The Tracy-Widom Distribution and its Application to Statistical Physics

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In this paper, we aim to present a few of the many applications of the Tracy-Widom distribution to problems in statistical physics. This probability distribution describes the largest eigenvalue of a Hermitian matrix whose entries are independently Gaussian-distributed. While a derivation of this distribution is beyond the scope of this paper, we describe its properties, and highlight its application in calculating the higher moments of surface growth processes and in the totally asymmetric simple exclusion process.

The theory of random matrices has found many and varied applications in statistical physics since Wigner introduced it in the 1950s to investigate the energy levels of heavy nuclei. In particular, he showed that the distribution of eigenvalues of a $N \times N$ Hermitian matrix whose upper triangular entries are independent and identically distributed, normalized to have mean zero and variance one, converges as $N \to \infty$ to a probability distribution that we call the Wigner semicircle distribution, although it is actually a semiellipse:

$$\rho(\lambda) = \frac{1}{\pi N}\sqrt{2N - \lambda^2},$$

where ρ is the unit-normalized density of eigenvalues. However, in many applications we are interested in the statistics of the largest eigenvalue.

In the case of independent random variables, extreme value statistics have been well understood for decades, and fall into three universality classes. The eigenvalues of such random matrices, however, are very strongly correlated, and so calculating the probability distribution of the largest eigenvalue is somewhat more difficult. The mean of the largest eigenvalue can simply be read off from the semicircle law: $\langle \lambda_{\max} \rangle = \sqrt{2N}$. In 1992, Forrester [2] showed that the standard deviation of this distribution scales as $N^{-1/6}$, and Tracy and Widom further showed that if

$$\xi \equiv \sqrt{2} N^{1/6}(\lambda_{\max} - \sqrt{2N}),$$

then the cumulative probability distribution of ξ is

$$F(\xi) = \exp\left(-\int_{\xi}^{\infty} (x - \xi) q^2(x)\, \mathrm{d}x\right).$$

Here $q(x)$ is the solution to the nonlinear differential equation

$$\frac{\mathrm{d}^2 q}{\mathrm{d}x^2} = x\frac{\mathrm{d}q}{\mathrm{d}x} + 2q^3,$$

subject to the boundary condition at infinity $q(x) \sim \mathrm{Ai}(x)$, where Ai is the Airy function

$$\mathrm{Ai}(x) = \frac{1}{\pi}\int_0^{\infty} \cos\left(\frac{t^3}{3} + xt\right)\, \mathrm{d}t.$$

The desired probability density function is then $\mathrm{d}F/\mathrm{d}s$. Unfortunately, deriving this functional form is quite far beyond the scope of this paper; the original derivation in [7] was quite technical and relied on the theory of Fredholm determinants, and there is a somewhat simpler (although still lengthy) derivation in [6], in which Nadal and Majumdar relate the distribution to the partition function of a 2-dimensional Coulomb gas constrained to a line and placed in a harmonic trap.

This is a lot to take in, so let us describe some of the properties of this distribution. First of all, the expected value of this distribution is actually negative. Second of all, it is highly asymmetric, with the tail on the left decaying as $\exp(-|x|^3/12)$ and the tail on the right decaying as $\exp(-4x^{3/2}/3)$. It is also worth noting that the range of validity of this distribution goes to zero as $N \to \infty$, since we scaled by $N^{1/6}$. We thus need a different tool to examine the part of the distribution where $|\lambda_{\max} - \sqrt{2N}| > O(N^{-1/6})$. These tails for large but finite N have a dependence that goes as e^{-N^2} on the left side and e^{-N} on the right.

The first application of the extreme-value statistics of random matrices was in evaluating the robustness of the stability of dynamical systems to a small linear perturbation, in [5]. Given any stable fixed point of a first-order dynamical system with N variables, we can linearize around it and diagonalize the relevant matrix. For simplicity, we will set all the eigenvalues to -1. Then our equations of motion are $\dot{x}_i = -x_i$. Now turn on a linear perturbation $\alpha \sum_j J_{ij} x_j$, which we will model as a real symmetric random matrix J with a constant α setting the strength of the interaction. It is clear that this new system is stable at $x_i = 0$ if and only if all the eigenvalues λ_i of J satisfy $\alpha\lambda_i \leq 1$, or equivalently that $\lambda_{\max} \leq 1/\alpha$. In the $N \to \infty$ limit, May noticed (through computer simulation) that this probability undergoes a sharp "phase transition" at $\alpha = 1/\sqrt{2}$: the probability that the system remains stable (again, in the $N \to \infty$ limit) is 0 if $\alpha > 1/\sqrt{2}$ and 1 if $\alpha < \sqrt{2}$. Recently, Majumdar and Schehr [4] have shown that N is large but finite, the Tracy-Widom distribution describes the crossover function between stable and unstable regimes of a random dynamical system, and furthermore that there is a third-order phase transition between these regimes, which also occurs in the Coulomb gas, two-dimensional QCD, and

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several other systems of physical interest.

The Tracy-Widom distribution also appeared in the solution of a longstanding problem from combinatorics, first raised by Ulam in the early 60s. Consider a permutation of the numbers $1, 2, \ldots, N$, picked uniformly at random from the $N!$ such permutations. What is the probability distribution of the longest increasing subsequence of this permutation? In 1999, Baik, Deift, and Johansson proved that if l_n is the longest increasing subsequence, then as $N \to \infty$ the distribution of $\frac{l_n - 2\sqrt{N}}{N^{1/6}}$ converges to the Tracy-Widom distribution [1]. Majumdar and Nechaev then showed that this problem maps exactly onto a particular model of discrete-time ballistic deposition in one spatial dimension, so that the height fluctuations of this growth process are exactly described by the Tracy-Widom distribution. This is significant, because many growth processes are known to belong to the same universality class as that of the well-known KPZ equation, but this universality is so far only known to describe the variance of height fluctuations. Since this model's height fluctuations agree with the Tracy-Widom distribution at all moments, this universality class might be much broader than previously believed. Since then, it has also been shown to describe free energy fluctuations in one-dimensional directed polymers in a random δ-correlated field, current fluctuations in a Fermi-Pasta-Ulam chain, and fluctuations in the critical temperature of a spin glass.

We invite the reader to survey some of the vast literature on random matrix theory in general, the Tracy-Widom law in particular, and their applications to statistical physics; we here have only scratched the surface.

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