8.322: Quantum Theory II

Problem Set #2 Solutions
February 23, 2007

1. **The Dirac Equation with a Central Potential**

   (a) The properties of the parity operator give

$$\Pi \vec{r} \Pi = -\vec{r}$$
$$\Pi \vec{p} \Pi = -\vec{p}.$$

We also know that $\beta \vec{\alpha} \beta = -\vec{\alpha}$, so

$$\begin{aligned} \Pi H_{\rm D} \Pi &= -\vec{\alpha} \cdot -\vec{p} + \beta m + \beta V(-\vec{r}) \\ &= H_D. \end{aligned}$$

   (b) First note that

$$\left\{\gamma^5, \beta\right\} = 0$$
$$\left[\gamma^5, \vec{\alpha}\right] = 0.$$

Thus

$$\gamma^5 H_{\rm D} \gamma^5 = \vec{\alpha} \cdot \vec{p} - \beta \left(m + V(r)\right).$$

Which gives

$$\gamma^5 \Pi H_{\rm D} \gamma^5 \Pi = -\vec{\alpha} \cdot \vec{p} - \beta \left(m + V(r)\right) = -H_{\rm D}$$

Thus

$$H_{\rm D} \left(\gamma^5 \Pi \psi_E(\vec{r})\right) = -E \left(\gamma^5 \Pi \psi_E(\vec{r})\right)$$

If $H_{\rm D} = \vec{\alpha} \cdot \vec{p} + \beta m + \phi(r)$ then clearly

$$\gamma^5 \Pi H_{\rm D} \gamma^5 \Pi \neq -H_{\rm D},$$

and the spectrum is no longer symmetric in energy.

(c) We first note that $[L_i, p_j] = i\epsilon_{ijk}p_k$.

$$\begin{aligned}\left[\vec{L}, H_{\rm D}\right] &= \left[\vec{L}, \vec{\alpha}\cdot\vec{p}\right] \\ &= i\vec{\alpha}\times\vec{p} \neq 0.\end{aligned}$$

Now we consider

$$\begin{aligned}[\vec{\sigma}, H_{\rm D}] &= [\vec{\sigma}, \vec{\alpha}\cdot\vec{p}] \\ &= -2i\vec{\alpha}\times\vec{p}.\end{aligned}$$

This follows from $[\sigma_i, \alpha_j] = 2i\epsilon_{ijk}\alpha_k$.

Thus $\left[\vec{J}, H_{\rm D}\right] = 0$.

(d) Dirac states are labeled by (j, π, m_j) and non-relativistic states by (l, j, m_j). We will drop the m_j label since it has an obvious 1 to 1 correspondence. Note that given a non-relativistic state l the corresponding eigenvalue of the parity operator $\pi = (-1)^l$.

The third column here is the spectroscopic label.

(j, π)	(l, j)	
$\frac{1}{2}, +$	$0, \frac{1}{2}$	$\mathrm{s}_{1/2}$
$\frac{1}{2}, -$	$1, \frac{1}{2}$	$\mathrm{p}_{1/2}$
$\frac{3}{2}, +$	$2, \frac{3}{2}$	$\mathrm{d}_{3/2}$
$\frac{3}{2}, -$	$1, \frac{3}{2}$	$\mathrm{p}_{3/2}$
$\frac{5}{2}, +$	$2, \frac{5}{2}$	$\mathrm{d}_{5/2}$
$\frac{5}{2}, -$	$3, \frac{5}{2}$	$\mathrm{f}_{5/2}$

and so on.

2. **Relativistic Corrections in a Central Potential**

(a)

$$\left[\vec{\alpha}\cdot\vec{p}+\beta(m+V(r))\right]\psi = E\psi$$

$$\begin{pmatrix} m+V & \vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -m-V \end{pmatrix}\begin{pmatrix}\varphi \\ \chi\end{pmatrix} = E\begin{pmatrix}\varphi \\ \chi\end{pmatrix}.$$

Solving these two equations in the limit $|V| \ll m$ gives

$$\chi = \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\varphi.$$

To lowest order in v/c, $E \approx m$ thus

$$\chi = \frac{\vec{\sigma}\cdot\vec{p}}{2m}\varphi.$$

(b) The magnetic moment operator is given by

$$\vec{\mu} = \frac{1}{2}\vec{r}\times\vec{j} = \frac{e}{2}\vec{r}\times\vec{\alpha}.$$

$$\begin{aligned}\psi^\dagger\vec{\mu}\psi &= e\begin{pmatrix}\varphi^\dagger & \frac{\vec{\sigma}\cdot\vec{p}}{2m}\varphi^\dagger\end{pmatrix}\frac{\vec{r}\times\vec{\alpha}}{2}\begin{pmatrix}\varphi \\ \frac{\vec{\sigma}\cdot\vec{p}}{2m}\varphi\end{pmatrix} \\ &= \frac{e}{4m}\varphi^\dagger\{\vec{\sigma}\cdot\vec{p},\vec{r}\times\vec{\sigma}\}\varphi\end{aligned}$$

We now must evaluate this anti-commutator.

$$\begin{aligned}\{\vec{\sigma}\cdot\vec{p},\vec{r}\times\vec{\sigma}\} &= \sigma_l p_l \epsilon_{ijk} r_j \sigma_k + \epsilon_{ijk} r_j \sigma_k \sigma_l p_l \\ &= \epsilon_{ijk} p_l r_j(\delta_{lk} + i\epsilon_{lkm}\sigma_m) + \epsilon_{ijk} r_j p_l(\delta_{kl} + i\epsilon_{klm}\sigma_m) \\ &= 2L_i + i\epsilon_{ijk} p_l r_j \epsilon_{lkm}\sigma_m + i\epsilon_{ijk} r_j p_l \epsilon_{klm}\sigma_m \\ &= 2L_i + i(\delta_{mi}\delta_{jl} - \delta_{mj}\delta_{il}) p_l r_j \sigma_m + i(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) r_j p_l \sigma_m \\ &= 2L_i + i\sigma_i p_j r_j - i p_i r_j \sigma_j + i r_j p_i \sigma_j - i r_j p_j \sigma_i \\ &= 2L_i - \sigma_i + 3\sigma_i\end{aligned}$$

Thus

$$\langle\vec{\mu}\rangle = \frac{e}{2m}\int d\vec{r}\,\varphi^\dagger(\vec{L}+\vec{\sigma})\varphi$$

(c) We let $\varphi(\vec{r}) = f(\vec{r})u$ where u is a two component constant spinor.

$$
\begin{aligned}
2\varphi^\dagger \vec{A}\varphi &= \varphi^\dagger \gamma^0 \vec{\gamma}\gamma^5 \varphi \\
&= \begin{pmatrix} \varphi^\dagger & \chi^\dagger \end{pmatrix} \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix} \begin{pmatrix} \varphi \\ \chi \end{pmatrix} \\
&= \varphi^\dagger \vec{\sigma}\varphi + \varphi^\dagger \frac{\vec{\sigma}\cdot\vec{p}}{2m}\vec{\sigma}\frac{\vec{\sigma}\cdot\vec{p}}{2m}\varphi \\
&= \varphi^\dagger \vec{\sigma}\varphi + \frac{1}{4m^2}\varphi^\dagger \left(-(\vec{\sigma}\cdot\vec{p})^2\vec{\sigma} + \vec{\sigma}\cdot\vec{p}\{\vec{\sigma}, \vec{\sigma}\cdot\vec{p}\}\right)\varphi \\
&= \varphi^\dagger \vec{\sigma}\varphi + \frac{1}{4m^2}\varphi^\dagger(-\vec{p}^2\vec{\sigma} + 2\vec{p}(\vec{\sigma}\cdot\vec{p}))\varphi \\
&= f^* f u^\dagger \vec{\sigma} u + \frac{1}{4m^2} f^* u^\dagger(-\vec{p}^2\vec{\sigma} + 2\vec{p}(\vec{\sigma}\cdot\vec{p}))uf.
\end{aligned}
$$

Using $\langle p_i p_j\rangle = \frac{1}{3}\delta_{ij}p^2$, we find

$$
\left\langle \vec{A} \right\rangle = \frac{1}{2}u^\dagger \vec{\sigma} u \left(\int d^3r f^* f - \int d^3r f^* \frac{\vec{p}^2}{12m^2} f \right).
$$

We note that we need to renormalize ψ.

$$
1 = \int d^3r f^* f + \frac{1}{4m}\int d^3r f^* p^2 f,
$$

giving

$$
\left\langle \vec{A} \right\rangle = \frac{1}{2}u^\dagger \vec{\sigma} u \left(1 - \int d^3r f^* \frac{\vec{p}^2}{3m^2} f \right).
$$

3. **Anomalous Magnetic Moment**

(a) We know that the non-relativistic limit of

$$(i\gamma^\mu \partial_\mu - ie\gamma^\mu A_\mu - m)\,\psi = 0$$

is

$$H\varphi = \frac{1}{2m}\left(\vec{p} - \frac{e}{c}\vec{A}\right)^2 \varphi - \frac{e}{2m}\left(\vec{L} + 2\vec{s}\right)\cdot \vec{B}\varphi = E\varphi,$$

which is obtained by multiplying by γ^0 and taking $\psi = \begin{pmatrix} \varphi \\ \chi \end{pmatrix}$. Thus

$$H \to H + \Delta H \text{ where}$$
$$\Delta H = -\frac{\kappa e}{4m}\gamma^0 \sigma^{\mu\nu} F_{\mu\nu}.$$

We note

$$F_{ij} = \epsilon_{ijk} B_k = -\epsilon_{ijk} B^k$$
$$F_{0i} = -E^i$$
$$F_{i0} = -E_i$$
$$\sigma^{0i} = \frac{i}{2}[\gamma^0, \gamma^i] = i\alpha^i$$
$$\sigma^{ij} = \frac{i}{2}[\gamma^i, \gamma^j] = \frac{-i}{2}[\sigma^i, \sigma^j] = \epsilon^{ijk}\sigma_k$$

Thus

$$\begin{aligned}\Delta H &= -\frac{\kappa e}{4m}\gamma^0(-2i\alpha^i E^i - \epsilon^{ijk}\sigma_k \epsilon_{ijl} B^l) \\ &= -\frac{\kappa e}{4m}\gamma^0(-2i\vec{\alpha}\cdot\vec{E} + 2\vec{\sigma}\cdot\vec{B})\end{aligned}$$

Assuming $\vec{E} = 0$, we look at the equation for φ since we are in the non-relativistic limit. Thus

$$\Delta H = -\frac{e}{2m}(2\kappa)\vec{S}\cdot\vec{B}.$$

In other words it takes the g factor from $g = 2 \to g = 2(1+\kappa)$.

(b) Reading off from (a)

$$H_{\mathrm{D}} \to H_{\mathrm{D}} - \beta\frac{\kappa e}{2m}(\vec{\sigma}\cdot\vec{B} - i\vec{\alpha}\cdot\vec{E})$$

(c) We note again that $\beta\vec{\alpha}\beta = -\vec{\alpha}$, and that the parity operation is $\Pi_{\mathrm{D}} = \beta\Pi$ where Π takes $\vec{r}$ to $-\vec{r}$. Thus under parity $\vec{\alpha}\cdot\vec{E}$ goes to itself as does $\vec{B}$ and the new term is invariant.

(d) We note that $\{\gamma^\mu, \gamma_5\} = 0$. Thus we use the results above to get

$$\Delta H = \gamma_5 \frac{\tilde{\kappa} e}{2m}\beta(\vec{\sigma}\cdot\vec{B} - i\vec{\alpha}\cdot\vec{E})$$

$$\Delta H = \frac{\tilde{\kappa} e}{2m}\begin{pmatrix} i\vec{\sigma}\cdot\vec{E} & -\vec{\sigma}\cdot\vec{B} \\ \vec{\sigma}\cdot\vec{B} & -i\vec{\sigma}\cdot\vec{E} \end{pmatrix}$$

Since β and γ_5 anti-commute it is clear that the above argument for being invariant under parity fails. In particular $\Pi_{\mathrm{D}}\Delta H\Pi_{\mathrm{D}} = -\Delta_H$.

If we now assume that $\vec{E} = 0$ as in (a), we see that the change in the equation for φ is

$$\Delta_H = -\frac{\tilde{\kappa} e}{2m}(\vec{\sigma}\cdot\vec{B})\chi$$

which to lowest order does not enter the non-relativistic limit.

In the case that $\vec{E} \neq 0$, this term corresponds to an electric dipole moment.

4. **MIT Bag Model**

(a) The current is given by $j_\mu = \bar{\psi}\gamma_\mu\psi$. Starting with the Dirac equation when $m = 0$,

$$i\gamma_\mu\partial^\mu\psi = 0$$
$$-i\psi^\dagger\gamma_\mu^\dagger\partial^\mu$$

and using $\gamma_\mu^\dagger = \gamma_0\gamma_\mu\gamma_0$, we derive the equation for $\bar{\psi} = \psi^\dagger\gamma^0$,

$$-i\partial_\mu\bar{\psi}\gamma_\mu = 0.$$

Now using the two equations we see that

$$\partial^\mu j_\mu = (\partial^\mu\bar{\psi})\gamma_\mu\psi + \bar{\psi}\gamma_\mu\partial^\mu\psi = 0.$$

Now at $r = R$, we begin with the boundary condition

$$i\hat{r}\cdot\vec{\gamma}\psi = \psi \rightarrow i\bar{\psi}\hat{r}\cdot\vec{\gamma}\psi = \bar{\psi}\psi.$$

Now looking at the conjugate,

$$-i\bar{\psi}\hat{r}\cdot\vec{\gamma} = \bar{\psi} \rightarrow -i\bar{\psi}\hat{r}\cdot\vec{\gamma}\psi = \bar{\psi}\psi.$$

Then

$$\hat{r}\cdot\vec{j} = \bar{\psi}\hat{r}\cdot\vec{\gamma}\psi = 0$$

on the surface $r = R$. Thus the current is conserved.

(b) From partial integration,

$$\int d^3r\psi^\dagger(-i\vec{\alpha}\cdot\vec{\nabla})\psi = \int d^3r(-i\vec{\alpha}\cdot\vec{\nabla}\psi)^\dagger\psi + \oint d^2s(-i\psi^\dagger\hat{r}\cdot\vec{\alpha}\psi).$$

The boundary condition sets the surface integral to be 0. Thus H is Hermitian.

(c) From the Dirac equation we find

$$-i\vec{\sigma}\cdot\vec{\nabla}\chi = E\varphi$$
$$-i\vec{\sigma}\cdot\vec{\nabla}\varphi = E\chi.$$

Combining these two equations, we find

$$-\nabla^2\chi = E^2\chi$$
$$-\nabla^2\varphi = E^2\varphi.$$

These two equations are identical to the free Schrödinger equation. Assuming $E > 0$. We let $\varphi = Aj_0(Er)$ which is merely the s-wave solution. Then

$$\begin{aligned}
\chi &= \frac{1}{E}(-i\vec{\sigma}\cdot\vec{\nabla}\varphi) \\
&= \frac{-i\vec{\sigma}\cdot\hat{r}}{E}\partial_r j_0(Er) \\
&= E\frac{i\vec{\sigma}\cdot\hat{r}}{E} j_1(Er)
\end{aligned}$$

We now plug this solution into the B.C. and find

$$-i\vec{\gamma}\cdot\hat{r}\psi = \psi \rightarrow j_1(ER) = j_0(ER)$$

Solving we find $ER = 2.0428$.

(d) Letting $a \approx 2.0428$, we start with

$$E(R) = \frac{3a}{R} + \frac{4\pi BR^3}{3}.$$

Setting $\frac{\partial E}{\partial R} = 0$ gives

$$R_0 = \left(\frac{3a}{4\pi B}\right)^{1/4} \Rightarrow E(R_0) = \frac{4}{3}(3a)^{3/4}(4\pi B)^{1/4}.$$

We then set $E(R_0) = 939MeV$. Solving for B gives $B = (96MeV)^4$. Restoring $\hbar$ and c and converting to atmospheres gives

$$B \sim 1.75 \times 10^{28}\text{atm}.$$

5. **Helicity and Chirality**

(a)

$$
\begin{aligned}
[h, H_D] &= [\vec{\sigma}\cdot\hat{p}, \vec{\alpha}\cdot\vec{p}] + [\vec{\sigma}\cdot\hat{p}, \beta m] \\
&= \hat{p}_i p_j [\sigma_i, \alpha_j] \\
&= \hat{p}_i p_j (2i)\epsilon_{ijk}\alpha_k \\
&= 2i\vec{\alpha}\cdot(\hat{p}\times\vec{p}) \\
&= 0.
\end{aligned}
$$

$$
h = \begin{pmatrix} \vec{\sigma}\cdot\hat{p} & 0 \\ 0 & \vec{\sigma}\cdot\hat{p} \end{pmatrix} \Rightarrow h^2 = I.
$$

Thus h has eigenvalues ± 1.

(b) Suppose we are looking at the eigenstate with momentum $\vec{p}$, energy E, and helicity $+1$. Let $|\vec{p}| = p$.

$$
\begin{aligned}
&H_D\psi = E\psi \\
&(\vec{\alpha}\cdot\vec{p} + \beta m)\,\psi = E\psi \\
&\big(\gamma^5(\vec{\sigma}\cdot\hat{p})p + \beta m\big)\,\psi = E\psi \\
&\big(\gamma^5 p + \beta m - E\big)\,\psi = 0
\end{aligned}
$$

We note that

$$
\begin{aligned}
\gamma^5(\beta u_{\mathrm{R}}) &= -\beta u_{\mathrm{R}} \\
\gamma^5(\beta u_{\mathrm{L}}) &= \beta u_{\mathrm{L}}.
\end{aligned}
$$

Thus $\beta u_{\mathrm{R}} = u_{\mathrm{L}}$ and $\beta u_{\mathrm{L}} = u_{\mathrm{R}}$. We let $\psi = c_{\mathrm{R}} u_{\mathrm{R}} + c_{\mathrm{L}} u_{\mathrm{L}}$.

$$
p(c_{\mathrm{R}} u_{\mathrm{R}} - c_{\mathrm{L}} u_{\mathrm{L}}) = E(c_{\mathrm{R}} u_{\mathrm{R}} + c_{\mathrm{L}} u_{\mathrm{L}}) - m(c_{\mathrm{R}} u_{\mathrm{L}} + c_{\mathrm{L}} u_{\mathrm{R}}).
$$

Thus

$$
\frac{c_{\mathrm{R}}}{c_{\mathrm{L}}} = \frac{p+E}{m}
$$

Normalizing, one finds that up to a p and E independent constant

$$
\psi_+ \sim \frac{m}{\sqrt{m^2 + (p+E)^2}} u_{\mathrm{L}} + \frac{E+p}{\sqrt{m^2 + (p+E)^2}} u_{\mathrm{R}}
$$

One can do the same procedure for $h = -1$ and find

$$\psi_- \sim \frac{E+p}{\sqrt{m^2+(p+E)^2}} u_{\rm L} + \frac{m}{\sqrt{m^2+(p+E)^2}} u_{\rm R}$$

If $E \to \infty$, then can set $m = 0$ giving

$$\psi_+ \sim u_{\rm R}$$
$$\psi_- \sim u_{\rm L}.$$

In other words the helicity eigenstate with $h = 1(-1)$ is equivalent to the right(left) chirality eigenstate.

If $E \to m$, then in both cases ψ is an equal mixture of $u_{\rm R}$ and $u_{\rm L}$.

6. **Duffin-Kemmer Formulation of Maxwell's Equations**

(a)

$$\begin{aligned}\frac{1}{c}\frac{\partial \vec{B}}{\partial t} &= -\vec{\nabla}\times\vec{E}\\ &= -\epsilon_{ijk}\partial_j E_k\\ &= i\Sigma^j_{ik}\partial_j E_k\\ &= i(\vec{\nabla}\cdot\vec{\Sigma})\vec{E}\\ &= -\frac{1}{\hbar}(\vec{p}\cdot\vec{\Sigma})\vec{E}.\end{aligned}$$

Thus

$$\begin{aligned}\hbar\frac{\partial E}{\partial t} &= c(\vec{\Sigma}\cdot\vec{p})\vec{B}\\ \hbar\frac{\partial B}{\partial t} &= -c(\vec{\Sigma}\cdot\vec{p})\vec{E}.\end{aligned}$$

Thus

$$i\hbar\partial_t\vec{v}_\pm = \pm c(\vec{\Sigma}\cdot\vec{p})\vec{v}_\pm.$$

(b) Under the parity operation $\vec{E}\to-\vec{E}$ but $\vec{B}\to\vec{B}$. Thus

$$\begin{aligned}\vec{v}_+ &\to -\vec{v}_-\\ \vec{v}_- &\to -\vec{v}_+\end{aligned}$$

Since $\vec{p}\to-\vec{p}$

$$i\hbar\partial_t\vec{v}_\pm = \pm c(\vec{\Sigma}\cdot\vec{p})\vec{v}_\pm \to i\hbar\partial_t\vec{v}_\mp = \mp c(\vec{\Sigma}\cdot\vec{p})\vec{v}_\mp$$

(c) One way of doing this is to take

$$f = \begin{pmatrix} v_+ + v_- \\ v_+ - v_- \end{pmatrix} \qquad\qquad \tilde{\Sigma}^k = \begin{pmatrix} 0 & \Sigma^k \\ \Sigma^k & 0 \end{pmatrix}$$

(d) From (9) we find

$$-\hbar^2\frac{\partial^2 f}{\partial t^2} = c^2(\vec{\tilde{\Sigma}}\cdot\vec{p})^2 f$$

Plugging in from (c)

$$-\hbar^2\frac{\partial^2}{\partial t^2}\begin{pmatrix}\vec{E}\\ i\vec{B}\end{pmatrix} = c^2\begin{pmatrix}(\vec{\Sigma}\cdot\vec{p})^2 & 0\\ 0 & (\vec{\Sigma}\cdot\vec{p})^2\end{pmatrix}\begin{pmatrix}\vec{E}\\ i\vec{B}\end{pmatrix}$$

$$
\begin{aligned}
(\vec{\Sigma}\cdot\vec{p})^2_{jm} &= (i\epsilon_{jik}p_i)(i\epsilon_{klm}p_l) \\
&= -\epsilon_{kji}\epsilon_{klm}p_ip_l \\
&= p^2\delta_{jm} - p_jp_m
\end{aligned}
$$

Thus

$$
\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\vec{E} + \vec{\nabla}(\vec{\nabla}\cdot\vec{E}) = 0
$$

and similarly for $\vec{B}$. Thus we have the wave equation for the transverse fields

Starting with (8), we find

$$
i\hbar\partial_t(\vec{\nabla}\cdot\vec{v}_\pm) = \pm c\vec{\nabla}\cdot(\vec{\Sigma}\cdot\vec{p})\vec{v}_\pm
$$

It then follows that

$$
\partial_t(\vec{\nabla}\cdot\vec{E}) \sim \vec{\nabla}\cdot(\vec{\nabla}\times\vec{E}) = 0.
$$

This is also true for $\vec{B}$.

(e) Consider (8) with solution

$$
\vec{v}_\pm = \vec{\varepsilon}_\pm(p)e^{iEt/\hbar - i\vec{p}\cdot\vec{x}/\hbar} \Rightarrow \frac{E}{c}\vec{\varepsilon}_\pm(p) = \pm c\left(\vec{\Sigma}\cdot\vec{p}\right)\vec{\varepsilon}_\pm(p)
$$

If $E = cp$ then

$$
\vec{\varepsilon}_\pm = \pm i(\hat{k}\times\vec{\varepsilon}_\pm).
$$

Which are just the conditions for left and right circularly polarized light.