MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

8.323: Relativistic Quantum Field Theory I

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INFORMAL NOTES
DIRAC DELTA FUNCTION AS A DISTRIBUTION

Why the Dirac Delta Function is not a Function:

The Dirac delta function $\delta(x)$ is often described by considering a function that has a narrow peak at $x = 0$, with unit total area under the peak. In the limit as the peak becomes infinitely narrow, keeping fixed the area under the peak, the function is sometimes said to approach a Dirac delta function. One example of such a limit is

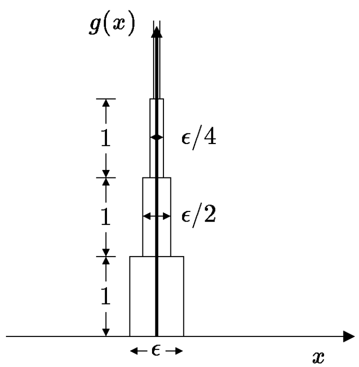
$$g(x) \equiv \lim_{\sigma \to 0} g_\sigma(x) \ , \tag{4.1}$$

where

$$g_\sigma(x) \equiv \frac{1}{\sqrt{2\pi}\,\sigma}\, e^{-\frac{1}{2}x^2/\sigma^2} \ . \tag{4.2}$$

The area under $g_\sigma(x)$ is 1, for any value of $\sigma > 0$, and $g_\sigma(x)$ approaches 0 as $\sigma \to 0$ for any x other than $x = 0$.

However, it was pointed out long ago that the delta function cannot be rigorously defined this way. The function $g(x)$ is equal to zero for any $x \neq 0$, and is infinite at $x = 0$; it can be shown that any such function integrates to zero. To see this, define the integral as the area under the curve, and consider the construction:



In this picture the vertical axis is entirely encased in rectangles, each of which has height 1. The width of the rectangles vary, with the lowest rectangle having width ϵ, for some $\epsilon > 0$, and each successive rectangle has half the width of the rectangle below. Note that the outline of the boxes is everywhere above the curve $g(x)$, so

the area under $g(x)$ must be less than the total area of the boxes. But the total area of the boxes is given by a geometric series,

$$A = \epsilon + \frac{1}{2}\epsilon + \frac{1}{4}\epsilon + \ldots \leq 2\epsilon \ . \tag{4.3}$$

Since ϵ can be chosen as small as one likes, the area under the limit function $g(x)$ must be zero.

This result does not contradict the statement that the area under $g_\sigma(x)$ is 1 for any $\sigma > 0$. Rather, this is a case where the limit of an integral is not the same as the integral of the limit of the integrand. The integral has the value 1 for every $\sigma > 0$, so the limit of the integral as $\sigma \to 0$ is 1. However, if one takes the limit of the integrand first, and then integrates, the answer is zero.

Dirac Delta Function as a Distribution:

A Dirac delta function is defined to have the property that

$$\boxed{\int_{-\infty}^{\infty} \mathrm{d}x\, \varphi(x)\, \delta(x-a) \equiv \varphi(a) \ .} \tag{4.4}$$

But we have just seen that there is no function $\delta(x)$ which has this property, as long as integration is defined by the area under a curve. However, there is no problem defining a distribution that behaves this way. Using the notation we defined earlier for distributions, we define

$$\boxed{T_{\delta(x-a)}[\varphi] \equiv \varphi(a) \ .} \tag{4.5}$$

Here $T_{\delta(x-a)}$ is the name that we will use for the distribution that acts like integration over the delta function $\delta(x-a)$, and $\varphi(x)$ is the test function. This is certainly a linear functional that is defined on all Schwartz functions, and therefore is a tempered distribution. We interpret Eqs. (4.4) and (4.5) as meaning the same thing, where the second form emphasizes the definition as a distribution, and the first form emphasizes that distributions can be viewed as a generalized kind of function, with a notation that makes them look like functions. That is, we define

$$\boxed{\int_{-\infty}^{\infty} \mathrm{d}x\, \varphi(x)\, \delta(x-a) \equiv T_{\delta(x-a)}[\varphi] \equiv \varphi(a) \ .} \tag{4.6}$$

To put it differently, we must remember that an integral over a delta function, such as Eq. (4.4), is not defined as a standard integral— instead it is symbolic integral, which is defined as a distribution which maps the function that multiplies the delta function to its value at the point where the argument of the delta function vanishes.

What about $\int \mathrm{d}p\, e^{-ip(x-a)}$?

If we interpret this integral in the sense of Riemann or Lebesgue, it simply does not exist — it diverges. As a distribution, however, we can interpret

$$\boxed{I(x) \equiv \int_{-\infty}^{\infty} \mathrm{d}p\, e^{-ip(x-a)}} \tag{4.7}$$

as the Fourier transform of

$$f(p) = e^{ipa} \ . \tag{4.8}$$

Since $I(x)$ is the Fourier transform of the distribution $T_f[\varphi]$, it must itself be a well-defined distribution. Using the notation of distributions, the distribution corresponding to I is then given by

$$T_I[\varphi] = \tilde{T}_f[\varphi] \ . \tag{4.9}$$

To evaluate the distribution $T_I[\varphi]$, we first write the distribution corresponding to function $f(p)$ in the usual way:

$$T_f[\varphi] = \int_{-\infty}^{\infty} \mathrm{d}p\, f(p)\, \varphi(p) = \int_{-\infty}^{\infty} \mathrm{d}p\, e^{ipa} \varphi(p) \ . \tag{4.10}$$

The Fourier transform of this distribution is then defined by applying the same distribution to the Fourier transform of the test function, so

$$\tilde{T}_f[\varphi] \equiv T_f[\tilde{\varphi}] = \int_{-\infty}^{\infty} \mathrm{d}p\, e^{ipa}\, \tilde{\varphi}(p) \quad . \tag{4.11}$$

But the inverse Fourier transform is given by

$$\varphi(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}p\, e^{ipx}\, \tilde{\varphi}(p) \ , \tag{4.12}$$

so by comparing the two formulas above, one sees that

$$\tilde{T}_f[\varphi] = 2\pi \varphi(a) \ . \tag{4.13}$$

But this is exactly the definition of the delta function distribution, as given in Eq. (4.6), so

$$\boxed{\tilde{T}_f[\varphi] = 2\pi\, T_{\delta(x-a)}[\varphi] \ .} \tag{4.14}$$

If we write these distributions as symbolic integrals, then the above equation becomes

$$\int_{-\infty}^{\infty} \mathrm{d}x\, I(x)\varphi(x) = 2\pi \int_{-\infty}^{\infty} \mathrm{d}x\, \delta(x-a)\varphi(x) \ , \tag{4.15}$$

where we remembered from Eq. (4.9) that $\tilde{T}_f[\varphi]$ is equal to the distribution corresponding to $I(x)$. Since Eq. (4.15) holds for an arbitrary test function $\varphi(x)$, we can equate the distributions without showing their arguments:

$$\boxed{I(x) = \int_{-\infty}^{\infty} \mathrm{d}p\, e^{-ip(x-a)} = 2\pi\delta(x-a) \ .} \tag{4.16}$$

The Derivative of a Delta Function:

If a Dirac delta function is a distribution, then the derivative of a Dirac delta function is, not surprisingly, the derivative of a distribution. We have not yet defined the derivative of a distribution, but it is defined in the obvious way. We first consider a distribution corresponding to a function, and ask what would be the distribution corresponding to the derivative of the function.

Starting with a well-behaved (i.e., piecewise continuous and bounded by some power of t) function $f(t)$, we defined the corresponding distribution by

$$T_f[\varphi] \equiv \int_{-\infty}^{\infty} \mathrm{d}t\, f(t)\varphi(t) \quad . \tag{4.17}$$

Then if we write the distribution corresponding to $\mathrm{d}f/\mathrm{d}t$, we get

$$T_{\mathrm{d}f/\mathrm{d}t}[\varphi] = \int_{-\infty}^{\infty} \mathrm{d}t\, \frac{\mathrm{d}f}{\mathrm{d}t}\, \varphi(t) \tag{4.18}$$

Since $f(t)$ is bounded for large $|t|$ by a power of t, and $\varphi(t)$ falls off faster than any power, we can integrate by parts without encountering a surface term:

$$T_{\mathrm{d}f/\mathrm{d}t}[\varphi] = -\int_{-\infty}^{\infty} \mathrm{d}t\, f(t)\, \frac{\mathrm{d}\varphi}{\mathrm{d}t} = -T_f\left[\frac{\mathrm{d}\varphi}{\mathrm{d}t}\right] \quad . \tag{4.19}$$

This result can then be taken as the general definition of the derivative of a distribution:

$$\boxed{T'[\varphi] \equiv -T\left[\frac{\mathrm{d}\varphi}{\mathrm{d}t}\right] \quad .} \tag{4.20}$$

Applying this result to a delta function in the notation of Eq. (4.6), it looks exactly like integration by parts:

$$\boxed{\int_{-\infty}^{\infty} \mathrm{d}t\, f(t) \frac{\mathrm{d}\delta(t-a)}{\mathrm{d}t} = -\left.\frac{\mathrm{d}f}{\mathrm{d}t}\right|_{t=a} \quad .} \tag{4.21}$$