A brief introduction to Anderson Localization

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This paper is aimed to make a thorough review of the famous phenomenon called Anderson Localization, both analytically and numerically. Properties of localization in 1, 2 and 3D would be discussed through various approaches including classical anology, transfer matrix methods, locator expansions and even scaling theory. At the end of this paper, the properties and results reviewed would extend to cutting-edge research topics, showing the significance of Anderson localization, as has been witnessed by over 50 years.

I. INTRODUCTION

P. W. Anderson stated in his 1958 paper[1] the phenomena of localization in random potential. And the localization phenomena leads to insights on Metal-Insulator Transition (MIT) and various kinds of application. The following more than 50 years have witnessed its importance with innovative derivations and numerical results, as well as experimental observation that stand the cutting-edge of every age.

- **Strong disorder:** In Anderson's seminal paper, he proposed that if the disorder is strong enough, the localization of the states will happen no matter which the dimension of the system d is. However, delocalization and crtical behaviour are not well studied in this article.
- **low dimension** $d = 1, 2$**:[5][4]** Motivated by Anderson's work, people tried to solve the problem when the disorder is finite.

  If $d = 1$, it can be exactly proved that the localization must happen whenever how small the disorder is. As for $d = 2$, there is no exact solution. However, we can solve it approximately by Renormalization Group (RG) analysis, which shows that localization must happen in $2d$ system no matter how small the disorder is. Many solid numeric experiment supported the RG results.
- **high dimension** $d = 3$**:** Decolization phenomenon only exist when dimension d is higher than 3. In addition, the critical disorder strength is dependent on the energy of the eigenstate.

II. BASIC METHODS[6]

Anderson localization is the result of quantum reflections in the lattice that make the wave function halt. Localization means the wave would decay and would not be scattered out of the lattice.

To determine whether localization happens in the lattice, usually we can have four different methods.

- **Dynamical transport** Place an electron, set its initial state and then wait. For delocalized states like a free particle, the return probability of the electron would decay, and thus its wave function propagates. Therefore, a non-decaying return probability would indicate localization. This is also Anderson's method in his article where he used the method called locator expansion[2] for the resolvent, using Green functions as well as analogy to random walking to figure out that, sufficiently large randomness would cause localization. Besides, one can also set the initial state (like a Gaussian wave packet) and use other time-dependent observables to study localization.
- **Localization of eigenstates** Localization of wavefunction can also be characterized by the inverse participation ratio.This fits well with matrix diagonalization and eigen-system calculation using computers. Also this method would be easy to visualize and to study the localization length of the system.
- **Spectrum continuity** The energy spectrum of localized states and delocalized states are different, which are absolute continuous spectrum and pure point spectrum respectively.
- **Spectral statistics** The spectral statistic study the statistics of the normalized gap between neigboring energy levels $s_i = \frac{E_{i+1}-E_i}{\langle E_{i+1}-E_i\rangle}$.

In this paper, we study Anderson Localization in dynamical analysis and also eigenstates calculation, dealing with locator expansion and transfer matrices. Besides, easier ways like classical analogy and scaling theory can also give intuitive explanations on the properties of localization in different dimensions. The spectrum continuity and spectral statistics will not be introduced here.

III. CLASSICAL ANALOGY OF RANDOM WALK

Anderson localization gives the theory of MIT (Metal-Insulator Transition), which would only exist in 3D situations. In 1D and 2D systems, however, even tiny disorders lead to localization. We can try to make sense of it

using a naive but intuitive analogy from classical random walk.

Consider symmetric random walk on an one-dimensional lattice. At each time step, the particle jumps to the right with probability $\frac{1}{2}$ and left with probability $\frac{1}{2}$. The distance of each jump is l.

$$\begin{aligned}\left\langle x_N^2\right\rangle &= \left\langle \left(\sum_{i=1}^{N} x_i\right)\left(\sum_{j=1}^{N} x_j\right)\right\rangle = \sum_{i,j=1}^{N}\langle x_i x_j\rangle \\ &= \sum_{i=1}^{N}\left\langle x_i^2\right\rangle + 2\sum_{1\le i<j\le N}\langle x_i\rangle\langle x_j\rangle \\ &= Nl^2\end{aligned} \tag{1}$$

It follows $\sigma_N^2 \equiv \left\langle x_N^2\right\rangle - \langle x_N\rangle^2 = 4Nl^2q(1-q)$ since $\langle x_i\rangle = 0$.

One can also compute the probability mass function of x_N . The probability that the particle jumps right k times among the first N steps is

$$p(k,N) = \begin{pmatrix} N \\ k \end{pmatrix}\frac{1}{4^k} \tag{2}$$

Perhaps it is more illuminating to consider the $N\to\infty$ limit. According to the central limit theorem, x_N tends to distribute normally $x_N \sim \mathcal{N}\left(\mu_N, \sigma_N^2\right)$:

$$p(x,N) = \frac{1}{\sqrt{2\pi}\sigma_N} e^{-\frac{(x-\mu_N)^2}{2\sigma_N^2}} \tag{3}$$

The probability that the particle returns to the origin after $2n$ steps is:

$$P\left(x_{2n}=0\right) = \frac{1}{2^{2n}}\begin{pmatrix} 2n \\ n \end{pmatrix} \sim \frac{1}{\sqrt{\pi n}} \tag{4}$$

Define the indicator random variable $X_n = 1$ if $x_n = 0$. $X = \sum_{i=1}^{\infty} X_i$ counts the total number of returns. Then:

$$\langle X\rangle = \sum_{i=1}^{\infty}\langle X_i\rangle = \sum_{i=1}^{\infty} P\left(x_{2i}=0\right) \sim \frac{1}{\sqrt{\pi}}\sum_{i=1}^{\infty}\frac{1}{\sqrt{i}} = \infty, \tag{5}$$

which diverges.

The expected number of returns $\langle X\rangle$ is related to the probability of return. Denote ρ as the probability that the particle ever returns to the origin, and ρ_k as the probability that the particle returns to the origin exactly k times. We have:

$$\rho_k = \rho^k(1-\rho) \tag{6}$$

We have:

$$\langle X\rangle = \frac{\rho}{1-\rho} \tag{7}$$

Therefore,

$$\begin{aligned}\langle X\rangle = \infty &\Leftrightarrow \rho = 1 \\ \langle X\rangle < \infty &\Leftrightarrow \rho < 1\end{aligned} \tag{8}$$

It indicates that random walk starting from the origin would surely return in 1D situations.

As for 2D situations, similarly, we have $P\left(\mathbf{x}_{2n}=0\right) = \frac{1}{4^{2n}}\begin{pmatrix} 2n \\ n \end{pmatrix}^2$

We can get:

$$\langle X\rangle = \sum_{i=1}^{\infty} P\left(\mathbf{x}_{2n}=0\right) \sim \frac{1}{\pi}\sum_{i=1}^{\infty}\frac{1}{i} = \infty \tag{9}$$

So the return probability of 2D systems is also 1.

The story would become different yet interesting in 3D systems. $\langle X\rangle_{3D} \propto \sum_{i=1} 1/i^{3/2} < \infty$. So the return probability of 3D random walk $\rho_{3\mathrm{D}} < 1$. This naive classical random walk analogy yields the results consistent with MIT theory from Anderson Localization.

IV. BEHAVIOUR OF DISORDER-FREE AND INFINITE DISORDER[8][6]

We have used the analogy of classiacl random walk to intuitively sketch the Anderson Localization of different dimension. Before we move to the rigorous and solid methods such as 2^{nd}-order pertubation theory or transfer matrix, we can discuss the disorder-free case as well as the infinite disorder case.

A. Tight-binding Hamiltonian

The original model is tight-binding model in the lattice system, which we would also use here in this paper.

1. *Spatial discretization*

$$(-\frac{1}{2m}\nabla^2 + \sum_k V_k(\mathbf{r}-\mathbf{r_n}))\Psi(\mathbf{r}) = E\Psi(\mathbf{r}) \tag{10}$$

where r_k is the position of k^{th} atom, and the potential is the superposition of each atom's central potential.

In the tight-binding approximation, the electron wave-function is a linear combination of orbitals of adjacent atoms.

$$\Psi((r)) = \sum \phi_j\chi_j(\mathbf{r}-\mathbf{r_n}) \tag{11}$$

where $\chi_j(\mathbf{r}-\mathbf{r_j})$ denotes the orbital of the j^{th} atom, and ϕ_n is corresponding amplitude. Therefore, this

method is also known as "Linear Combination of Atom Orbitals (LCAO)".

If we substitute the tight-binding wavefunction into the Schödinger equation and integrate the equation, we can get a compact algebric equation:

$$\sum_j [(\epsilon_j - E)A_{n,j} + \sum_{k\neq j} B_{n,k,j}]\phi_j = 0 \tag{12}$$

where

$$\begin{aligned} A_{n,j} &= \int d\mathbf{r}\chi_n^*(-\mathbf{r_n})\chi_j(\mathbf{r}-\mathbf{r_j}) \\ B_{n,k,j} &= \int d\mathbf{r}\chi_n^*(\mathbf{r}-\mathbf{r_n})v_k^*(\mathbf{r}-\mathbf{r_k})\chi_j^*(\mathbf{r}-\mathbf{r_j}) \end{aligned} \tag{13}$$

We neglect the overlap between the orbitals of neighboring atoms $A_{n,j} = \delta_{n,j}$. And we assume that the non-vanishing element of the hopping integrals are only the nearest-neigboring $B_{n,n,n+z}$, and the their values don't fulctuate $B_{n,n,n+z} = -g$.

$$\epsilon_n \Psi_n - g\sum_z \Psi_{n+z} = E\Psi_n \tag{14}$$

Therefore, the wavefunction in continuous space can be transformed into a discretized problem on lattce.

2. second-quantization

The Hamiltonian can be written in second-quantization representation.

$$H = \sum_n \{\epsilon_n |n\rangle\langle n| - g[|n\rangle\langle n+1| + |n+1\rangle\langle n|]\} \tag{15}$$

$$= \sum_n \{\epsilon_n c_n^\dagger c_n - g[c_{n+1}^\dagger c_n + c_n^\dagger c_{n+1}]\} \tag{16}$$

$\sum_n \epsilon_n |n\rangle\langle n|$ terms refer to on-site energy of a particle and the sum of $-g[|n\rangle\langle n+1| + |n+1\rangle\langle n|]$ is called hopping term, which moves the particle to its neighboring sites.

In the following part, we will discuss the problem mainly on square lattice with periodic boundary condition.

B. Disorder-free Case

In the absence of disorder $\epsilon_n = 0$, the Hamiltonian comes back to the free electron gas, which is translational invariant. Therefore, the eigenstates of Disorder-free case are naive plane waves.

$$\Psi_k(j) = \frac{1}{\sqrt{L^d}} e^{ik_i \cdot j_i},\ i = 0, 1, ..., d \tag{17}$$

where $k_i = \frac{2\pi}{L}n,\ n = 0, 1, ..., L-1$ is the i^{th} component of the wave-vector on discretized lattice.

Corresponding eigenvalues can be easily computed:

$$E_k = -2g\sum_{i=1}^d \cos(k_i) \tag{18}$$

1. Return probability

If the initial state is localized at site j, the wavefunction will be a δ-function.

$$\Psi(t=0, j) = \frac{1}{L^d}\sum_k e^{ik_i j_i} \tag{19}$$

The different k component of the wavefunction will evolve with time independently.

$$\Psi(t, j) = \frac{1}{L^d}\sum_k e^{ik_i j_i + 2itg\sum_{i=1}^d \cos(k_i)} \tag{20}$$

In the thermodynamic limit, the series summation will be substituted by integration.

$$\Psi(t, j) = \int_{k\in[0,2\pi]^d} e^{ik_i j_i + 2itg\sum_{i=1}^d \cos(k_i)} = i^d \prod_{i=1}^d J_{j_i}(2tg) \tag{21}$$

Therefore

$$\Psi(t=0, j) \approx \frac{e^{-2idtg}}{t^{d/2}} \int_{x\in R^d} \frac{d^d x}{2\pi^d} e^{-igx^2} \tag{22}$$

we can then write the return probability and mean square displacement explicitly:

$$p(t) = |\Psi(t=0, j)|^2 \sim 1/t^d \tag{23}$$

and

$$\Delta x_t^2 = 2dgt^2 \tag{24}$$

We can see that when the time evolves to infinity, the return probability decays to zero. Therefore, the state will be delocalized in disorder-free case.

In addition, what we should note that is the energy spectrum of the disorder-free case is continuous in the thermodynamic limit, which is an important criterion of the localization transition.

C. Infinite disorder case

when disorder is large enough or hopping is small enough, $\frac{g}{W}$ can be seen as zero. we can write the eigenstates of this case without calculation.

$$\Psi_i(j) = \delta_{j,i},\ E_i = \epsilon_i \tag{25}$$

The corresponding return probability and mean square displacement are trivial.

$$p(t) = 1,\ \Delta x_t^2 = 0 \tag{26}$$

what we should note that is the energy spectrum of infinite disorder case is pure point spectrum.

V. 1D ONE-DIMENSION

To show localization in 1D lattice, here **we use the transfer matrix method analytically to prove that the resistivity grows exponentially with the total number of sites in the lattice**.

A. Transfer Matrix in the lattice

Firstly, we setup relations between neighboring sites, which are of vital importance for further derivation to connect the sites as well as sites and boundaries.

Note a_n as the wavefunction amplitude of the N-th lattice site.

For eigen-energy E, we have

$$Ea_n = \epsilon_n a_n + V_{n,n+1}a_{n+1} + V_{n-1,n}a_{n-1} \tag{27}$$

It can be written in the form of matrix equation:

$$\begin{pmatrix} a_{n+1} \\ a_n \end{pmatrix} = \begin{pmatrix} \frac{E-\epsilon_n}{V_{n,n+1}} & \frac{-V_{n-1,n}}{V_{n,n+1}} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_n \\ a_{n-1} \end{pmatrix} \tag{28}$$

The matrix $\begin{pmatrix} \frac{E-\epsilon_n}{V_{n,n+1}} & \frac{-V_{n-1,n}}{V_{n,n+1}} \\ 1 & 0 \end{pmatrix}$ is called transfer matrix and noted as p_n. With these transfer matrices, once we are given the wavefunctions of a pair of neighboring sites, we can know that of the sites next to them. Iteratively, we can know the wavefunctions of all sites in the random lattice. Note $P_N = \prod_{i=0}^{n} p_i$:

$$\begin{pmatrix} a_{n+1} \\ a_n \end{pmatrix} = \prod_{i=0}^{n} p_i \begin{pmatrix} a_1 \\ a_0 \end{pmatrix} = P_N \begin{pmatrix} a_1 \\ a_0 \end{pmatrix} \tag{29}$$

This leads to the recursion relations as follows:

$$\begin{aligned} P_n^{11} &= \frac{E-\epsilon_n}{V_{n,n+1}} P_{n-1}^{11} - \frac{V_{n-1,n}}{V_{n,n+1}} P_{n-2}^{11} \\ P_n^{21} &= P_{n-1}^{11} \\ P_n^{12} &= \frac{E-\epsilon_n}{V_{n,n+1}} P_{n-1}^{12} - \frac{V_{n-1,n}}{V_{n,n+1}} P_{n-2}^{12} \\ P_n^{22} &= P_{n-1}^{12} \end{aligned} \tag{30}$$

And if only diagonal disorders are considered, the transfer matrix would just be

$$p_n = \begin{pmatrix} \frac{E-\epsilon_n}{V} & -1 \\ 1 & 0 \end{pmatrix} \tag{31}$$

B. Resistivity

1. Goal and Workflow

In this approach, **our goal is to calculate the resistivity of wave and prove that it (its expectation value actually) grows exponentially with the number of sites in lattice (N)**. We use transfer matrices and S-matrices to connect disordered sites and boundary conditions. Then we get expression of $\langle\rho\rangle$, the expected value of resistivity.

Transfer matrices come from the Hamiltonian of the system while S-matrices bring their physical insight of reflection and transmission. The combination of them can thus yield the expression of the resistivity of the systems expressed by the certain Hamiltonian.

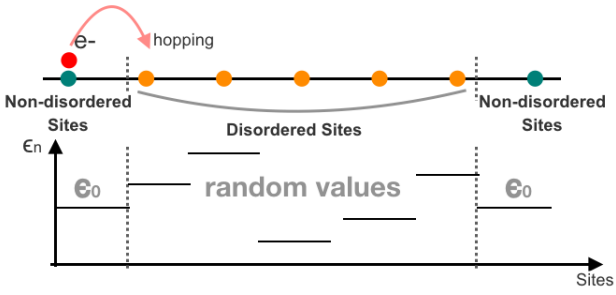

FIG. 1: Disordered lattice and non-disordered boundary condition

Consider electron flow shooting from the left into the random lattice. In this approach, we would show 1D localization by proving that the resistivity of the electron wave function, defined as $\rho = \frac{R}{T}$ where R and T refer to transmission and reflection rate, grows exponentially with N, the total number of random lattice sites.

The key in this approach is its boundary condition that out of the lattice, there is no disorders so the wavefunctions there are just plane waves.

2. Derivation

Here n is the label of a site in the lattice of disorder and d is the width between neighboring sites:

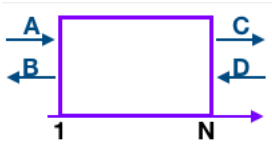

FIG. 2: Plane waves outside the lattice

$$\begin{aligned} a_n &= Ae^{ikdn} + Be^{-ikdn} && -\infty < n \le 1 \\ a_n &= Ce^{ikdn} + De^{-ikdn} && n \ge N \end{aligned} \tag{32}$$

From the wavefunctions above, the relation between E and k can also be determined as $E = 2V_0\cos(kd)$, which comes from dispersion in 1D lattice with no disorder.

Next, we relate two boundaries of the random lattice using the matrix T and express the resistivity ρ using the matrix elements:

Define T:

$$T\begin{pmatrix}B\\A\end{pmatrix} = \begin{pmatrix}D\\C\end{pmatrix} \tag{33}$$

Scattering matrix (S-matrix), which also relates two boundaries of the random lattice and thus looks pretty similar to the transfer matrix T, would help us build the relationship between T and ρ, where r, t and $\bar{r}, \bar{t}$ respectively give the reflected and transmitted amplitudes for waves incident from the left and right.

$$S = \begin{pmatrix}r & \bar{t}\\ t & \bar{r}\end{pmatrix} \tag{34}$$

$$S\begin{pmatrix}A\\D\end{pmatrix} = \begin{pmatrix}B\\C\end{pmatrix} \tag{35}$$

Our reflection and transmission rate can be expressed using the elements of S-matrices: $R = |r|^2$ and $T = |t|^2$.

Relating T and S we have:

$$\begin{aligned} r = S_{11} &= -\frac{T_{12}}{T_{11}} \\ t = S_{21} &= \frac{1}{T_{11}} \\ \Rightarrow \rho = \frac{R}{T} &= (-T_{12}^*)(-T_{12}) = |T_{12}|^2 \end{aligned} \tag{36}$$

To know how the boundaries relate to the sites inside the random lattice, we are supposed to find the relationship between T and P_N. In the following equations we note θ as $\begin{pmatrix}e^{ikNd} & 0\\ 0 & e^{-ikNd}\end{pmatrix}$ and Λ as $\begin{pmatrix}e^{-ikd} & e^{ikd}\\ 1 & 1\end{pmatrix}$:

$$\begin{aligned} \begin{pmatrix}a_{N+1}\\ a_N\end{pmatrix} &= \begin{pmatrix}e^{-ikd} & e^{ikd}\\ 1 & 1\end{pmatrix}\begin{pmatrix}e^{-ikNd} & 0\\ 0 & e^{ikNd}\end{pmatrix}\begin{pmatrix}D\\C\end{pmatrix} \\ &= \Lambda\theta^{-1}\begin{pmatrix}D\\C\end{pmatrix} \end{aligned} \tag{37}$$

Similarly, we can also derive the relation between a_1, a_0 and A, B

$$\begin{pmatrix}a_1\\a_0\end{pmatrix} = \begin{pmatrix}e^{-ikd} & e^{ikd}\\ 1 & 1\end{pmatrix}\begin{pmatrix}B\\A\end{pmatrix} = \Lambda\begin{pmatrix}B\\A\end{pmatrix} \tag{38}$$

Recalling Equations 33, 37 and 38, we can get

$$T = \theta\Lambda^{-1}P_N\Lambda \tag{39}$$

Setting $E = 0$ here for simplicity and convinience, since we have known that ρ is just T_{12}, ρ can be now expressed using the elements of P_N. And the expectation value of ρ (also the iterative relations for F_N, F_{N-1} and f_N, f_{N-1}) can be derived from the recursion relations by squaring and taking the expactation values of Equation 30. Defining $F_n = \langle (P_N^{11})^2\rangle$ and $f_n = \langle (P_N^{12})^2\rangle$ would give $\langle\rho\rangle$ a neat look:

$$\langle\rho\rangle = \frac{1}{4}[F_N + F_{N-1} + f_N + f_{N-1}] - \frac{1}{2} \tag{40}$$

The relation yielded for F_N and F_{N-1} (Similar for f_N and f_{N-1}) is written as

$$\begin{pmatrix}F_N\\F_{N-1}\end{pmatrix} = \begin{pmatrix}\langle\epsilon^2\rangle\langle\frac{1}{V^2}\rangle & \langle V^2\rangle\langle\frac{1}{V^2}\rangle\\ 1 & 0\end{pmatrix}\begin{pmatrix}F_{N-1}\\F_{N-2}\end{pmatrix} \tag{41}$$

Let $R = \begin{pmatrix}\langle\epsilon^2\rangle\langle\frac{1}{V^2}\rangle & \langle V^2\rangle\langle\frac{1}{V^2}\rangle\\ 1 & 0\end{pmatrix}$. We would use its eigenvalues and corresponding eigen vectors, as well as the boundary conditions to calculate the expressions of F_n, F_{n-1} in the expectation values of ϵ^2 and V^2 together with the eigenvalues of R.

The properties of the eigenvalues of R are the key for further derivation:

$$\lambda_\pm = \frac{1}{2}\langle\epsilon^2\rangle\langle\frac{1}{V^2}\rangle \pm \sqrt{(\frac{1}{2}\langle\epsilon^2\rangle\langle\frac{1}{V^2}\rangle + \langle V^2\rangle\langle\frac{1}{V^2}\rangle)} \tag{42}$$

The acuatual expression for the expectation value of ρ is as following. But we do not have to pay much attention to its details since we only want to research on its exponentially growing behavior when $N \to +\infty$:

$$\begin{aligned} \langle\rho\rangle = \frac{1}{4(\lambda_+ - \lambda_-)}\Bigg\{&\left(1 + \frac{\langle\epsilon^2\rangle}{V_0^2}\right)(\lambda_+^N - \lambda_-^N) \\ &+ \left(\frac{\langle V^2\rangle}{V_0^2} + V_0^2\langle\frac{1}{V^2}\rangle + \langle\epsilon^2\rangle\langle\frac{1}{V^2}\rangle\right)\cdot(\lambda_+^{N-1} - \lambda_-^{N-1}) \\ &+ \langle V^2\rangle\langle\frac{1}{V^2}\rangle\cdot(\lambda_+^{N-2} - \lambda_-^{N-2})\Bigg\} - \frac{1}{2} \end{aligned} \tag{43}$$

According to Schwartz Inequality of expectationvalues:

$$E(X^2)E(Y^2) \geq (E(XY))^2 \quad \Rightarrow \quad \langle V^2\rangle\langle\frac{1}{V^2}\rangle \geq 1 \tag{44}$$

Together with the condition that $\langle\epsilon^2\rangle > 0$, we have:

$$\lambda_+ > 1, \quad |\lambda_-| < \lambda_+ \tag{45}$$

Therefore, when $N \to +\infty$:

$$\boxed{\langle\rho\rangle \propto \lambda_+^N = e^{N\ln\lambda_+}} \tag{46}$$

Since $\lambda_+ > 1$, $\langle\rho\rangle$ **grows exponentially with N as $N \to +\infty$, leading to the localization of wave, no matter how big or tiny the randomness is.**

As for the situation of diagonal disorders only, $\langle V^2\rangle\langle\frac{1}{V^2}\rangle = 1$. Still we have the same properties for the eigenvalues, which lead to localization.

C. Numeric results of 1D system

After analyzing the 1D model analytically, we can solve the problem using three different numeric methods including Exact Diagonalization (ED), transfer matrix and time propagation of wavepacket.

1. *Exact Diagonalization*

If the system is located on the orthogonal lattice with Periodic Boundary Condition (PBC), the Hamiltonian of the system can be written in the matrix representation.

$$E\begin{pmatrix} a_1 \\ a_2 \\ ... \\ a_{L-1} \\ a_L \end{pmatrix} = \begin{pmatrix} \epsilon_1 & -g & ... & 0 & -g \\ -g & \epsilon_2 & ... & 0 & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & ... & \epsilon_{L-1} & -g \\ -g & 0 & ... & -g & \epsilon_L \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \\ ... \\ a_{L-1} \\ a_L \end{pmatrix} \tag{47}$$

Then the Anderson Localization becomes an eigenvalue problem.

Here we set $-g$ as 1, and sample the on-site energy ϵ_n from an uniform distribution $[-W/2, W/2]$. Then we can get eigenstates with different disorder strength.

If the disorder strength $W = 0$, the eigenstates are simple Bloch waves.

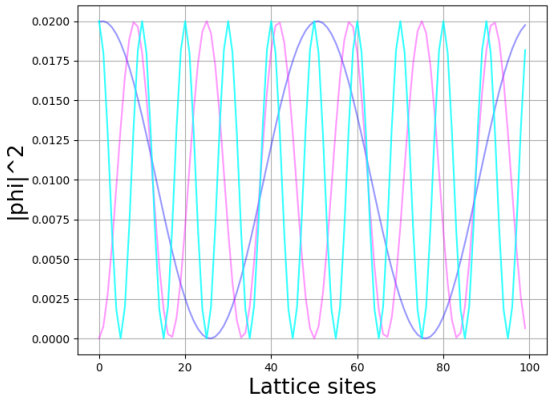

FIG. 3: Bloch waves: $W = 0$

If the disorder strength $W = 0.5$, the states are localized.

If we further improved the disorder strength $W = 2.0$, the localization of the wavefunction were strengthened.

Apart from the naive eigenstates, we can use Inverse Participation Ratio (IPR) to characterize the localization. Here, we define IPR as:

$$I = \sum_j |\Psi(j)|^4 \tag{48}$$

we can find that when the system is approaching thermodynamic limit, the correlation length versus energy curve will converges to a stabe one.

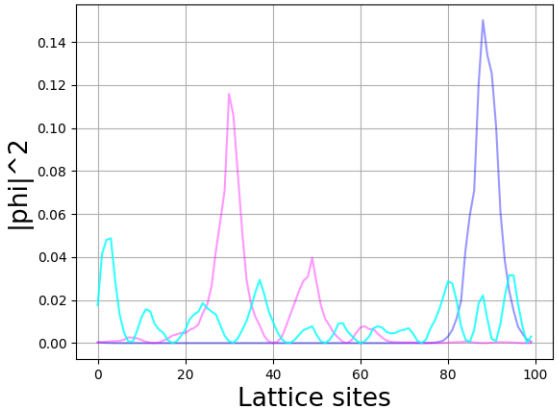

FIG. 4: Localized States: $W = 0.5$

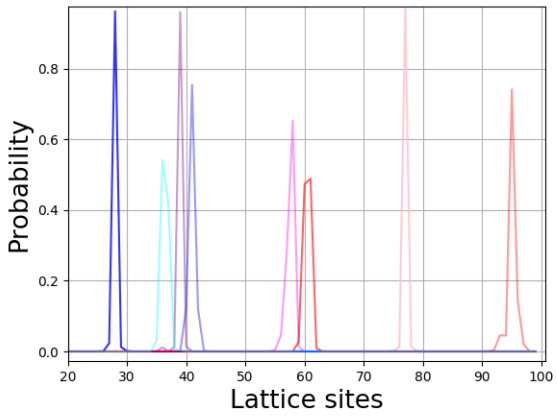

FIG. 5: Localized States: $W = 2.0$

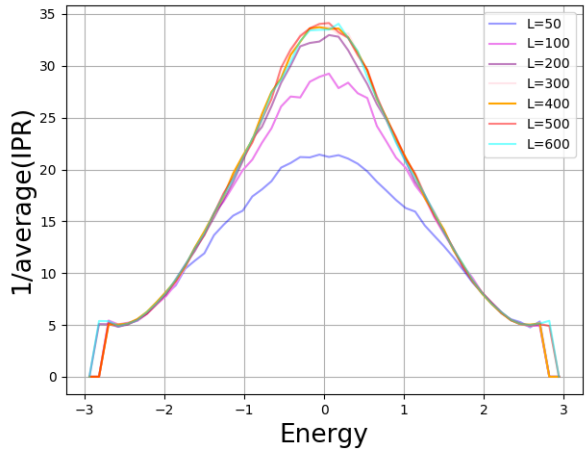

FIG. 6: Inverse participation ratio versus energy with different system size: $W = 2.0$

If the IPR is higher, the states are more localized and vice versa. Realizing the model twenty times to avoid fluctuation, we can calculate the IPR distribution over eigen energies with different disorder strength

Furthermore, IPR can be used to construct a phase diagram in disorder strength W and energy E parameter space. Unfortunately, because there is no delocalization in 1D system, there is merely one phase in 1D case.

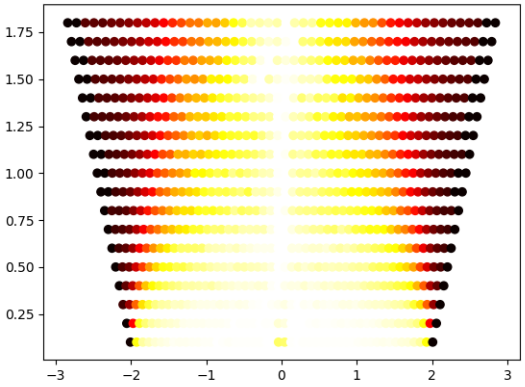

FIG. 7: Phase diagram in disorder strength W and energy E parameter space

2. *Transfer matrix*

The correlation length is another important information in the Anderson localization.

By applying transfer matrix on the initial vector (a_1, a_0) thousands of times, we can get the average ampltitude a_n , which can be used to calculate the correlation length.

we can find that when the system is approaching thermodynamic limit, the correlation length versus energy curve will differ.

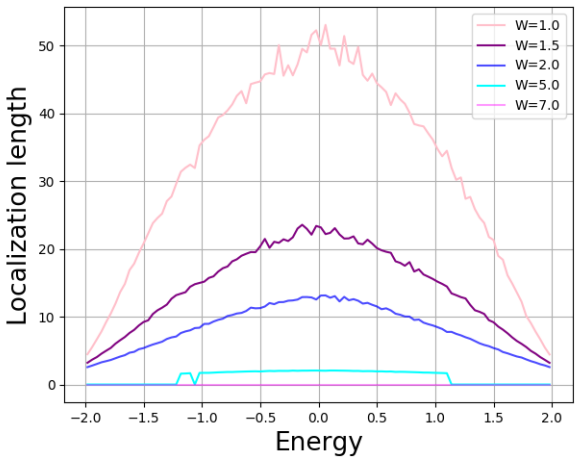

FIG. 8: Correlation length versus energy with different system size

If we can calculate the correlation length with more different disorder strength. We can find that if the energy of the eigenstate is fixed, the correlation length scales as $\xi = \alpha \cdot W^2$.

3. *Time propogation*

Dynamical process can also support this problem. Here, we use the quantum dynamics to investigate the time evolution of an wave packet.

Starting with initial state as δ function, we can calculate the time-dependent Schödinger equation. When $t = 100s$, we can get the wavepackets with different disorder strength.

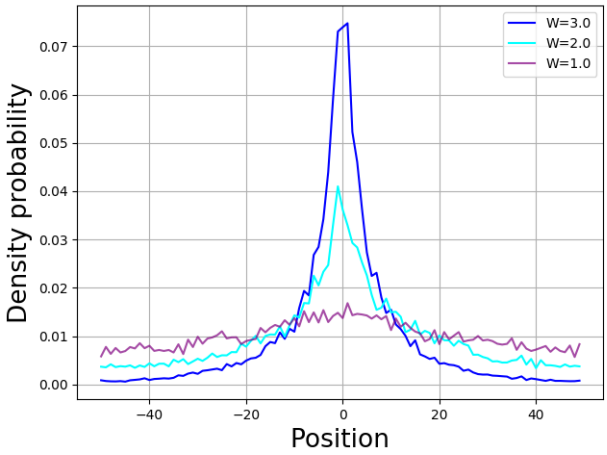

FIG. 9: Time evolution at $t = 100s$ with different disorder strength

If the disorder strength is stronger, the wavepacket at $t = 100s$ is more localized.

VI. SCALING THEORY

Now we finished the derivation of 1D Anderson localization using transfer matrices. However, using similar approaches in 2D or 3D situations would be conceivably difficult. To deal with such difficulties, here we introduce a smart and easy way to investigate 2D and 3D lattices.

In this part L refers to the length of the lattice in one direction (sidelength) and A is the cross section: $A = L^{d-1}$, where d is the dimension of the lattice. Define the conductance g as the inverse of resistivity we have defined before: $g = \frac{1}{\rho}$, which is related to the transmission in one direction as

$$g = \frac{1}{\rho} = \frac{T(L)}{R(L)} = \frac{T(L)}{1 - T(L)} \tag{49}$$

Scaling Theory methods applying to Anderson Localization dates back to "Gang of four": Abrahams, Anderson, Licciardello, and Ramakrishnan[4], where the crucial step was to claim that the conductance at changed scale is the function of its previous scale and the scale itself, which can be expressed as

$$g(nL) = F(n, g(L)) \tag{50}$$

Define $\beta = \frac{d\ln(g)}{d\ln(L)}$ to show how g would change when tuning L. And we would research on the limiting behavior of β when g is extremely large or small to get the properties of the conductance in 1/2/3D situations.

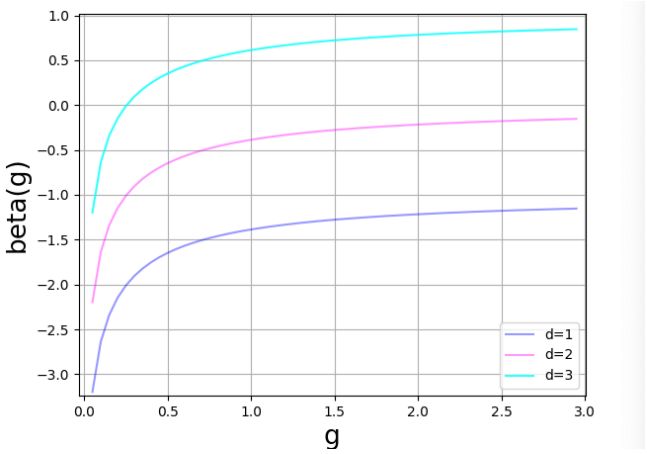

FIG. 10: Scaling Theory

$g \to +\infty$: In the case of a good conductor $g \gg 1$, according to Ohm's law

$$R = \rho\frac{L}{A} = \rho\frac{L}{L^{d-1}} \quad \Rightarrow \quad g = \sigma_0 L^{d-2} \tag{51}$$

Then we can easily obtain

$$\frac{d\ln(g)}{d\ln(L)} = d - 2 \quad \Rightarrow \quad \lim_{g\to+\infty}\beta(g) = d - 2 \tag{52}$$

$g \to 0$: In the limit of very strong disorder, wavefunctions will behave as exponentially localized

$$g(L) \propto e^{-L/\xi} \quad \Rightarrow \quad \frac{d\ln(g)}{d\ln(L)} = -\frac{L}{\xi} = \ln(g) \tag{53}$$

Thus we have

$$\lim_{g\to 0}\beta(g) = \ln(g) \tag{54}$$

Having the limiting behavior of $\beta(g)$ for $g \to 0$ and $g \to +\infty$,a plot can be made (FIG10), where the function we used in plotting is $\beta(g) = (d-1) - (1+g)\ln(1+g^{-1})$.

According to the limiting behavior and the plot of $\beta(g)$:

For 1D and 2D systems, β never exceeds x-axis so the conductane would always decrease under the extension in one direction since $\beta(g) = \frac{d\ln g}{d\ln L}$ is always negative. In the limit of $L \to +\infty$, there would be localization since its conductance would approach 0.

For 3D systems, however, it is special that there is an turning point g_c where $\beta(g_c) = 0$, indicating the behavior of transition from insulator to metal. Therefore, localization would only exist with large disorder. As for tiny disorder, the density of wavefunctions may not decay to cause localization. g_c refers to a certain degree of disorder connecting insulator and metal.

VII. PERTUBATIVE METHODS ON HIGHER DIMENSION

Apart from the transfer matrix, some advanced pertubative methods such as locator expansion and forward approximation were proposed in past several years.

A. Locator expansion for the resolvent

The locator expansion was firstly proposed by Anderson, which used Random Walk representation to expand pertubatively the resolvent at the vicinity of infinite disorder case (compeletly localized state).

The non-pertubative resolvent is

$$G(E) := \frac{1}{E - H} \tag{55}$$

where $G(a, b, E) = \langle a|\frac{1}{E-H}|b\rangle$ is the localized matrix element.

If we expand the resolvent at the vicinity of the compeletely localized state ($W \longrightarrow \infty$), we can get expansion series similar with the Dyson series.

$$G = G_0 + \sum_{n=1}^{+\infty} g^n \left(G_0 T\right)^n G_0 \tag{56}$$

Using different approximations/representations, we can get different expansions.

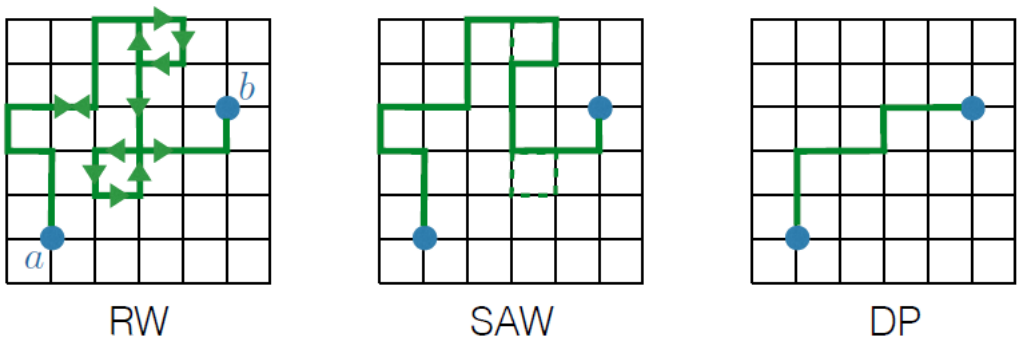

FIG. 11: RW: naive pertubation expansion for the resolvent vetween a and b; SAW: Self-avoid random walk representation using renormalized pertubation theory; DP: Forward approximation merely considering shortest path [2]

Naive Random Walk

$$G(b,a,E) = \frac{\delta_{a,b}}{E-\epsilon_a} + \frac{1}{E-\epsilon_a}\sum_{n=1}^{+\infty}\sum_{\pi:a\to b,|\pi|=n}(-g)^n\prod_{s=1}^{n}\frac{1}{E-\epsilon_{\pi(s)}} \tag{57}$$

Self-avoiding Random Walk

$$G(b,a,E) = \sum_{\text{SAW}}\sum_{\pi:a\to b,|\pi|=n}(-g)^n\prod_{s=1}^{n}\frac{1}{E-\epsilon_{\pi(s)}-\sum_{\pi(s)}^{\{\pi(0),\cdots,\pi(s-1)\}}(E)} \tag{58}$$

B. Forward Approximation

If we only keep the shortest path in our expansion, we can get a more rough expression.

$$\psi_\alpha(b) \simeq \sum_{DP\pi:a\to b}(-g)^n\prod_{s=1}^{n}\frac{1}{\epsilon_\alpha-\epsilon_{\pi(s)}} \tag{59}$$

It is called the Forward Approximation.

This was a approxiamted expansion of the resolvent, therefore the localization might be under-estimated during our calculation. For example, the critical disorder strength in $2D$ case calculated by Forward Approximation is not zero $W_c|_{2d} \neq 0$.

C. Numerci results of Higher dimension

1. 2D lattice eigenstates

Similarly for 2D, a 20×20 lattice, periodic boundary conditions are applied to make the 2D lattice a torus. We still choose $E_0 = 10$. Again, two eigen-states are randomly chosen and presented in FIG 12 and 13.

The shape of resulting eigen-states looks like Bugles, showing the behavior of localization.

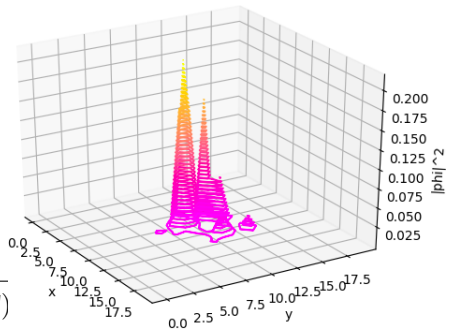

FIG. 12: An eigenstate of 2d lattice

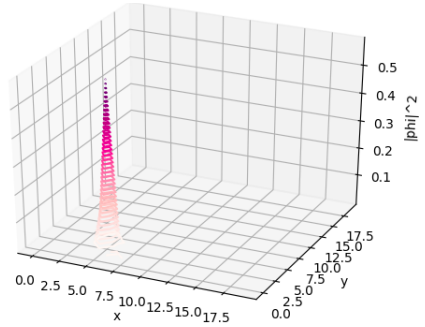

FIG. 13: An eigenstate of 2d lattice

2. 2D lattice wavepacket evolution

Apart from the eigenstates, the time-evolution of the wavepacket can also be calculated.

3. 3D lattice eigenstates

From the Scaling Theory part we have seen that 3D situation is special because of the existence of turning point indicating that there would be a transition from insulator to metal. It is stated[**?**] that the critical disorder

value depends on the band energy E. So in the simulation where we tune the randomness trying to show the transition, we chose the eigenstates of nearly the same energy (eigenvalues),which is the one nearest to 0 in different randomness.

FIG. 17 shows the situations where E_0 is respectively 1,10 and 25. We can see the eigenstates transition from delocalization to localization. But the specific turning point cannot be determined here.

VIII. EXPERIMENTAL OBSERVATION

Billy et al.[3] are the first to directly observe exponential spatial localization of matter-waves. They reported the observation of exponential localization of a Bose-Einstein condensate released into a one-dimensional waveguide with controlled disorder created by laser speckle. Here we briefly introduce their experiments.

They operated weak disorder such that localization would result from quantum reflections instead of bounded by huge potential basin and walls. Besides, they controlled the atomic density to be very small thus the interaction can be neglected.

In a horizontal optical waveguide providing strong transverse confinement, a magnetic longitudinal trap and a weak disorderd optical potential are applied. In the beginning, a Bose-Einstein condensate is formed in the trap. When the longitudinal magnetic trap is turned off, the Bose-Einstein condensate begins to expand then becomes localized.

In FIG18, a) is the beginning state of trapped Bose-Einstein condensate and b) is when the magnetic trap goes off and it is localized. c) and d) show the original and fitting curve of the atomic density, where pink lines refer to the initial state and red lines refer to observed localization. Logarithmic y coordinate is applied in d), showing that the density decays exponentially.

Apart from the matter wave observation, anderson localization of solitons are also overserved, which is a very interesting phenomenon.[11]

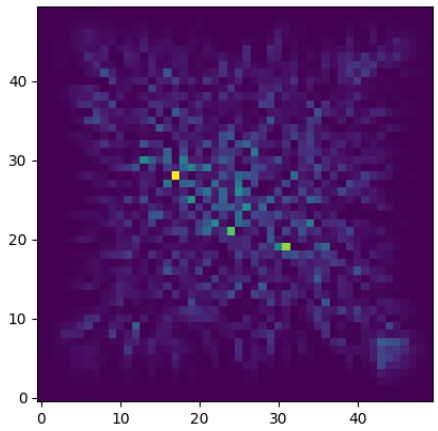

FIG. 14: Wave packet time evolution of 2d lattice whith $W = 2.0$ at $t = 50$

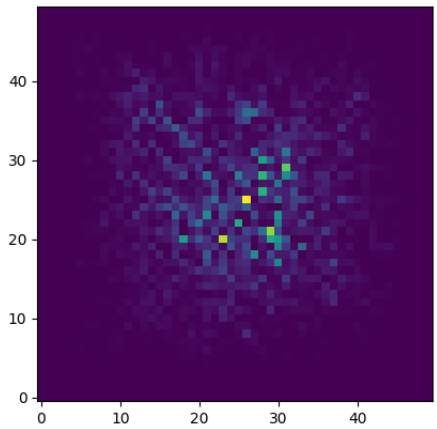

FIG. 15: Wave packet time evolution of 2d lattice whith $W = 3.0$ at $t = 50$

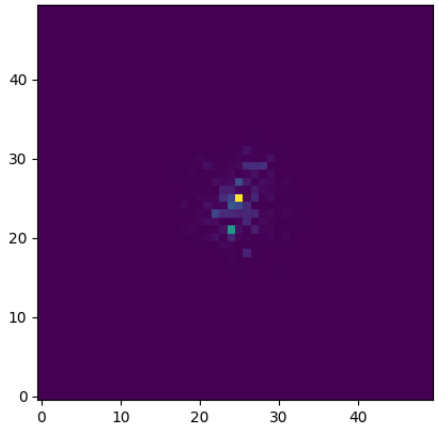

FIG. 16: Wave packet time evolution of 2d lattice whith $W = 10.0$ at $t = 50$

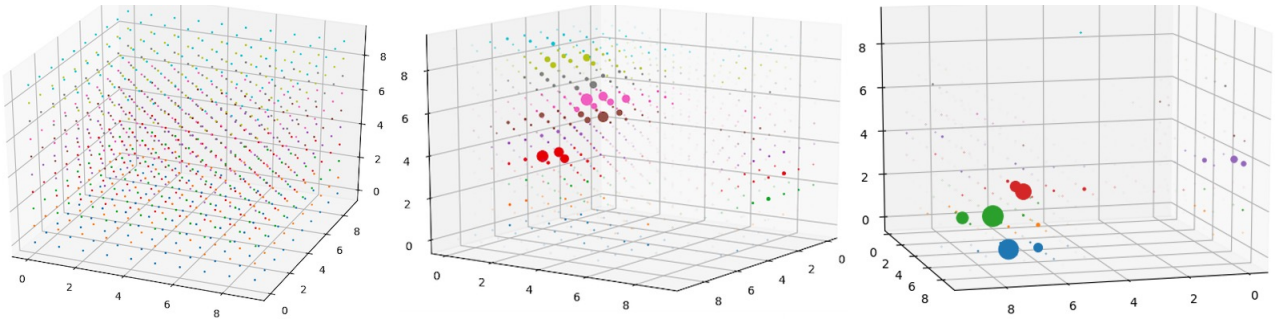

FIG. 17: 3D lattice eigenstate: Left: $E_0 = 1$. No localization is seen. Middle: $E_0 = 10$. Weak wavefunction localization can be seen. Right: $E_0 = 25$. We can see strong localization.

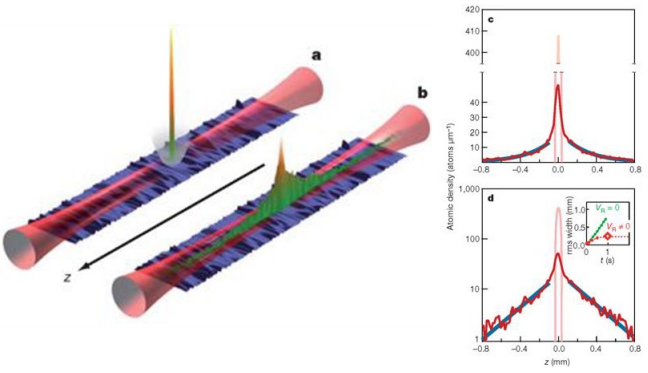

FIG. 18: Anderson localization of a Bose-Einstein condensate, taken from [3]

IX. SUMMARY

In the whole paper we made exploration of Anderson Localization. We have done the derivation of 1D using transfer-matrices and used scaling theory to show the difference of 1/2D and 3D systems: Only in 3D lattices we can have transition from metal to insulator. Besides, we did simulation of 1, 2 and 3D localization to visualize our results of scaling theory. Finally we introduced an experimental observation of Anderson localization in BEC. Actually there are far more fancy content related to Anderson localization. For example, anderson localization can be observed even in light[10]. Sixty years have witnessed the importance of Andserson localization, yet it would still yield brand new discoveries.

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