Massachusetts Institute of Technology
Physics Department

Physics 8.322 — **Spring 2007**
Quantum Theory II — **April 2, 2007**

Assignment 7

Due April 13, 2007, at the end of the day

Announcements

- None

Reading topics for this period

Many particle quantum mechanics.

- Identical and indistinguishable particles, bosons and fermions.
- Second Quantization
- Degenerate Fermi Gas
- Thomas Fermi Approximation

Reading Recommendations 7

- On identical indistinguishable particles: Sakurai §6.
- On second quantization: Gottfried and Yan, §11.2.
- Degenerate fermi gas: Notes *Degenerate Fermion Systems* posted on line §1.1 — 1.3.
- Thomas Fermi approximation: Notes *Degenerate Fermion Systems* posted on line §4, and Bethe and Jackiw, *Intermediate Quantum Mechanics*, §5.

Problem Set 7

Topics covered in the problems

- Isospin and nuclear forces
- Spectroscopic terms and Fermi-Dirac statistics.
- Enumerating fermion states in the context of the nuclear shell model
- The Bethe Ansatz

Problems

1. **States of two hadrons with isospin**

Isospin is an "internal" $SU(2)$ symmetry label carried by quarks and the hadrons they compose. The word "internal" here means that the Hilbert space of isospin states has nothing to do with the space-time configuration space of the particle. The quantum mechanics of isospin is *exactly analogous to spin* except, of course, it is not related to angular momentum: When a spin-1/2 particle is rotated, its spin state changes, but its isospin state does not. At the fundamental level it arises from the unitary transformations acting on the doublet formed by the up, u, and down, d, quarks:

$$\begin{pmatrix} u \\ d \end{pmatrix} \rightarrow \begin{pmatrix} u' \\ d' \end{pmatrix} = \mathbf{U} \begin{pmatrix} u \\ d \end{pmatrix}$$

where $\mathbf{U}$ is a 2×2 matrix with $\mathbf{U}^\dagger \mathbf{U} = \mathbf{I}$ and $\det \mathbf{U} = 1$. The u and d quarks are chosen to be the eigenstates of I_3: $I_3|u\rangle = \frac{1}{2}|u\rangle$ and $I_3|d\rangle = -\frac{1}{2}|d\rangle$. If the u and d quarks had the same mass, isospin would be an exact symmetry of the strong interactions. Since $|m_d - m_u| \sim 4$ MeV and the typical mass scale of the strong interactions is ~ 200 MeV, isospin violation is usually at the $< 2\%$ level. Electromagnetic effects also violate isospin because the u and d quarks have different charges, but they are $\mathcal{O}(\alpha)$, where $\alpha = e^2/\hbar c = 1/137.036...$, which is also small.

So isospin is an approximate symmetry of the hadrons that are made of u and d quarks. To a good approximation $[I_j, H_{\rm QCD}] = 0$, for j=1,2,3, where $\vec{I}$ are the generators of isospin transformations, and $H_{\rm QCD}$ is the hamiltonian that describes the interactions of quarks (governed by Quantum Chromo Dynamics). Therefore hadrons fall into multiplets that are eigenstates of $\vec{I}^2$. Because states of different I_3 have different electric charge, and because the electric charge operator does not commute with I_1 and I_2, only I_3 among the generators $\{I_j\}$ can be diagonalized in the space of physical states (with definite charge). The multiplets are bases for irreducible representations of $SU(2)_I$ Table [1] give the names, masses, and isospins of some familiar hadrons.

Isospin is very useful in enumerating and combining the states of hadrons. For example the "nucleon" has isospin-1/2. It comes in two isospin states, the proton with isospin "up" and the neutron with isospin "down". Including spin, there are four internal states of the nucleon,

$$|p^\uparrow\rangle,\ |p^\downarrow\rangle,\ |n^\uparrow\rangle,\ |n^\downarrow\rangle$$

(a) What are the quantum numbers (total spin, total isospin, total angular momentum, and parity) of the states of two nucleons with orbital angular momentum $\ell = 0$ and $\ell = 1$?

(b) What are the quantum numbers (total spin, total isospin, total angular momentum, and parity) of the states of two pions with orbital angular momentum $\ell = 0$ and $\ell = 1$?

(c) The Rho decays into two pions. What are the allowed charge states for the decay $\rho^0 \rightarrow \pi\pi$?

Table 1: Names, isospins, and masses of some prominent hadrons. $[I(I+1)$ is the eigenvalue of $\vec{I}^2]$. Note: some masses are much better known than others.

Name	I	Spin	Charge state	I_3	A Mass×c^2 [MeV]
Nucleon	1/2	1/2	p	+1/2	938.27200 ± 0.00004
			n	-1/2	939.56533 ± 0.00004
Delta	3/2	3/2	Δ^{++}	+3/2	~ 1232
			Δ^{+}	+1/2	~ 1232
			Δ^{0}	-1/2	~ 1232
			Δ^{-}	-3/2	~ 1232
Pion	1	0	π^+	+1	139.57018 ± 0.00035
			π^0	0	134.9766 ± 0.0006
			π^-	-1	139.57018 ± 0.00035
Rho	1	1	ρ^+	+1	771.1 ± 0.9
			ρ^0	0	771.1 ± 0.9
			ρ^-	-1	771.1 ± 0.9

(d) When two particles scatter, the initial state $|\psi\rangle_i$ is transformed into the final state, $|\psi\rangle_f$ by the scattering operator $\mathcal{S}$. Because isospin is conserved by the strong interactions, $[\mathcal{S}, I_j] = 0$. Consider the scattering of two pions, ${}_f\langle\pi\pi|\mathcal{S}|\pi\pi\rangle_i$. At low energies only the first few values of orbital angular momentum, $\ell = 0, 1, 2$, show significant scattering. Find the allowed values of total isospin for each value of ℓ. Then for each ℓ, 0,1, and 2, and total I, for $I_3 = 0$ give the relative amplitudes for the different charge states. The restriction to $I_3 = 0$ means that the sum of the charges of the scattering pions is zero. Thus you are asked for the ratio $\pi^+\pi^- \to \pi^+\pi^-$: $\pi^+\pi^- \to \pi^0\pi^0$: $\pi^0\pi^0 \to \pi^+\pi^-$: $\pi^0\pi^0 \to \pi^0\pi^0$.

2. **Spin and isospin dependence of nuclear forces**

Consider states of two nulceons, each with spin and isospin-1/2. The total spin and isospin operators for the two nucleon system are:

$$\vec{S} = \vec{S}_1 + \vec{S}_2 \text{ and } \vec{I} = \vec{I}_1 + \vec{I}_2$$

(a) Prove that

$$P_{12} = \frac{1}{2}(1 + 4\vec{S}_1 \cdot \vec{S}_2) \text{ and } R_{12} = \frac{1}{2}(1 + 4\vec{I}_1 \cdot \vec{I}_2)$$

are *exchange operators* for spin and isospin respectively. That is, show $P_{12}|\pm\rangle = \pm|\pm\rangle$, where $|\pm\rangle$ is symmetric (+) or antisymmetric (−) under exchange of the particle's spin. Likewise for R_{12}.

(b) The force between nucleons due to exchange of pions, which dominates the strong force at long distances, has the form

$$H_{\text{pion}} = V_0 \vec{S}_1 \cdot \vec{S}_2 \vec{I}_1 \cdot \vec{I}_2 \tag{1}$$

Rewrite H_{pion} in terms of the operators P_{12} and R_{12} and find its expectation value in all possible states of the two nucleon system with no angular momentum.

(c) The only two nucleon state that is bound is the deuteron, a proton-neutron bound state with total angular momentum one. Is this fact consistent with an interaction of the form of eq. (1)? If not, why not? If so, what is the sign of V_0?

3. **Two semi-quantitative questions**

(a) Why does the requirement than an electron in this room be antisymmetrized with respect to an electron on the moon have no observable consequences? Why is the exchange interaction energy negligible? Are the electron's allowed states restricted by the requirement of antisymmetry?

(b) The deuteron, ^{2}H, is a bound state of a proton and a neutron with spin one. Consider the molecular ion composed of two deuterons bound by a single electron. What are allowed total spin of the two deuterons in the molecular ground state? Consider the wavefunction of the lowest spin state as a function of the deuteron-deuteron separation, $\vec{R}$. How does it differ from the wavefunction of an hypothetical molecule made of two *elementary, pointlike* bosons with the same mass and spin as the deuteron?

4. **A spin-3/2 electron**

Sakurai, §6, problem 6:

Suppose the electron were a spin-3/2 particle obeying Fermi-Dirac statistics. Write the configuration of a hypothetical Ne ($Z = 10$) atom made up of such "electrons". [If the electron had spin-1/2, the configuration would be $(1s)^2(2s)^2(2p)^6$.] Show that the configuration is highly degenerate. What is the ground state of the hypothetical Ne atom in spectroscopic notation, $^{2S+1}L_J$, where S, L and J are the total spin, total orbital angular momentum and total angular momentum, respectively, when exchange splitting and spin-orbit interactions are taken into account.

5. **The energy levels of calcium isotopes**

The the late 1940's Maria Goeppert-Mayer and Hans Jensen discovered that the single particle energy levels of nuclei show a shell structure somewhat similar to atoms. In atoms the central potential is generated by the coulomb field of the nucleus. In nuclei each nucleon sees an average central field due to the interactions with all the other nucleons. The shell structure is not as pronounced in nuclei as in atoms, but it is important near closed shells (the analogues of noble gases). The key to unravelling nuclear shell structure was the realization that *spin-orbit* coupling is much more important in nuclei than in atoms. So states of definite j (the eigenvalue of the total angular momentum) are well separated in energy from one another. For example, a nucleon with orbital angular momentum $\ell = 1$ and spin-1/2 ("p-states") can have $j = 1/2$ or $j = 3/2$. The four states with $j = 3/2$ are considerably lower in energy than those with $j = 1/2$. So they "fill" first as one builds nuclei, and thus the nucleus with a filled $p_{3/2}$ shell are anomalously stable.

The $d_{3/2}/s_{1/2}$ shell closes at $N = 20$, making 20 one of the most prominent "magic numbers" (*ie.* one of the shell closings in nuclei). Thus ^{40}Ca ($Z = 20$ protons and $N = 20$ neutrons) is a "doubly magic" nucleus, and relatively stable (and common).

[In fact, ^{40}Ca is the most massive stable nucleus with an equal number of protons and neutrons. Heavier nuclei are neutron rich because the coulomb repulsion (an effect that grows like Z^2) between protons makes proton rich nuclei less stable. Even the preceding $N = Z$ nuclei, ^{38}K and ^{36}Ar are not stable. This is a typical example of a "shell effect" in nuclei.]

Another important effect in nuclear stability is called the "pairing interaction". Two neutrons or two protons in the same shell bind strongly when their total angular momenta are paired to $J = 0$. This effect can sometimes overcome the effects of shell closures.

The next available level after 20 is the $f_{7/2}$ shell (*ie.* $\ell = 3$, $j = 7/2$). Remarkably, *all calcium isotopes from* ^{40}*Ca to* ^{48}*Ca are either stable or long lived.* The shortest lived is ^{47}Ca with a lifetime of about 5 days (it decays by β^- decay to ^{47}Sc).

(a) Work out the possible quantum numbers of n neutrons in the $f_{7/2}$ shell. Note: You *do not* have to compute the wavefunctions. Explain why you *do not* have do n =5,6,7, and 8 if you have done n =0,1,2,3.

(b) The spins and parities of the ground states and first few excited states of the calcium isotopes from 41 to 47 are shown in Fig. 1 (along with some other irrelevant information). Based on the information in the introduction to the problem and on part (a), make as many observations as you can about the phenomenology calcium isotope quantum numbers. For example: Are the isotopes with 20+x and 28-x neutrons related? Are there low lying energy levels that you didn't expect? What could positive parity levels correspond to? [Hint: To answer the last question you will need to consider the effect of the pairing interaction described above.]

(c) What do you think the ground state quantum numbers of ^{41}Sc and ^{42}Ti are? ^{43}V and all higher elements with 20 neutrons are quite unstable. For example, ^{47}Co has never been observed. However, ^{48}Ni lives at least half a microsecond. Any thoughts?

6. **The Bethe Ansatz**

A model problem that serves as an example for many topics in many-body theory is a one-dimensional system of N particles interacting via δ-function forces, with the Hamiltonian

$$H = -\frac{\hbar^2}{2m}\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2} + g\sum_{i>j}^{N}\delta(x_i - x_j)$$

a) Show that both $\hbar^2/2m$ and $|g|$ may be removed from the problem by appropriate scaling of the length and energy. Use this dimensionless form for the rest of the problem. Given this, do you think perturbation theory can be useful for this problem? For uniform translationally invariant systems, show that $\frac{mg}{\hbar^2\rho}$ is the

41Ca

E_{level} (keV)	XREF	Jπ
0.0	ABCDEFGHIJKLMNOPQRSTUVWXYZabcdefghijklm	7/2-
1942.76 *15*	CD FGHI M OPQRSTUVWXYZa efghi	3/2-
2009.9 *2*	BCD FGHIJ MNOPQRST VWXYZ h	3/2+
2462.3 *2*	CD FGHI M OP R TUVWX Zab ef hi	3/2-
2576.5 *6*	A C FGHI LM OP R TU X Z b e	5/2-

42Ca

E_{level} (keV)	XREF	Jπ
0	ABCDEFGHIJKLMNOPQRSTUVWXYZabcdefg	0+
1524.70 *3*	ABCDEFGHIJKLMNOPQRSTUVWXYZabcdefg	2+
1837.3 *3*	AB GHIJK NOP S U WXYZ bcdefg	0+
2424.17 *4*	A CD GHIJKL NOPQRS UVWXYZ bcdef	2+
2752.41 *4*	A CDEFGHI K MNO QRS U WXYZ bcdef	4+
3189.44 *8*	CDEF HI K MNO QRS WX Z bcd f	6+

43Ca

E_{level} (keV)	XREF	Jπ
0	ABCDEFGHIJKLMNOPQRSTU	7/2-
372.761 *5*	ABCDE HIJKLMNOPQRSTU	5/2-
593.394 *5*	ABCDE HIJKLMNOPQR TU	3/2-
990.257 *5*	CDEF HIJK MNOP R T	3/2+
1394.472 *8*	A CDEF HIJK MNOP R T	5/2+
1677.8 *2*	CDE IJKLMNOPQ TU	11/2-

44Ca

E_{level} (keV)	XREF	Jπ
0.0	ABCDEF HIJKLM OPQ ST VW YZabc	0+
1157.030 *15*	ABCDEF HIJKLMNOPQ ST VW YZabc	2+
1883.52 *2*	A EF IJKLMNO Q S W Zabc	0+
2283.12 *2*	A CDEF IJ LMNO Q S W Z bc	4+
2656.51 *2*	AB EF IJKLMNO Q S W Zabc	2+
3044.26 *3*	A EF IJ NO S W bc	4+
3285.0 *1*	CD F HIJ NO	6+

45Ca

E_{level} (keV)	XREF	Jπ
0.0	ABCDEFGH	7/2-
174.266 *24*	A CDEFGH	5/2-
1434.71 *7*	A CDEFGH	3/2-
1554.37 *8*	B H	(11/2-)
1558 *10*	E	

46Ca

E_{level} (keV)	XREF	Jπ	$T_{1/2}$
0.0	ABCDEFGHIJKLM	0+	STABLE
1346.0 *3*	ABCDE GHIJKLM	2+	3.6 ps *3*
2423.1 *8*	CD G K M	0+	> 4.5 ps
2574.7 *5*	A C E G JKLM	4+	
2973.9 *6*	C E G JKL	6+	10.4 ns *5*

47Ca

E_{level} (keV)	XREF	Jπ
0.0	ABCD FGH	7/2-
2013.54 *9*	ABCD FGH	3/2-
2578.34 *9*	A CD FGH	3/2+
2599.55 *11*	A CD FGH	1/2+

Figure 1: Low lying levels of the calcium isotopes. Energies above the ground state (in KeV) and spins and parities are given. The other information can be ignored. Source: National Nuclear Data Center, Brookhaven National Lab, <http://www.nndc.bnl.gov/>

only dimensionless parameter, where ρ is the density. Note that high density corresponds to the weak coupling limit.

b) Solve for the exact totally symmetric bound states for an attractive interaction as follows. Note that the symmetric function $\exp\left(-\alpha \sum_{i<j} |x_i - x_j|\right)$ satisfies a free Schrödinger equation except when the x's coincide. By appropriate matching at the δ-functions, obtain the following symmetric ground state wave functions and energies for N particles:

$$\psi_{\text{Symm}}(x_1 \cdots x_N) = Ce^{-\frac{1}{4}\sum_{i<j}^{N}|x_i - x_j|}$$

and

$$E_N = -\tfrac{1}{48}(N^3 - N)$$

where C is a normalization constant. Note the behavior of the size and energy as $N \to \infty$. This Hamiltonian was first solved exactly by H.A. Bethe (1931), and the wavefunction is often referred to as the Bethe *Ansatz*.