8.322: Quantum Theory II

Problem Set #3 Solutions
March 6, 2007

1. **Anharmonic Oscillator**

(a)

$$\begin{aligned} V &= \frac{1}{2}m\omega x^2 + v_1 x + \frac{1}{2}v_2 x^2 \\ &= \frac{1}{2}m(\omega^2 + \frac{v_2}{m})x^2 + v_1 x \\ &= \frac{1}{2}m\tilde{\omega}^2 x^2 + v_1 x \\ &= \frac{1}{2}m\tilde{\omega}^2 \left(x + \frac{v_1}{m\tilde{\omega}^2}\right)^2 - \frac{v_1^2}{2m\tilde{\omega}^2} \end{aligned}$$

Thus, with $\tilde{\omega} = \sqrt{\omega^2 + v_2/m}$

$$\begin{aligned} E_n &= \hbar\tilde{\omega}\left(n + \frac{1}{2}\right) - \frac{v_1^2}{2m\tilde{\omega}^2} \\ \Delta E_n &= \hbar(\tilde{\omega} - \omega)\left(n + \frac{1}{2}\right) - \frac{v_1^2}{2m\tilde{\omega}^2} \end{aligned}$$

(b) Note that here we use the unshifted frequency ω, but it is also acceptable to do the problem with the shifted frequency $\tilde{\omega}$ from (a)

$$\langle 0|x^3|0\rangle = 0$$

Thus there is no shift to the ground state energy due to v_3.

$$\begin{aligned} \langle 0|x^4|0\rangle &= \frac{1}{4}\left(\frac{\hbar}{m\omega}\right)^2 \langle 0|(a + a^\dagger)^4|0\rangle \\ &= \frac{1}{4}\left(\frac{\hbar}{m\omega}\right)^2 \langle 0|aa^\dagger aa^\dagger + aaa^\dagger a^\dagger|0\rangle \\ &= \frac{1}{4}\left(\frac{\hbar}{m\omega}\right)^2 (3) \end{aligned}$$

Thus the shift in the ground state is

$$\Delta E_0 = \frac{v_4}{32}\left(\frac{\hbar}{m\omega}\right)^2$$

(c) For second order perturbation theory we look at $\langle n|x^3|0\rangle$. Clearly this is only nonzero for $n = 1$ and $n = 3$.

$$\begin{aligned}
\Delta E_0 &= \frac{|\left\langle 1|\frac{1}{6}v_3 x^3|0\right\rangle|^2}{E_0 - E_1} + \frac{|\left\langle 3|\frac{1}{6}v_3 x^3|0\right\rangle|^2}{E_0 - E_3} \\
&= \frac{v_3^2}{288}\left(\frac{\hbar}{m\omega}\right)^3\left(\frac{9}{-\hbar\omega} + \frac{6}{-3\hbar\omega}\right) \\
&= -\frac{11}{288}v_3^2\frac{\hbar^2}{m^3\omega^4}
\end{aligned}$$

2. **Quadrupole perturbation of the Hydrogen p-states**

(a)

$$\begin{aligned}
\lambda xy &= \lambda r^2 \sin^2\theta \sin\phi\cos\phi \\
&= \frac{\lambda}{2}r^2\sin^2\theta\sin(2\phi) \\
&= \frac{\lambda}{4i}r^2\sin^2\theta(e^{2i\phi} - e^{-2i\phi}) \\
&= -\lambda i\sqrt{\frac{2\pi}{15}}\left(T_2^{(2)} - T_{-2}^{(2)}\right)
\end{aligned}$$

(b) Here we are trying to find the right eigenvectors to diagonalize the perturbation for the *degenerate* states so we only need to consider states of the same N. Using the Wigner-Eckart theorem gives

$$\left\langle N,1,m_2|T_{\pm 2}^{(2)}|N,1,m_1\right\rangle = \frac{1}{\sqrt{3}}\langle 1,m_2|2,\pm 2,1,m_1\rangle \cdot \langle N,1||T^{(2)}||N,1\rangle$$

Looking at a Clebsch–Gordan table we see that

$$\begin{aligned}
\langle 1,1|2,2,1,-1\rangle &= \sqrt{\frac{3}{5}} \\
\langle 1,-1|2,-2,1,1\rangle &= \sqrt{\frac{3}{5}}
\end{aligned}$$

Thus

$$\langle N,1,m_2|V|N,1,m_1\rangle = -i\lambda\sqrt{\frac{2\pi}{75}}\langle N,1||T^{(2)}||N,1\rangle \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

The eigenvectors are

$$|\psi_0\rangle = |N,1,0\rangle$$
$$|\psi_\pm\rangle = \frac{1}{\sqrt{2}}\left(|N,1,1\rangle \mp i|N,1,-1\rangle\right)$$

(c) We have to compute

$$\langle 2,1,-1|V|2,1,1\rangle =$$
$$\int_0^\infty \int_0^\pi \int_0^{2\pi} R_{21}(r)Y_1^{-1}(\theta,\phi)^*(\lambda r^2 \sin^2\theta \sin\phi\cos\phi)R_{21}(r)Y_1^1(\theta,\phi)\ r^2\sin\theta\ d\phi d\theta dr$$

where

$$R_{21}(r) = \left(\frac{1}{2a_0}\right)^{3/2}\frac{r}{\sqrt{3}a_0}e^{-r/2a_0}$$

and

$$Y_1^{\pm 1}(\theta,\phi) = \mp\sqrt{\frac{3}{8\pi}}\sin\theta e^{\pm i\phi}.$$

The integration gives

$$\langle 2,1,-1|V|2,1,1\rangle = \lambda(30a_0^2)\left(\frac{-3}{8\pi}\right)\left(\frac{\pi i}{2}\right)\left(\frac{16}{15}\right) = -6i\lambda a_0^2.$$

Thus

$$\langle\psi_0|V|\psi_0\rangle = 0$$
$$\langle\psi_\pm|V|\psi_\pm\rangle = \mp 6\lambda a_0^2$$

3. **Form of perturbation expansions**

We know that to second order the shift in energy of the n^{th} eigenstate is

$$E_n - E_n^0 = V_{nn} + \sum_{m\neq n}\frac{V_{nm}V_{mn}}{E_n^0 - E_m^0},$$

and the first order shift to the eigenstates is given by

$$|n\rangle = |n^0\rangle + \sum_{m\neq n}\frac{V_{mn}}{E_n^0 - E_m^0}|m^0\rangle.$$

We have

$$\begin{aligned}\langle m^0|n\rangle &= \frac{\langle m_0|V|n\rangle}{E_n - E_m^0} \\ &= \frac{1}{E_n^0 + V_{nn} - E_m^0}\left(V_{mn} + \sum_{k\neq n}\frac{V_{mk}V_{kn}}{E_n^0 - E_k^0}\right) \\ &= \frac{V_{mn}}{E_n^0 - E_m^0}\left(1 - \frac{V_{nn}}{E_n^0 - E_m^0}\right) + \sum_{k\neq n}\frac{V_{mk}V_{kn}}{(E_n^0 - E_k^0)(E_n^0 - E_m^0)}\end{aligned}$$

Thus

(a)

$$E_n - E_n^0 = V_{nn} + \sum_{m\neq n}\left(\frac{V_{nm}V_{mn}}{E_n^0 - E_m^0} - \frac{V_{nm}V_{mn}V_{nn}}{(E_n^0 - E_m^0)^2}\right) + \sum_{k\neq n;m\neq n}\frac{V_{nm}V_{mk}V_{kn}}{(E_n^0 - E_k^0)(E_n^0 - E_m^0)}$$

(b)

$$|n\rangle = |n^0\rangle + \sum_{m\neq n}\left(\frac{V_{mn}}{E_n^0 - E_m^0} - \frac{V_{mn}V_{nn}}{(E_n^0 - E_m^0)^2}\right)|m^0\rangle + \sum_{k\neq n;m\neq n}\frac{V_{mk}V_{kn}}{(E_n^0 - E_k^0)(E_n^0 - E_m^0)}|m^0\rangle$$

4. **Removing the degeneracy beyond leading order**

(a) First after diagonalization, we note that to third order

$$E_n^3 = E_n^0 + V_{nn} + \sum_{m\notin D}\left(\frac{V_{nm}V_{mn}}{E_n^0 - E_m^0} - \frac{V_{nm}V_{mn}V_{nn}}{(E_n^0 - E_m^0)^2}\right) + \sum_{k\notin D;m\notin D}\frac{V_{nm}V_{mk}V_{kn}}{(E_n^0 - E_k^0)(E_n^0 - E_m^0)}$$

Using the results of (3), we can now build the shift in energy to fourth order in V. Clearly one of the terms is

$$E_n^4 = \sum_{k,l\notin D_1,p\neq n}\frac{V_{nk}V_{kp}V_{pl}V_{ln}}{(E_n^0 - E_k^0)(E_n^0 - E_p^0)(E_n^0 - E_l^0)}.$$

In this term p can be in D_1 since $k, l \notin D_1$. However then all three $V \neq 0$, but if $n, p \in D_1$, the subspace that is not diagonalized then $E_n^0 - E_p^0$ vanishes, but the numerator remains finite. Thus perturbation theory breaks down at this order. All the other terms to fourth order do not have this problem because if they contain $E_n^0 - E_m^0$ in the denominator then they have V_{nm} in the numerator so that you do not get an infinite term.

(b) Looking at the problematic term which occurs for $p, n \in D_1$, we see that using, for $|n\rangle \in D_1$,

$$(H_0 + V)|n^0\rangle = \bar{E}^1|n^0\rangle$$

we have

$$E_n^4 = \sum_{k,l \notin D_1, p \neq n} \frac{V_{nk} V_{kp} V_{pl} V_{ln}}{(\bar{E}^1 - E_k^0)(\bar{E}^1 - \bar{E}^1)(\bar{E}^1 - E_l^0)}$$

$$= \sum_{p \neq n} \frac{\langle n^0 | V^2 | p^0 \rangle \langle p^0 | V^2 | n^0 \rangle}{\bar{E}^1 - \bar{E}^1}$$

If we now diagonalize V^2 it is clear that we have $\langle n^0 | V^2 | p^0 \rangle = 0$ if $n \neq p$ which is the case to which we are restricted. Thus the problematic term never arises and there is no division by zero and the problem in the fourth order expansion is fixed.

5. **Bounding an exotic particle**

(a) We are considering the perturbation by

$$V = \frac{m_1 e_1}{M} \cdot \frac{m_2 e_2}{M} \cdot \frac{e^{-r/\lambda}}{r}$$

The first particle will be the proton, the second one the electron. V is a function of r only, therefore it doesn't mix states with different azimuthal quantum numbers. The energy shift for the two states will be

$$\langle 2s_{1/2} | V | 2s_{1/2} \rangle = -\frac{m_e m_p e^2}{M^2} \int_0^\infty r^2 dr \frac{e^{-r/\lambda}}{r} R_{20}^2$$

$$\langle 2p_{1/2} | V | 2p_{1/2} \rangle = -\frac{m_e m_p e^2}{M^2} \int_0^\infty r^2 dr \frac{e^{-r/\lambda}}{r} R_{21}^2$$

where

$$R_{20}^2 = 4 \left(\frac{1}{2a_0} \right)^3 \left(1 - \frac{r}{2a_0} \right)^2 e^{-r/a_0}$$

$$R_{21}^2 = \frac{1}{3} \left(\frac{1}{2a_0} \right)^3 \left(\frac{1}{a_0} \right)^2 r^2 e^{-r/a_0}$$

and a_0 is the Bohr radius. Evaluating the integrals gives

$$\Delta E(2s_{1/2}) = -\frac{m_e m_p e^2}{M^2} \frac{\lambda^2 (2a_0^2 + \lambda^2)}{4a_0 (a_0 + \lambda)^4}$$

$$\Delta E(2p_{1/2}) = -\frac{m_e m_p e^2}{M^2} \frac{\lambda^4}{4a_0 (a_0 + \lambda)^4}$$

The difference is

$$\Delta E = \Delta E(2p_{1/2}) - \Delta E(2s_{1/2}) = \frac{m_e m_p e^2}{M^2} \frac{a_0 \lambda^2}{2(a_0 + \lambda)^4}$$

This should be less than the difference between the measured and the theoretical values

$$\delta = |\Delta E_{TH} - \Delta E_{EXP}| \approx 0.032 \text{ MHz}$$

we find

$$\lambda \approx 5.81 \cdot 10^{-14}\text{m}$$
$$a_0 \approx 5.29 \cdot 10^{-11}\text{m}$$

which gives a lower bound for M

$$M \gtrsim 1.5 \cdot 10^4 \text{m}_e \approx 7.6 \text{ GeV}/c^2$$

(b) We have to change the values for the muonic atom:

$$e \to 2e$$
$$\text{m}_e \to 207 \text{ m}_e$$

The reduced mass of the atom is $\frac{4\text{m}_p 207\text{m}_e}{4\text{m}_p + 207\text{m}_e} = 201 \text{ m}_e$ which enters in the expression for the new a_0 (Note $Z = 2$).

$$a_0 \to \frac{1}{2 \cdot 201} a_0$$

In the expression for V_{X}, $m_1 = 207 m_{\text{e}}$ and $m_2 = 2m_{\text{p}}$. The previous expression for ΔE is still valid with the above modifications and using that $\Delta E \approx 0.01\text{eV}$ gives the following bound

$$M \gtrsim 1.4 \cdot 10^6 \text{m}_e \approx 70 \text{ GeV}.$$

6. **The Stark effect in hydrogen**

(a)

$$\Delta E^{(1)} = \langle 100 | \mathcal{E} e z | 100 \rangle = 0$$

since z is parity–odd and $\Psi_{100}(r)$ is an even function.

(b)

$$\Delta E^{(2)} = (\mathcal{E} e)^2 \sum_{n>0} \frac{|\langle n|z|0\rangle|^2}{E_0^{(0)} - E_n^{(0)}} \geq (\mathcal{E} e)^2 \sum_{n>0} \frac{|\langle n|z|0\rangle|^2}{E_0 - E_1}$$

Since $\langle 0|z|0\rangle = 0$, we can write

$$\frac{(\mathcal{E} e)^2}{E_0 - E_1} \sum_{n\geq 0} \langle 0|z|n\rangle\langle n|z|0\rangle = \frac{(\mathcal{E} e)^2}{E_0 - E_1} \langle 0|z^2|0\rangle$$

$$\langle 0|z^2|0\rangle = a_0^2$$

$$E_0 - E_1 = \left(1 - \frac{1}{4}\right) E_0 = \frac{3}{4} E_0 = -\frac{3}{8}\frac{\hbar^2}{m a_0^2}$$

which gives

$$\Delta E^{(2)} \geq -\frac{8(\mathcal{E}e)^2}{3\hbar^2} m a_0^4 = -\frac{8}{3} a_0^3 \mathcal{E}^2$$

(c) Take

$$F = -\frac{m a_0}{\hbar^2}\left(\frac{r}{2} + a_0\right) z$$

Then,

$$[F, H_0] = \left[F, \frac{\vec{p}^2}{2m}\right] = -\frac{a_0}{2\hbar^2}\left[\left(\frac{r}{2} + a_0\right) z, \vec{p}^2\right]$$

Now,

$$\langle \vec{r}| \left[\left(\frac{r}{2} + a_0\right) z, \vec{p}^2\right] |0\rangle = \left(\left(\left(\frac{r}{2} + a_0\right) z\right)(-\hbar^2\nabla^2) - (-\hbar^2\nabla^2)\left(\left(\frac{r}{2} + a_0\right) z\right)\right) \frac{2}{a_0^{3/2}\sqrt{4\pi}} e^{-r/a_0}$$

In spherical coordinates

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial}{\partial \phi}$$

Hence,

$$\nabla^2(e^{-r/a_0}) = \left(1 - \frac{2a_0}{r}\right)\frac{e^{-r/a_0}}{a_0^2}$$

$$\nabla^2\left((r/2 + a_0)(r \cdot \cos\theta) e^{-r/a_0}\right) = \left(\frac{r}{2} - \frac{2a_0^2}{r} - 2a_0\right) z \frac{e^{-r/a_0}}{a_0^2}$$

Using these expressions, we obtain

$$\left[\left(\frac{r}{2} + a_0\right) z, \vec{p}^2\right] |0\rangle = -\frac{2\hbar^2}{a_0} z|0\rangle$$

$$[F, H_0]|0\rangle = z|0\rangle$$

Hence F satisfies the condition and we will use it to evaluate the sum

$$\Delta E^{(2)} = (\mathcal{E}e)^2 \sum_{n\geq 0} \frac{|\langle n|z|0\rangle|^2}{E_0^{(0)} - E_n^{(0)}} = (\mathcal{E}e)^2 \sum \langle 0|z|n\rangle\langle n|F|0\rangle = (\mathcal{E}e)^2 \langle 0|zF|0\rangle$$

Finally,

$$\langle 0|zF|0\rangle = \int_0^\infty \int_0^\pi \int_0^{2\pi} \left(\frac{1}{\pi a_0^3} e^{-2r/a_0}\right)\left(-\frac{m a_0}{\hbar^2}\right)\left(\frac{r}{2} + a_0\right)(r \cos\theta)^2 r^2 \sin\theta \, d\phi \, d\theta \, dr = -\frac{9}{4}\frac{m a_0^4}{\hbar^2}$$

Therefore

$$\Delta E^{(2)} = -\frac{9}{4}a_0^3\mathcal{E}^2$$

which satisfies the

$$\Delta E^{(2)} \geq -\frac{8}{3}a_0^3\mathcal{E}^2$$

bound from (b).

7. **Proton–Antiproton Atoms**

(a)

$$\langle 0|r^2|0\rangle = 4\int dr\ r^4 \left(\frac{1}{a_0}\right)^3 e^{-2r/a_0} = 3a_0^2$$

The reduced mass is $\mathrm{m}_{red} = \frac{\mathrm{m}_p}{2}$.

$$\sqrt{\langle r^2\rangle} \approx 10^{-13}\ \mathrm{m}$$

(b) The probability of annihilation of $p\bar{p}$ is proportional to the probability of being inside the sphere of radius R.

$$\frac{\Gamma_{2p}}{\Gamma_{1s}} = \frac{\int_0^R dr\ r^2|\Psi_{2p}|^2}{\int_0^R dr\ r^2|\Psi_{2s}|^2} = \frac{\int_0^R dr\ r^2|R_{21}|^2}{\int_0^R dr\ r^2|R_{10}|^2} = \frac{\frac{1}{24a_0^5}\int_0^R dr\ r^4 e^{-r/a_0}}{\frac{4}{a_0^3}\int_0^R dr\ r^2 e^{-2r/a_0}} \approx 7.7\times 10^{-6}$$

(c)

For the p-wave state

$$\Pi(p)\cdot\Pi(\bar{p})\cdot(-1)^l = \underbrace{(-1)}_{intrinsic}\cdot\underbrace{(-1)}_{orbital} = 1$$

i. e. the final meson has even parity. The angular momentum

$$\mathbf{1}\otimes\frac{\mathbf{1}}{\mathbf{2}}\otimes\frac{\mathbf{1}}{\mathbf{2}} = \mathbf{0}\oplus\mathbf{1}\oplus\mathbf{1}\oplus\mathbf{2}\ \rightarrow\ j = 0,1,1,2$$

For the s-wave state

$$\Pi(p)\cdot\Pi(\bar{p})\cdot(-1)^l = \underbrace{(-1)}_{intrinsic}\cdot\underbrace{(1)}_{orbital} = -1$$

i. e. the final meson has odd parity. The angular momentum

$$\mathbf{0}\otimes\frac{\mathbf{1}}{\mathbf{2}}\otimes\frac{\mathbf{1}}{\mathbf{2}} = \mathbf{1}\oplus\mathbf{0}\rightarrow j = 0,1$$

(d)

The energy of the X–ray is

$$E_{2p} - E_{1s} = \frac{e^2}{2a_0} - \frac{e^2}{8a_0} = 9.36 \text{ keV}$$

(e) Let us choose the z axis in the direction of the $\vec{E}$-field. Then experiments say that

$$\frac{|\langle 2p_{mixed}|2p_0\rangle|^2 \cdot \Gamma_{2p}}{(1 + |\langle 2p_{mixed}|1s_0\rangle|^2) \cdot \Gamma_{1s}} \leq 1\%$$

We use the following approximation

$$|\langle 2p_{mixed}|2p_0\rangle|^2 \approx 1$$

It follows that

$$|\langle 2p_{mixed}|1s_0\rangle|^2 \approx 7.7 \cdot 10^{-4}$$

Now we need to compute the mixing of the $2p$ states

$$|210\rangle,\ |211\rangle,\ |21\ {-1}\rangle$$

with the $|100\rangle$ 1s state. From perturbation theory,

$$\langle 100|2p_{mixed}\rangle = \frac{\langle 100|V|21m\rangle}{E_1^{(0)} - E_2^{(0)}} = \frac{4e\mathcal{E}}{3E_1}\langle 100|z|21m\rangle$$

where $V = e\mathcal{E}z$. Since $z = Y_{10}$ and $\langle 00|Y_{10}|1m\rangle = \delta_{m0}$, we obtain

$$\langle 100|z|21{\pm}1\rangle = 0$$

Now we need to compute

$$\langle 100|z|210\rangle = \int_0^\infty \int_0^\pi \int_0^{2\pi} R_{10}(r)R_{21}(r)Y_{00}(\theta,\phi)^* Y_{10}(\theta,\phi) r \cos\theta\, r^2 \sin\theta\, dr\, d\phi\, d\theta = \frac{128\sqrt{2}}{243}a_0$$

Hence,

$$\langle 100|2p_{mixed}\rangle = \frac{512\sqrt{2}}{729}\frac{e\mathcal{E}a_0}{E_1} = \frac{1024\sqrt{2}}{729}\frac{\mathcal{E}a_0^2}{e}$$

From this, we obtain

$$\mathcal{E} \approx 10^{14} \text{ V/m}.$$