|0\rangle|^2}{E_0 - E_1}$$

   Note that the sum over a complete set of states, $\{n\}$ includes both bound states and continuum states.

   (c) Evaluate the exact sum, $\sum_{n\neq 0} \frac{|\langle n|z|0\rangle|^2}{E_0 - E_n}$ by the following trick: Suppose it is possible to find an operator $F$ which satisfies

$$z|0\rangle = [FH_0 - H_0F]|0\rangle$$

   where $H_0$ is the unperturbed Hamiltonian. The we have

$$\langle n|z|0\rangle = (E_0 - E_n)\langle n|F|0\rangle$$

   so the required sum may be evaluated by completeness. Show that for the Coulomb problem

$$F = -\frac{ma}{\hbar^2}\left(\frac{r}{2} + a\right) z$$

   where $a = \hbar^2/me^2$. Compute $E^{(2)}$ exactly and compare with your lower bound from the previous section.

7. **Proton-Antiproton Atoms**

   A proton and an antiproton can form an ordinary atom, bound by Coulomb forces, analogous to the hydrogen atom. However the proton and antiproton can annihilate (mostly to mesons like pions, kaons, *etc.*) if they are within the range of the strong QCD interactions, $R$.

   $p\bar{p}$ atoms can be formed by stopping antiprotons in a hydrogen target. The antiproton is captured by one of the hydrogen atoms. Typically it is captured in a highly excited quasi-classical state and radiates away its energy as it cascades down into the ground state. Experiments are performed with capture targets that are either a diffuse hydrogen gas or liquid hydrogen. The reason for these alternative approaches is explained in this problem.

(a) What is the root-mean-square radius of the ground state ($1s$) of the $p\bar{p}$ atom? Don't forget to use the *reduced* mass for this bound state of equal mass particles.

(b) Assuming that the annihilation amplitude is independent of the $p\bar{p}$ angular momentum, calculate the ratio of annihilation rates from the $2p$ ($\ell = 1$) states to the $1s$ state ($\ell = 0$). Take $R = 2 \times 10^{-13}$ cm to get a numerical value for the ratio. [Answer: $\approx 7.7 \times 10^{-6}$.]

**Now comes the interesting part of the problem.**

When proton and antiproton annihilate, they produce complicated arrays of mesons, like 3, 4, or 5 pions, or kaons and pions, even $\eta$-mesons that are less well known. To help sort out these complex final states, experimenters would like very much to know the total angular momentum ($J$) and parity ($\Pi$) of the meson state. Because the strong interactions conserve angular momentum and parity, it suffices to know the quantum numbers of the $\bar{p}p$ just before annihilation.

(c) Assume that the $p\bar{p}$ annihilate from the $s$-wave or the $p$-wave. What is the $J^{\Pi}$ of the different possible states? [The "intrinsic parity" of a particle and antiparticle are opposite: $\Pi(p) \cdot \Pi(\bar{p}) = -1$.]

From this result, it's clear that there would be significant advantage in being able to know whether the system annihilates from the $s$-wave or the $p$-wave.

Naively, from the result of part (b), one would think that the annihilation occurs overwhelmingly from the $s$-wave. However in reality $p\bar{p}$ atoms in isolation frequently annihilate from the $2p$ orbital because *the annihilation rate from that level is a comparable to the* $2p \rightarrow 1s$ *transition rate.* [It is measured as 55% at NTP.[2]].

(d) One approach is to capture the antiprotons in dilute hydrogen gas. If the gas is dilute enough the $\bar{p}p$ is effectively isolated. Experimentalists can identify annihilation from the ground state by triggering on annihilations which occur coincident with an $X$-ray with energy equal to the $2p - 1s$ energy difference. What is the energy of that $X$-ray in KeV?

(e) Another approach makes use of Stark mixing *in liquid hydrogen.* Observations show that $p\bar{p}$-annihilation in liquid hydrogen occurs more than $\approx 99\%$ of the time from the $s$-wave. The small $p\bar{p}$ atom in liquid hydrogen is perturbed by the strong electric fields generated by the atoms of ordinary hydrogen. The consequent Stark effect mixes an $s$-wave component into the higher excited states of the $p\bar{p}$ atom, allowing the annihilation to occur very rapidly.

   Suppose a $p\bar{p}$ atom finds itself in the $2p$-state. It experiences a Stark mixing with the $1s$-ground state. Given the ratio of annihilation rates (from part b) estimate the strength of the electric field given the fact that the annihilation occurs from the $p$ state $\approx 1\%$ of the time. [You should average over the directions of the electric field.]

[2] Details may be found in the article by H. Poth in *Proceedings Conference: First Workshop on Antimatter Physics at Low Energy, 1st, Batavia, Ill., Apr 10-12, 1986*, (Fermilab, 1986), page 486. (Shelved in the Physics Reading Room.)