8.322: Quantum Theory II

Problem Set #9 Solutions
April 30, 2007

1. **Relativistic Fermi gas**

(a) Let g denote the degeneracy of levels and p_F denote the Fermi momentum. Then,

$$N = \frac{g}{h^3}\int_{\text{occupied}} d^3V d^3p = \frac{Vg}{h^3}4\pi\int_0^{p_F} p^2 dp = \frac{Vg}{h^3}\frac{4\pi}{3}p_F^3$$

The energy is

$$E = \frac{Vg}{h^3}4\pi\int_0^{p_F} dp\ p^2\sqrt{p^2c^2+m^2c^4}$$

Evaluating the integral gives

$$\epsilon = \frac{E}{V} = \frac{9c}{8\pi^2\hbar^3}\left\{p_F(p_F^2+m^2c^2)^{3/2} - \frac{1}{2}m^2c^2p_F(p_F^2+m^2c^2)^{1/2} - \frac{1}{2}m^4c^4\ln\left(\frac{p_F+|\sqrt{p_F^2+m^2c^2}|}{mc}\right)\right\}$$

From $n = \frac{gp_F^3}{6\pi^2\hbar^3}$, we have

$$p_F = \left(\frac{6\pi^2\hbar^3 n}{g}\right)^{1/3}$$

Substituting p_F and $g = 2$ and setting $\hbar = c = 1$ yields the energy density as a function of the number density

$$\epsilon = \frac{1}{4\pi^2}\left\{(3\pi^2n)^{1/3}\left((3\pi^2n)^{2/3}+m^2\right)^{3/2} - \frac{m^2}{2}(3\pi^2n)^{1/3}\left((3\pi^2n)^{2/3}+m^2\right)^{1/2} - \right.$$

$$\left. -\frac{m^4}{2}\ln\left[\frac{(3\pi^2n)^{1/3}+\left((3\pi^2n)^{2/3}+m^2\right)^{1/2}}{mc}\right]\right\}$$

(b) By definition,

$$P = -\left(\frac{\partial E}{\partial V}\right)_N$$

and now

$$E = \frac{V}{\pi^2\hbar^3}\int_0^{p_F} dp\ p^2\sqrt{p^2c^2+m^2c^4}$$

Hence,

$$P = -\epsilon - \frac{V}{\pi^2\hbar^3}\left(\frac{\partial p_F}{\partial V}\right)_N p_F^2(p_F^2c^2+m^2c^4)^{1/2}$$

The partial derivative is

$$p_F = \left(6\pi^2\hbar^3 N\right)^{1/3} V^{-1/3} \quad \Rightarrow \quad \left(\frac{\partial p_F}{\partial V}\right)_N = -\frac{1}{3V} p_F$$

So ($\hbar = c = 1$)

$$P = -\epsilon + n\left((3\pi^2 n)^{2/3} + m^2\right)^{1/2}$$

(c) The index is

$$\gamma = \frac{n}{P}\frac{dP}{dn} = \frac{n}{P}\frac{dP}{dp_F}\frac{dp_F}{dn}$$

Now,

$$\frac{dP}{dp_F} = \frac{gc^2}{6\pi^2\hbar^3}\frac{p_F^4}{\sqrt{p_F^2c^2 + m^2c^4}}$$

and

$$\frac{dp_F}{dn} = \frac{2\pi^2\hbar^3}{gp_F^2}$$

Combining,

$$\frac{dP}{dn} = \frac{1}{3}\frac{p_F^2c^2}{\sqrt{p_F^2c^2 + m^2c^4}}$$

In the *non–relativistic limit*, $mc \gg p_F$. Then, by Taylor expansion

$$P(p_F) \approx \frac{gp_F^3}{30m\pi^2\hbar^3}$$

In this limit

$$\frac{dP}{dn} = \frac{1}{3}\frac{p_F^2}{m}$$

This gives for the index

$$\gamma \approx \frac{5}{3}$$

In the *relativistic limit*, $mc \ll p_F$. Hence,

$$P(p_F) \approx \frac{gp_F^4}{24m\pi^2\hbar^3}$$

Again,

$$\frac{dP}{dn} = \frac{1}{3}p_F c$$

Therefore, the index is

$$\gamma \approx \frac{4}{3}$$

2. **Surface tension of a Fermi gas**

(a) In the original counting, we consider a sphere in the octant of n–space. The sphere has radius

$$n_F = \frac{lk_F}{\pi}$$

Then the "volume" is

$$N_0 = g \cdot \frac{1}{8} \cdot \frac{4\pi n_F^3}{3} = \frac{gk_F^3}{6\pi^2} V$$

and thus

$$\rho_0(k) = \frac{dN_0}{dk} = \frac{gk^2}{2\pi^2} V$$

Note that the counting of states on the boundary of the octant is ambiguous. We counted half of the lattice points on the boundary. The total number of lattice points on the boundary

$$g \cdot \underbrace{3}_{\#\text{boundaries}} \cdot \underbrace{\frac{1}{4} n_F^2 \pi}_{\text{area of boundary}}$$

We should subtract half of these as they must be excluded from the counting. Hence,

$$N = \frac{gk_F^3}{6\pi^2} V - \frac{1}{2} \left(\frac{gk_F^2 S}{8\pi} \right)$$

where S is the area of the bounding surface. Finally, the density of states is

$$\rho(k) = \frac{g}{2\pi^2} \left(Vk^2 - \frac{\pi}{4} Sk \right)$$

in accordance with the counting by Green's functions.

(b) Here we simply include the external lattice nodes, since they correspond to non–vanishing wavefunctions with Neumann boundary conditions. Then, by repeating the same counting,

$$\rho(k) = \frac{g}{2\pi^2} \left(Vk^2 + \frac{\pi}{4} Sk \right)$$

(c) The density of states

$$\rho(k) = \frac{gk^2 V}{2\pi^2}$$

The total number of states

$$N = \int_0^{k_F} \rho(k) dk = \frac{gk_F^3 V}{6\pi^2}$$

and the energy is

$$E(k) = \frac{\hbar^2 k^2}{2m}$$

Hence,

$$N = \frac{gV}{6\pi^2}\left(\frac{2m\epsilon}{\hbar^2}\right)^{3/2}$$

Solving for the volume,

$$V = \left(\frac{\hbar^2}{2m\epsilon}\right)^{3/2}\frac{6\pi^2 N}{g}$$

Equivalently, the radius is

$$r = \left(\frac{\hbar^2}{2m\epsilon}\right)^{1/2}\left(\frac{9\pi N}{2g}\right)^{1/3}$$

The total energy of the system is

$$E = \int_0^{k_F}\epsilon(k)\rho(k)dk = \int_0^{k_F} dk\ \frac{\hbar^2 k_F^2}{2m}\frac{gk^2V}{2\pi^2} = \frac{g\hbar^2}{4m\pi^2}\frac{k_F^5}{5}V$$

Hence,

$$E = \frac{3}{5}\epsilon_F N$$

The energy is proportional to N.

(d)

$$N = \frac{1}{\pi^2}\int_0^{k_F} dk(Vk^2 - \frac{\pi}{4}Sk) = \frac{1}{\pi^2}\left(\frac{4\pi}{9}(k_F r)^3 - \frac{4\pi^2}{8}(k_F r)^2\right)$$

Let $x = k_F r$. Then,

$$N + \frac{1}{2}x^2 = \frac{4}{9\pi}x^3$$

First order

$$x_0 = \left(\frac{9\pi}{4}\right)^{1/3}N^{1/3}$$

Second order

$$x_1 = \left(\frac{9\pi}{4}\right)^{1/3}\left[N + \frac{1}{2}\left(\frac{9\pi}{4}\right)^{2/3}N^{2/3}\right]^{1/3} = \left(\frac{9\pi}{4}\right)^{1/3}N^{1/3}\left[1 + \frac{1}{2}\left(\frac{9\pi}{4}\right)^{2/3}\frac{1}{3}N^{-1/3} + \dots\right]$$

Hence,

$$k_F r = \left(\frac{9\pi}{4}\right)^{1/3}N^{1/3} + \frac{3}{8}\pi$$

and

$$r = \left(\frac{\hbar^2}{2m\epsilon}\right)^{1/2}\left[\left(\frac{9\pi N}{4}\right)^{1/3} + \frac{3}{8}\pi\right]$$

where we see the constant shift. The energy,

$$E = \frac{\hbar^2}{2m}\frac{1}{\pi^2}\int_0^{k_F} dk\ k^2(Vk^2 - \frac{\pi}{4}Sk) = \frac{\epsilon_F}{\pi^2}\left(\frac{4\pi}{15}x^3 - \frac{\pi^2}{4}x^2\right)$$

$$\approx \frac{3}{5}\epsilon_F N + \frac{\epsilon_F}{20}\left(\frac{9\pi}{4}N\right)^{2/3}$$

Expressing this in terms of V and S yields

$$E = \underbrace{\left(\frac{\hbar^2 k_F^5}{2\pi^2 m}\right)}_{c_V} V - \underbrace{\left(\frac{\hbar^2 k_F^4}{32\pi m}\right)}_{c_S} S$$

Note that the c_S surface tension is negative. For a Neumann gas, we obtain

$$c_S^{\text{Neumann}} = -c_S^{\text{Dirichlet}} = \frac{\hbar^2 k_F^4}{32\pi m}$$

3. **Scattering in one dimension I**

(a) The condition

$$\int \psi^*\psi = 1$$

gives

$$|R|^2 + |T|^2 = 1 \qquad \text{and} \qquad |\bar{R}|^2 + |\bar{T}|^2 = 1$$

Also,

$$2 = |(R+\bar{T})e^{-ikx}|^2 + |(\bar{R}+T)e^{ikx}|^2 = RR^* + \bar{T}\bar{T}^* + R\bar{T}^* + R^*\bar{T} + T\bar{R}^* + T^*\bar{R} + TT^* + \bar{R}\bar{R}^*$$

Hence

$$T^*\bar{R} + R^*\bar{T} = 0$$

(b) Define the S–matrix by

$$S = \begin{pmatrix} T & \bar{R} \\ R & \bar{T} \end{pmatrix}$$

Then,

$$S^\dagger S = \begin{pmatrix} |R|^2 + |T|^2 & T^*\bar{R} + R^*\bar{T} \\ \bar{R}^*T + \bar{T}^*R & |\bar{R}|^2 + |\bar{T}|^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

results in the same conditions.

(c) The Lippmann–Schwinger equation

$$|\psi^{(\pm)}\rangle = |\phi\rangle + \frac{1}{E - H_0 \pm i\epsilon} V|\psi^{(\pm)}\rangle$$

In position space,

$$\psi^{(\pm)}(x) = e^{\pm ikx} + \int G^{(\pm)}(x,x')U(x')\psi^{(\pm)}(x')dx'$$

where

$$G^{(\pm)}(x,x') = \frac{\hbar^2}{2m}\langle x|\frac{1}{E-H_0 \pm i\epsilon}|x'\rangle$$

and

$$U(x) = \frac{2mV(x)}{\hbar^2}$$

Now,

$$G^{(\pm)}(x,x') = \frac{\hbar^2}{2m}\int\int \frac{dk'}{2\pi}\frac{dk''}{2\pi}\langle x|k'\rangle\langle k'|\frac{1}{E-H_0 \pm i\epsilon}|k''\rangle\langle k''|x'\rangle$$

$$= \int \frac{dk'}{2\pi}\frac{dk''}{2\pi}e^{-ik'x}\left(\frac{2\pi\delta(k'-k'')}{k'^2-k^2 \pm i\epsilon}\right)e^{ik''x'} = -\int \frac{dk'}{2\pi}\frac{e^{ik'(x'-x)}}{k'^2-k^2 \mp i\epsilon}$$

The poles of the integrand are at $k' = k\sqrt{1 \pm \frac{i\epsilon}{k^2}} \equiv k \pm i\bar{\epsilon}$ and at $k' = -k \mp i\bar{\epsilon}$. For $x'-x > 0$ ($x'-x < 0$), we close the contour in the upper (lower) half plane. Evaluating the residue gives

$$G^{(\pm)}(x,x') = -(2\pi i)\left(\frac{e^{\pm ik|x'-x|}}{2\pi(\pm 2k)}\right) = \pm\frac{1}{2ik}e^{\pm ik|x'-x|}$$

Then,

$$\psi^{(\pm)}(x) = e^{\pm ikx} - \frac{im}{\hbar^2 k}\int e^{ik|x-x'|}\, V(x')\psi^{(\pm)}(x')dx'$$

(d) Taking

$$V(x) = -\frac{\gamma}{2m}\delta(x)$$

gives

$$\psi^{(+)}(x) = e^{ikx} + \frac{m}{i\hbar^2 k}e^{ik|x|}\left(-\frac{\gamma}{2m}\right)\psi^{(+)}(0)$$

From $x=0$, we obtain

$$\psi^{(+)}(0) = \frac{1}{1+\gamma/2i\hbar^2 k}$$

Hence,

$$\psi^{(+)}(x) = e^{ikx} - \frac{\gamma}{2i\hbar^2 k}\left(\frac{1}{1+\gamma/2i\hbar^2 k}\right)e^{ik|x|}$$

As $x \to -\infty$,

$$\psi^{(+)}(x) \to e^{ikx} - \frac{\gamma}{2i\hbar^2 k}\left(\frac{1}{1+\gamma/2i\hbar^2 k}\right)e^{-ikx}$$

therefore

$$R(k) = -\frac{\gamma}{2i\hbar^2 k}\left(\frac{1}{1+\gamma/2i\hbar^2 k}\right) = \frac{i\gamma}{2\hbar^2 k - i\gamma}$$

As $x \to +\infty$,

$$\psi^{(+)}(x) \to e^{ikx} - \frac{\gamma}{2i\hbar^2 k}\left(\frac{1}{1+\gamma/2i\hbar^2 k}\right)e^{ikx}$$

and

$$T(k) = \frac{1}{1 - \frac{i\gamma}{2\hbar^2 k}}$$

(e) The Schrödinger equation

$$-\frac{\hbar^2}{2m}\psi''(x) - \frac{\gamma}{2m}\delta(x)\psi(x) = E\psi(x)$$

and thus

$$\left.\frac{d\psi}{dx}\right|_{\epsilon} - \left.\frac{d\psi}{dx}\right|_{-\epsilon} = -\frac{\gamma}{\hbar^2}\psi(0)$$

For $x \neq 0$, the bound state has the wavefunction

$$\psi(x) = e^{-\kappa|x|}$$

Plugging this into the above equation gives

$$\kappa = \frac{\gamma}{2\hbar^2}$$

Therefore,

$$E = -\frac{\hbar^2\kappa^2}{2m} = -\frac{\gamma^2}{8m\hbar^2}$$

Clearly, the poles of R and T are at

$$k = \frac{i\gamma}{2\hbar^2}$$

This corresponds to

$$E = \frac{\hbar^2 k^2}{2m} = -\frac{\gamma^2}{8m\hbar^2}$$

which precisely agrees with the previous result.

(f) Recall that

$$\psi^{(\pm)}(x) = e^{\pm ikx} - \frac{im}{\hbar^2 k}\int e^{ik|x-x'|}\, V(x')\psi^{(\pm)}(x')dx'$$

By changing variable $x' \to -x'$ and using $V(x) = V(-x)$, we obtain the same integral equations for $\psi^{(+)}(x)$ and $\psi^{(-)}(-x)$. The uniqueness of solutions guarantees that they are equal

$$\psi^{(+)}(x) = \psi^{(-)}(-x)$$

Now from

$$\lim_{x\to\infty}\psi^{(+)}(x) = T(k)e^{ikx} \qquad \text{and} \qquad \lim_{x\to\infty}\psi^{(-)}(-x) = \bar{T}(k)e^{ik(-x)}$$

we conclude that $T(k) = \bar{T}(k)$.

Similarly,

$$\lim_{x\to-\infty}\psi^{(+)}(x) = e^{ikx} + R(k)e^{-ikx} \qquad \text{and} \qquad \lim_{x\to-\infty}\psi^{(-)}(-x) = e^{ikx} + \bar{R}(k)e^{ik(-x)}$$

Hence, $R(k) = \bar{R}(k)$.

For the S–matrix, we have

$$S = \left(\begin{array}{cc} T & R \\ R & T \end{array} \right)$$

The eigenvectors are

$$\frac{1}{\sqrt{2}}\binom{1}{1} \qquad \frac{1}{\sqrt{2}}\binom{1}{-1}$$

with eigenvalues

$$T(k) + R(k) \qquad T(k) - R(k)$$

respectively. These correspond to the symmetric and antisymmetric wavefunctions.

In this basis,

$$S = \left(\begin{array}{cc} T+R & 0 \\ 0 & T-R \end{array} \right)$$

By unitarity,

$$|T \pm R|^2 = |T|^2 + |R|^2 \pm (R^*T + T^*R) = 1$$

Hence,

$$T(k) + R(k) = e^{2i\delta_1(k)}$$
$$T(k) - R(k) = e^{2i\delta_2(k)}$$

and

$$S = \left(\begin{array}{cc} e^{2i\delta_1(k)} & 0 \\ 0 & e^{2i\delta_2(k)} \end{array} \right)$$

We now call the two eigenfunctions $\phi_{1,2}(x)$.

$$x < -a \qquad \phi_1 = \frac{1}{\sqrt{2}}\left(e^{ikx} + e^{2i\delta_1(k)}e^{-ikx}\right)$$

$$x > a \qquad \phi_1 = \frac{1}{\sqrt{2}}\left(e^{2i\delta_1(k)}e^{ikx} + e^{-ikx}\right)$$

and

$$x < -a \qquad \phi_1 = \frac{1}{\sqrt{2}}\left(e^{ikx} - e^{2i\delta_2(k)}e^{-ikx}\right)$$

$$x > a \qquad \phi_1 = \frac{1}{\sqrt{2}}\left(e^{2i\delta_2(k)}e^{ikx} - e^{-ikx}\right)$$

Thus we see that $2\delta_{1,2}$ are the phase shifts associated with the symmetric and antisymmetric wavefunctions.

4. **Scattering in one dimension II**

(a) The Lippmann-Schwinger equation reads

$$\psi^{(+)}(x) = e^{ikx} - \frac{im}{\hbar^2 k}\int e^{ik|x-x'|}\, V(x')\psi^{(+)}(x')dx'$$

Taking the $x \to -\infty$ limit,

$$\lim_{x\to-\infty}\psi^{(+)}(x) = e^{ikx} + \underbrace{\left(-\frac{im}{\hbar^2 k}\int e^{ikx'}\, V(x')\psi^{(+)}(x')dx'\right)e^{-ikx}}_{R(k)}$$

Taking the $x \to +\infty$ limit,

$$\lim_{x\to+\infty}\psi^{(+)}(x) = \underbrace{\left(1-\frac{im}{\hbar^2 k}\int e^{-ikx'}\, V(x')\psi^{(+)}(x')dx'\right)}_{T(k)}e^{ikx}$$

(b) The first Born approximation for $\psi^{(+)}(x)$ is

$$\psi_1^{(+)}(x) = e^{ikx} - \frac{im}{\hbar^2 k}\int e^{ik|x-x'|}\, V(x')\underbrace{\psi_0^{(+)}(x')}_{e^{ikx'}}dx'$$

Thus,

$$R_B(k) = -\frac{im}{\hbar^2 k}\int e^{2ikx'}V(x')dx'$$

$$T_B(k) = 1-\frac{im}{\hbar^2 k}\int V(x')dx'$$

(c) Taking

$$V(x) = V_0 e^{-\alpha x^2/2}$$

gives

$$T_B = 1 - \frac{imV_0}{\hbar^2 k}\sqrt{\frac{2\pi}{\alpha}}$$

$$R_B = -\frac{imV_0}{\hbar^2 k}\left(\sqrt{\frac{2\pi}{\alpha}}e^{-2k^2/\alpha}\right)$$

Hence,

$$|R_B|^2 = \frac{2\pi m^2 V_0^2}{\hbar^4 k^2 \alpha}e^{-4k^2/\alpha}$$

$$|T_B|^2 = 1 + \frac{2\pi m^2 V_0^2}{\hbar^4 k^2 \alpha}$$

(d) The unitary bound is *not* satisfied

$$|R_B|^2 + |T_B|^2 = 1 + \frac{2\pi m^2 V_0^2}{\hbar^4 k^2 \alpha}(1 + e^{-4k^2/\alpha})$$

This can be expected, since R_B and T_B are only approximations to the real values of R and T. Note that the difference from the $|R|^2 + |T|^2 = 1$ bound is of second order in V_0 since we used a first order approximation.

(e) For the potential

$$V(x) = -\frac{\gamma}{2m}\delta(x)$$

we obtain

$$T_B = 1 - \frac{im}{\hbar^2 k}\int e^{2iky}\left(-\frac{\gamma}{2m}\delta(y)\right)dy = 1 + \frac{i\gamma}{2\hbar^2 k}$$

and

$$R_B = -\frac{im}{\hbar^2 k}\int\left(-\frac{\gamma}{2m}\delta(y)\right)dy = \frac{i\gamma}{2\hbar^2 k}$$

Recall the exact solutions

$$T = \frac{1}{1 - i\gamma/2\hbar^2 k} = \sum_{n=0}^{\infty}\left(\frac{i\gamma}{2\hbar^2 k}\right)^n = 1 + \frac{i\gamma}{2\hbar^2 k} + \dots$$

$$R = \frac{1}{1 - i\gamma/2\hbar^2 k}\left(\frac{i\gamma}{2\hbar^2 k}\right) = \frac{i\gamma}{2\hbar^2 k} + \dots$$

They apparently match with the Born approximation to first order.

Since the Born approximated values are polynomials in γ, they don't exhibit poles on the complex k plane. This is also true for the n-th approximation and hence no finite n would be enough to identify the poles.