Massachusetts Institute of Technology
Physics Department

Physics 8.322 **Spring 2007**
Quantum Theory II **April 16, 2007**

Assignment 9

DUE APRIL 27, 2007, AT THE END OF THE DAY

Announcements

- Please keep an eye on the website, http://web.mit.edu/8.322/, for updates on the problem set.

Reading topics for this period

Many particle quantum mechanics.

- Second quantization
- Degenerate Fermi gas
- Beginning of scattering theory

Reading Recommendations 9

- On second quantization: Gottfried and Yan, §11.2. You can find more discussion of second quantization in Baym (§19); Merzbacher (§20); and Schiff (§55). If you read only one of these references, Baym or Schiff would be best.
- Degenerate fermi gas: Notes *Degenerate Fermion Systems* posted on line §1.1 — 1.3.
- Green's functions and the density of states: Notes *Degenerate Fermion Systems* posted on line §3.
- Scattering Theory: Gottfried and Yan §4.4 gives a very thorough discussion of scattering in one dimension.

Problem Set 9

Topics covered in the problems

- Relativistic Fermi Gas
- Scattering in one dimension.

Problems

1. **Relativistic Fermi gas**

   Consider a degenerate, non-interacting, but *relativistic* Fermi gas. That is, a system of spin-1/2 fermions with the energy, momentum relation,

   $$E(p) = \sqrt{p^2c^2 + m^2c^4}$$

   (a) Calculate the energy density as a function of the number density in the ground state.

   (b) Calculate the pressure as a function of the number density in the ground state.

   (c) Calculate the "adiabatic index",

   $$\gamma = \frac{n}{P}\frac{dP}{dn}$$

   (where n is the number density and P is the pressure). How does γ behave in the relativistic and non-relativistic limits?

2. **Surface tension of a fermi gas**

   Read about the density of states of a degenerate Fermi gas in the notes posted online. The density of states is discussed from an elementary point of view in pages 10 – 14 and from the point of view of the Green's function in pages 63 – 75.

   (a) A heuristic way to estimate the surface contribution to the density of states is as follows. Consider particle in a cube $0 \leq x_j \leq \ell$, $j = 1, 2, 3$, with $\psi = 0$ on the boundary. In the "derivation" of pages 10-14 in the notes, we mistakenly counted the planes with $n_1 = 0$, $n_2 = 0$, and $n_3 = 0$. The Schrödinger wavefunction, $\sin n_j \pi x_j/\ell$ vanishes if any $n_j = 0$, so we overcounted. Show that a careful counting of states excluding $n_j = 0$ gives the same result as derived from the analysis of the Green's function,

   $$\rho(k, V, S) = \frac{dn}{dk} = \frac{g}{2\pi^2}\left(Vk^2 - \frac{\pi}{4}Sk\right) \tag{1}$$

   where g is the "degeneracy factor", $g = 2$ for a spin-1/2 particle with no other internal degrees of freedom.

   (b) Use either the heuristic approach outlined above or the Green's function method to be found in pages 63–75 of the notes, to derive the surface correction to ρ for a Fermi gas of particles that obey the *Neumann* boundary condition $\hat{n} \cdot \vec{\nabla}\psi = 0$ on the boundary of a volume V.

   Consider a collection of N ($N \gg 1$) spin-1/2 particles confined to a sphere by their mutual interactions. A simple model of this system is a non-interacting, non-relativistic Fermi gas of N particles *with an N-independent Fermi energy*, $\epsilon = \hbar^2 k_F^2/2m$, confined by a boundary condition $\psi = 0$ at the surface of the sphere.

(c) First keep only the leading term in the density of states (the first term in eq. (1)). Find the radius of the sphere as a function of N and ϵ. Show that the total energy of the system (U) grows linearly with N and find the constant of proportionality.

(d) Now include the second, surface area dependent, term in the density of states. Find the leading correction to the results of the previous section. First, show that the radius is shifted by an N *independent* constant. Then find the leading correction to the internal energy. Re-express your result in the form $U = c_V V + c_S S$ where V and S are the volume and surface area of the Fermi gas. c_V is the pressure and c_S is the "surface tension". What can you say about the surface tension of a "Neumann" Fermi gas (*ie.* one confined by a the Neumann boundary condition).

3. **Scattering in one dimension I**

Consider "scattering" in one dimension: Let a plane wave be incident from the left ($\psi^{(+)}$) or right ($\psi^{(-)}$) on a localized potential, $V(x)$ ($V(x) = 0$ for $|x| > a$), giving rise to reflected and transmitted waves.

$$\begin{aligned} \lim_{x\to-\infty} \psi^{(+)}(x) &= e^{ikx} + R(k)e^{-ikx} \\ \lim_{x\to\infty} \psi^{(+)}(x) &= T(k)e^{ikx} \end{aligned} \qquad (2)$$

$$\begin{aligned} \lim_{x\to\infty} \psi^{(-)}(x) &= e^{-ikx} + \bar{R}(k)e^{ikx} \\ \lim_{x\to-\infty} \psi^{(-)}(x) &= \bar{T}(k)e^{-ikx} \end{aligned} \qquad (3)$$

Notice that to avoid double counting we restrict k to positive values.

(a) What constraints does unitarity place on T, $\bar{T}$, R and $\bar{R}$?

(b) Show that the constraints of part (a) correspond to requiring a certain 2×2 matrix, S, to be unitary, $S^\dagger S = 1$.

(c) Find the analog of the Lippmann-Schwinger equation (eg. 7.1.22 of Sakurai) in one dimension for $\psi^{(+)}$. Be careful to define the Green's function so that the solution has the proper behavior as $x \to \pm\infty$. You should be able to write the Green's function as a very simple function of k, and the two points, x and y.

(d) Solve the L-S equation from part (b) in the case of an attractive δ-function,

$$V(x) = -\frac{\gamma}{2m}\delta(x)$$

and find $R(k)$ and $T(k)$.

(e) The one dimensional δ-function with $\gamma > 0$ admits one (and only one) bound state. Show that T and R have poles at the (imaginary) value of k at which the bound state occurs.

(f) Return to the S-matrix that you found in part (b). Suppose that $V(x)$ is *symmetric*, $V(-x) = V(x)$. Show that $T(k) = \bar{T}(k)$ and $R(k) = \bar{R}(k)$. Then show that you can find a basis for the two dimensional space such that S is *diagonal*,

$$S(k) = \begin{pmatrix} e^{2i\delta_1(k)} & 0 \\ 0 & e^{2i\delta_2(k)} \end{pmatrix}$$

Notice that the unitarity of the S-matrix is quite obvious in this basis. Call the eigenfunctions in the two channels $\phi_{1,2}(x)$. Extract the behavior of $\phi_{1,2}$ when $|x| > a$ and interpret the significance of the phases $\delta_{1,2}(k)$.

4. **Scattering in one dimension II**

We will derive the Born Approximation for scattering in three dimensions a little later in the course. Here is the derivation for weak potentials in one dimension.

Start with the Lippmann-Schwinger equation that you derived in the previous problem,

$$\psi^{(+)}(x) = e^{ikx} - \frac{im}{\hbar^2 k} \int_{-\infty}^{+\infty} dy e^{ik|x-y|} V(y) \psi^{(+)}(y) \tag{4}$$

(a) Derive expressions for the transmission and reflection amplitudes, T and R in terms of $\psi^{(+)}$ from eq. (4).

(b) Find the *first Born approximation* for T and R (T_{Born} and R_{Born}) by constructing the iterative solution to eq. (4).

(c) Evaluate $|R_{\text{Born}}|^2$ and $|T_{\text{Born}}|^2$ when $V(x) = V_0 e^{-\alpha x^2/2}$.

(d) Does your result satisfy the constraint of unitarity? If not, why not?

(e) What are T_{Born} and R_{Born} for the δ-function potential of the previous problem? Do they exhibit poles at the bound states? Do they agree with the expansion of the exact result to lowest order in γ? The n^{th} Born approximation is defined as the approximation taken to n^{th} order in V. To what order, n, would you have to go in order to find the pole in T and R in the δ-function case?