8.322: Quantum Theory II

Problem Set #7 Solutions
April 17, 2007

1. **States of two hadrons with isospin**

(a)

Nucleons have both spin and isospin 1/2. Thus the total spin and isospin must be $S, I = 0, 1$. For nucleons the overall wavefunction should be antisymmetric.

$l = 0$. The spatial part is symmetric so the net effect of the spin and isospin should be antisymmetric. The total angular momentum J is the combination of the orbital angular momentum L and the spin S. Thus the possible quantum numbers are $|S, I, J^{\Pi}\rangle$

$$|0, 1, 0^+\rangle$$
$$|1, 0, 1^+\rangle$$

$l = 1$. Now the spatial part is antisymmetric so the possible quantum numbers are

$$|0, 0, 1^-\rangle$$
$$|1, 1, 0^-\rangle$$
$$|1, 1, 1^-\rangle$$
$$|1, 1, 2^-\rangle$$

(b)

Pions have isospin 1 and spin 0. Thus the total spin will be in the singlet state and the total isospin must be in the state $0, 1,$ or 2. The overall wavefuntion for the pions should be symmetric.

$l = 0$. Total isospin 0 and 2 are symmetric and since the overall wavefunction is symmetric these are the only options.

$$|0, 0, 0^+\rangle$$
$$|0, 2, 0^+\rangle$$

$l = 1$. The only option is that the total isospin is 1.

$$|0, 1, 1^-\rangle$$

(c)

Total angular momentum is conserved. The Rho has total angular momentum 1. Thus we need a state of two pions that has total angular momentum 1. Returning to (b) we see this is only possible if the two pions are in the $|0, 1, 1^-\rangle$ state. Thus the total isospin is one. The value of the isospin is also conserved. The Rho is in the state $I_3 = 0$ so we should be in the isospin state $|I, m_I\rangle = |1, 0\rangle$. Looking at the Clebsch-Gordan coefficients we see that this state has no component from the single particle $|0, 0\rangle$ states so thus only decay to the π^+, π^- channel is possible.

(d)

The allowed isospins for each l are

$$l = 0 \Rightarrow I = 0, 2$$
$$l = 1 \Rightarrow I = 1$$
$$l = 2 \Rightarrow I = 0, 2$$

According to standard Clebsch-Gordon decompositions:

$$|\pi^+\pi^-\rangle = \sqrt{\frac{1}{6}}|20\rangle + \sqrt{\frac{1}{2}}|10\rangle + \sqrt{\frac{1}{3}}|00\rangle$$
$$|\pi^0\pi^0\rangle = \sqrt{\frac{2}{3}}|20\rangle - \sqrt{\frac{1}{3}}|00\rangle$$
$$|\pi^-\pi^+\rangle = \sqrt{\frac{1}{6}}|20\rangle - \sqrt{\frac{1}{2}}|10\rangle + \sqrt{\frac{1}{3}}|00\rangle$$

where the states on the r.h.s. are labeled with $|II_3\rangle$. Let $\mathcal{M}_I$ be the amplitude for scattering with isospin I. Then the scattering of two pions can be decomposed into channels of definite isospin:

We now consider the two cases.

If $l = 0$ or 2, we know that $I = 0$ or 2. The initial states are

$$|\pi^+\pi^-\rangle_i = \frac{1}{\sqrt{2}}\left(|\pi^+\pi^-\rangle + |\pi^-\pi^+\rangle\right) = \sqrt{\frac{1}{3}}|20\rangle + \sqrt{\frac{2}{3}}|00\rangle$$
$$|\pi^0\pi^0\rangle_i = |\pi^0\pi^0\rangle = \sqrt{\frac{2}{3}}|20\rangle - \sqrt{\frac{1}{3}}|00\rangle$$

$$
\begin{aligned}
{}_f\langle\pi^+\pi^-|\mathcal{S}|\pi^+\pi^-\rangle_i &= \frac{1}{3}\mathcal{M}_{20} + \frac{2}{3}\mathcal{M}_{00} \\
{}_f\langle\pi^0\pi^0|\mathcal{S}|\pi^+\pi^-\rangle_i &= \frac{\sqrt{2}}{3}\mathcal{M}_{20} - \frac{\sqrt{2}}{3}\mathcal{M}_{00} \\
{}_f\langle\pi^+\pi^-|\mathcal{S}|\pi^0\pi^0\rangle_i &= \frac{\sqrt{2}}{3}\mathcal{M}_{20} - \frac{\sqrt{2}}{3}\mathcal{M}_{00} \\
{}_f\langle\pi^0\pi^0|\mathcal{S}|\pi^0\pi^0\rangle_i &= \frac{2}{3}\mathcal{M}_{20} + \frac{1}{3}\mathcal{M}_{00}
\end{aligned}
$$

Now suppose the scattering only takes place through $I = 0$. Then the ratio of the four charge processes requested can be read off these four equations, by setting $\mathcal{M}_{20} = 0$.

So for $I = 0$,

$$\frac{2}{3} : -\frac{\sqrt{2}}{3} : -\frac{\sqrt{2}}{3} : \frac{1}{3}$$

and for $I = 2$:

$$\frac{1}{3} : \frac{\sqrt{2}}{3} : \frac{\sqrt{2}}{3} : \frac{2}{3}$$

If $l = 1$, then $I = 1$ so that the initial states become

$$
\begin{aligned}
|\pi^+\pi^-\rangle_i &= \frac{1}{\sqrt{2}}\left(|\pi^+\pi^-\rangle - |\pi^-\pi^+\rangle\right) = |10\rangle \\
|\pi^0\pi^0\rangle_i &= 0.
\end{aligned}
$$

because you cannot be in the state $\pi^0\pi^0$ with $l = 1$. Thus

$$
\begin{aligned}
{}_f\langle\pi^+\pi^-|\mathcal{S}|\pi^+\pi^-\rangle_i &= \mathcal{M}_{10} \\
{}_f\langle\pi^0\pi^0|\mathcal{S}|\pi^+\pi^-\rangle_i &= 0 \\
{}_f\langle\pi^+\pi^-|\mathcal{S}|\pi^0\pi^0\rangle_i &= 0 \\
{}_f\langle\pi^0\pi^0|\mathcal{S}|\pi^0\pi^0\rangle_i &= 0
\end{aligned}
$$

which gives

$$1 : 0 : 0 : 0$$

2. **Spin and isospin dependence of nuclear forces**

(a) Note that

$$2\vec{S}_1 \cdot \vec{S}_2 = (\vec{S}_1 + \vec{S}_2)^2 - \vec{S}_1^2 - \vec{S}_2^2.$$

The states $|+\rangle$, $|-\rangle$ have total spin 1 and 0 respectively. Thus

$$2\vec{S}_1 \cdot \vec{S}_2|+\rangle = \left(2 - \frac{3}{4} - \frac{3}{4}\right)|+\rangle = \frac{1}{2}|+\rangle$$

$$2\vec{S}_1 \cdot \vec{S}_2|-\rangle = \left(-\frac{3}{4} - \frac{3}{4}\right)|+\rangle = -\frac{3}{2}|-\rangle$$

Thus

$$P_{12}|+\rangle = \frac{1}{2}\left(1 + 4\vec{S}_1 \cdot \vec{S}_2\right)|+\rangle = |+\rangle$$

$$P_{12}|-\rangle = \frac{1}{2}\left(1 + 4\vec{S}_1 \cdot \vec{S}_2\right)|-\rangle = -|-\rangle$$

Hence,

$$P_{12}|\pm\rangle = \frac{1}{2}\left(1 + 4\vec{S}_1 \cdot \vec{S}_2\right)|\pm\rangle = \pm|\pm\rangle$$

R_{12} follows identically (just substitute $P \to R$ and $S \to I$).

(b) Note that

$$\vec{S}_1 \cdot \vec{S}_2 = \frac{1}{4}(2P_{12} - 1)$$

and

$$\vec{I}_1 \cdot \vec{I}_2 = \frac{1}{4}(2R_{12} - 1)$$

Hence,

$$H_{\text{pion}} = \frac{V_0}{16}(2P_{12} - 1)(2R_{12} - 1) = \frac{V_0}{16}(4P_{12}R_{12} - 2R_{12} - 2P_{12} + 1)$$

The previously determined $l = 0$ states of two nucleons are

$$|0, 1, 0^+\rangle, \quad |1, 0, 1^+\rangle$$

Then,

$$\langle 0, 1, 0^+|H_{\text{pion}}|0, 1, 0^+\rangle = \frac{V_0}{16}(4(1)(-1) - 2(-1) - 2(1) + 1) = -\frac{3V_0}{16}$$

$$\langle 1, 0, 1^+|H_{\text{pion}}|1, 0, 1^+\rangle = \frac{V_0}{16}(4(-1)(1) - 2(1) - 2(-1) + 1) = -\frac{3V_0}{16}$$

(c) This fact is not consistent with an interaction of the form of eq. (1) because if $V_0 > 0$ so that the energy for the proton-neutron bound state with angular momentum 1,i.e. $|1, 0, 1^+\rangle$ is negative then the other state also has a negative binding energy of the same value. If one added a potential term that contained either spin or isospin exchange but not both, the one can account for the deuteron binding.

3. **Two semi-quantitative questions**

(a) The reason this anti-symmetrization has no observable consequences is that the overlap of the wavefunctions for an electron on the moon and on earth is basically zero. So that the exchange coupling is negligible. The requirement of antisymmetry though does restrict the electron's allowed states in that they cannot be in the same state; however, the probability of the two electrons being at the same place is very small.

(b) Since deuterons are bosons, the wavefunction must be symmetric and thus the total spin in the molecular ground state ($l = 0$) may be 0 or 2.

Moreover, for the deuteron–deuteron system with $S = 0$, the spatial wavefunction will be partially suppressed for $r \lesssim R_N \approx 10^{-13}$cm, due to the Fermi–Dirac statistics of the constituent fermions. The wavefunction of the system deviates from that of the elementary bosons (see Figure 1).

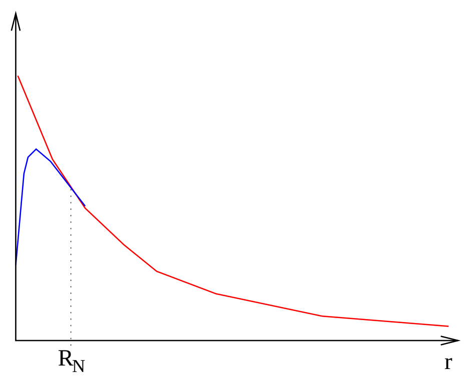

Figure 1: Red: Wavefunction for elementary bosons. Blue: Wavefunction for deuterons.

4. **A spin–3/2 electron**

Clearly, four electrons will occupy the 1s state. Another four electrons occupy the 2s state. The remaining two electrons will be in 2p. Hence,

$$\mathrm{Ne} = (1s)^4(2s)^4(2p)^2$$

There are $\frac{12\cdot 11}{2} = 66$ degenerate antisymmetric $(2p)^2$ states. The closed inner shells do not contribute to L, S or J. The two electrons repel each other (exchange interaction), and thus the spatially antisymmetric states are energetically preferred. Hence, the ground state spin is $S = 1$ or $S = 3$ and $L = 1$. Then, we have

$$^3\mathrm{P}_{0,1,2} \qquad \text{and} \qquad ^7\mathrm{P}_{2,3,4}$$

The spin–orbit coupling favors states with low total angular momentum J and large L and S. Hence the ground state is $^7\mathrm{P}_2$.

5. **The energy levels of calcium isotopes**

(a) Let n denote the number of neutrons.

n=0

1 state, ${}^1\mathrm{S}_0 = |0^+\rangle$

n=1

8 states, ${}^2\mathrm{F}_{7/2} = |1/2, 3, 7/2^-\rangle$

n=2

S and L are no longer good quantum numbers due to the interactions between nucleons.

There are $\binom{8}{2}$ states, $|J^\Pi\rangle = |0^+\rangle,\ |2^+\rangle,\ |4^+\rangle,\ |6^+\rangle$

n=3

There are $\binom{8}{3}$ states. The possible m_J values are shown in the following tables (only those with positive $\sum m_J$ are considered)

7/2	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$
5/2	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$					
3/2	$\times$						$\times$	$\times$	$\times$	$\times$	$\times$
1/2		$\times$					$\times$				
$-1/2$			$\times$					$\times$			
$-3/2$				$\times$					$\times$		
$-5/2$					$\times$					$\times$	
$-7/2$						$\times$					$\times$
$2 \cdot \sum m_J$	15	13	11	9	7	5	11	9	7	5	3

7/2	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$
5/2						
3/2						
1/2	$\times$	$\times$	$\times$	$\times$		
$-1/2$	$\times$				$\times$	$\times$
$-3/2$		$\times$			$\times$	
$-5/2$			$\times$			$\times$
$-7/2$				$\times$		
$2 \cdot \sum m_J$	7	5	3	1	3	1

7/2											
5/2	×	×	×	×	×	×	×	×	×		
3/2	×	×	×	×	×					×	×
1/2	×					×	×	×		×	×
−1/2		×				×			×	×	
−3/2			×				×		×		×
−5/2				×				×			
−7/2					×						
$2 \cdot \sum m_J$	9	7	5	3	1	5	3	1	1	3	1

Combining them gives:

15/2, 13/2, 11/2, 9/2, 7/2, 5/2, 3/2, 1/2

11/2, 9/2, 7/2, 5/2, 3/2, 1/2

9/2, 7/2, 5/2, 3/2, 1/2

7/2, 5/2, 3/2, 1/2

5/2, 3/2, 1/2

3/2, 1/2

Hence, the possible values of the total angular momentum are

$$J = \frac{15}{2}, \frac{11}{2}, \frac{9}{2}, \frac{7}{2}, \frac{5}{2}, \frac{3}{2}$$

and the parity is odd.

n=4

We have $\binom{8}{4}$ states. A similar counting gives

$$J = 8, 6, 5, 4, 4, 2, 2, 0$$

with even parity.

n=5...8

The possible states are the same as for $n = 3 \dots 0$, respectively, since counting the states is equivalent to counting the holes.

(b) Let us first look at the ground states. ^{41}Ca has $J = 7/2^-$ ground state, as it should, because of the single $f_{7/2}$ neutron added to the closed shells. Higher isotopes with even number of neutrons have $J = 0$ ground states. This suggests that neutrons pair up into

$J = 0$ configurations. This is due to interactions between nucleons that we neglected in (a).

We expect odd (even) parity states depending on whether the number of neutrons is odd (even). Surprisingly, we find even parity states for ^{41}Ca, ^{43}Ca and ^{47}Ca. This can be explained again by neutron pairing. From the $d_{3/2}$ shell nucleons get excited to $f_{7/2}$ to pair up with neutrons. Hence, the resulting configuration has $J = 3/2$ and positive parity.

We don't find the anticipated $J = 15/2$ and $J = 9/2$ states for ^{43}Ca and ^{45}Ca. These are at higher energy levels that have been truncated from the list.

The isotopes with $20 + x$ and $28 - x$ neutrons have similar spectrum (especially for the first levels), but they are not exactly mirror to each other.

(c) ^{41}Sc and ^{42}Ti are ^{41}Ca and ^{42}Ca with their protons changed to neutrons. Hence, we expect them to have the same ground states, i. e.

$$^{41}\text{Sc}: \quad |7/2^-\rangle$$

$$^{42}\text{Ti}: \quad |0^+\rangle$$

^{48}Ni has 20 neutrons and 28 protons. Thus, the neutrons can close the $d_{3/2}$ shell and the protons can close the $f_{3/2}$ shell which presumably gives more stability to the core.

6. **The Bethe Ansatz**

(a) The Hamiltonian is

$$H = -\frac{\hbar^2}{2m}\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2} + g\sum_{i>j}^{N}\delta(x_i - x_j)$$

Let us introduce a new variable

$$\alpha_i = \frac{x_i}{l}$$

where l is an appropriately chosen length scale. Then,

$$\delta(x_i - x_j) = \frac{\delta(\alpha_i - \alpha_j)}{l}$$

$$\frac{\partial}{\partial x_i} = \frac{1}{l}\frac{\partial}{\partial \alpha_i}$$

Hence, the Hamiltonian takes the following form

$$H = -\frac{\hbar^2}{2ml^2}\sum_{i=1}^{N}\frac{\partial^2}{\partial \alpha_i^2} + \text{sign}(g)\frac{|g|}{l}\sum_{i>j}^{N}\delta(\alpha_i - \alpha_j)$$

By setting

$$l = \frac{\hbar^2}{2m|g|}$$

and measuring energy in units $\frac{2m|g|^2}{\hbar^2}$, we can remove the unwanted factors and H becomes

$$H = -\sum_{i=1}^{N} \frac{\partial^2}{\partial \alpha_i^2} + \text{sign}(g) \sum_{i>j}^{N} \delta(\alpha_i - \alpha_j)$$

Regardless of how small g was in the original units, after rescaling the relevant parameter is sign(g) which is not small. Thus, perturbation theory cannot be used here.

For uniform translationally invariant systems, we have the natural length scale

$$l = \frac{1}{\rho}$$

where ρ is the density. The applicable measures of the system are

$$\hbar,\ m,\ g,\ \rho$$

The only dimensionless quantity made out of these is

$$\tilde{g} = \frac{mg}{\hbar^2 \rho}$$

which is the effective coupling. In fact, writing the Hamiltonian in terms of ρ gives

$$H = -\frac{\hbar^2 \rho^2}{2m} \sum_{i=1}^{N} \frac{\partial^2}{\partial \alpha_i^2} + g\rho \sum_{i>j}^{N} \delta(\alpha_i - \alpha_j) = \frac{\hbar^2 \rho^2}{2m} \left(-\sum_{i=1}^{N} \frac{\partial^2}{\partial \alpha_i^2} + \frac{2mg}{\hbar^2 \rho} \sum_{i>j}^{N} \delta(\alpha_i - \alpha_j) \right)$$

By rescaling the energy, we can get rid of the overall factor and we see that the effective coupling is given by the dimensionless $\tilde{g}$.

(b) We have an attractive interaction, so $\text{sign}(g) < 0$. Let us consider the trial (factorized) wavefunction

$$\psi = Ce^{-\alpha \sum_{i<j} |x_i - x_j|}$$

Note that ψ is non–negative, therefore it has no nodes. If it satisfies the Schrödinger equation, then it must be the ground state.

Now,

$$H\psi = \left(-\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} - \sum_{i>j}^{N} \delta(x_i - x_j) \right) \left(Ce^{-\alpha \sum_{i<j} |x_i - x_j|} \right) = -\sum_{k=1}^{N} \frac{\partial^2}{\partial x_k^2} Ce^{-\alpha \sum_{i<j} |x_i - x_j|}$$

$$H\psi = -\left[\alpha^2 \sum_k \left(\sum_{i \neq k} \text{sign}(x_k - x_i) \right)^2 - 2\alpha \sum_k \sum_{i \neq k} \delta(x_k - x_i) \right] \psi$$

Now

$$\sum_{i \neq k} \text{sign}(x_k - x_i) = k - (N + 1 - k)$$

where N is the number of particles. Hence,

$$\sum_k \left(\sum_{i \neq k} \text{sign}(x_k - x_i) \right)^2 = \frac{N(N^2 - 1)}{3}$$

Then,

$$H\psi = -\alpha^2 \frac{N(N^2 - 1)}{3} \psi + (4\alpha - 1) \sum_{i<j} \delta(x_k - x_i)\psi = E\psi$$

Setting $\alpha = \frac{1}{4}$ gives

$$E = -\frac{N^3 - N}{48}$$

Note that the energy of a particle scales as N^2 as $N \to \infty$. The average distance between two particles remains unchanged. Hence, the density increases as N increases.