Spin Coherent States and Statistical Physics

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I review the basic properties of spin coherent states, and explain how they may be used to relate 1 dimensional quantum chains to 2 dimensional classical statistical physical models. In particular, I review the map from quantum antiferromagnets to the $O(3)$ NLSM with a topological term, and discuss the Haldane conjecture.

I. INTRODUCTION

Because of the enhanced role of fluctuations, low-dimensional quantum and thermal systems often behave very differently than their high dimensional counterparts. An important example of this is the Mermin-Wagner theorem, which states that continuous symmetry-breaking order is not possible at finite temperature in two dimensions. While this rules out a symmetry-breaking phase transition in 2d, a more exotic phase transition driven by the proliferation of vortices, the BKT transition, *is* possible at finite temperature. Since the transfer matrix formalism may be used to relate classical statistical physics and quantum mechanics in one lower space dimension, it is interesting to ask what these results can tell us about one dimensional quantum systems. Studying this is the purpose of this note.

This connection appears in many ways. For instance, the one dimensional spin-1/2 XXZ model, with Hamiltonian:

$$H_{\mathrm{XXZ}} = J\sum_n S_n^x S_{n+1}^x + S_n^y S_{n+1}^y + \Delta S_n^z S_{n+1}^z$$

may be mapped to the classical six-vertex model [1], and the isotropic antiferromagnetic point $\Delta = 1, J > 0$ on the quantum side corresponds to the BKT critical point on the classical side [2]. But the result this note aims for is Affleck's argument [3] for the Haldane conjecture [4]. The Haldane conjecture [5] is that one dimensional Heisenberg antiferromagnets display drastically different behavior depending on whether the spins are integral or half-integral, with the former leading to a gapped ground state and the latter leading to a gapless ground state. A crucial step in the argument is mapping the quantum system onto a 2d $O(3)$ nonlinear sigma model.

This mapping may be accomplished in a variety of ways, but following [6], we will use the formalism of spin coherent states. To that end, after reviewing the quantum-classical correspondence provided by the transfer matrix method, we will introduce spin coherent states and their basic properties. Then, we will use spin coherent states to construct classical systems corresponding to a single spin in a magnetic field, and a pair of spins interacting via an isotropic exchange interaction. Next, we will show how the Berry phase term for a single spin becomes a topological term for the action of a spin chain, and we will obtain the NLSM for a 1d antiferromagnetic chain. Finally, we will discuss how to use this NLSM framework to establish the Haldane conjecture.

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II. TRANSFER MATRICES RELATE QUANTUM AND CLASSICAL PHYSICS

In this section, we review how the transfer matrix method provides a connection between d-space-dimensional quantum mechanics and $d+1$-space-dimensional classical statistical physics. This has been taken up by many authors (see, for instance, [1]), including past 8.334 projects [7]. We wish to emphasize the role of anisotropy on the classical side and the specific relationship between quantum and thermal fluctuations (and what that means for, e.g., the Mermin Wagner theorem).

Suppose we have a classical Hamiltonian which depends on some parameters g and the configuration σ as:

$$H(g,\sigma) = \sum_{i=1}^{N} \mathcal{H}[g;\sigma_i,\sigma_{i+1}],$$

where σ_i is the configuration of the i'th "row" of the system. Suppose also that $\mathcal{H}$ is symmetric in the σ's. Then if we introduce $T(g,\beta)$ via its matrix elements as:

$$\langle\sigma'|T(g,\beta)|\sigma\rangle = e^{-\beta\mathcal{H}[g;\sigma',\sigma]}$$

then we can write:

$$Z = \sum_{\{\sigma\}} e^{-\beta H} = \mathrm{tr}\left(T(g,\beta)^N\right) = \mathrm{tr}\left(e^{-\beta_q H_q}\right).$$

Thus, the equation:

$$e^{-\beta\mathcal{H}[g;\sigma',\sigma]} = \langle\sigma'|e^{-\beta_q H_q/N}|\sigma\rangle \tag{1}$$

provides a mapping from classical Hamiltonians $\mathcal{H}$ and quantum hamiltonians H_q which relates their partition functions. Computing expectation values for classical observables corresponds to inserting diagonal operators into the trace on the quantum side.

Let us first consider $0+1$ dimensional quantum systems. When the quantum mechanical system is zero-dimensional and the classical mechanical system is one dimensional, (1) is often satisfactory as is. For example, it may be used to provide a correspondence between a quantum-mechanical spin-1/2 particle in an external field and the classical Ising chain. For such a particle, we may compute:

$$\exp\left(\gamma\sigma_x\right) = \begin{pmatrix} \cosh\gamma & \sinh\gamma \\ \sinh\gamma & \cosh\gamma \end{pmatrix},$$

which tells us:

$$\beta\mathcal{H}[\sigma,\sigma'] = -\frac{1}{2}\log\left(\frac{1}{2}\sinh(2\gamma)\right) + \frac{1}{2}\log(\tanh(\gamma))\sigma\sigma',$$

which is a classical Ising chain. Notice that the bigger γ is, the smaller $\log(\tanh(\gamma))$ is, and thus the stronger the thermal fluctuations on the classical side become. This makes sense since on the quantum side $|\rightarrow\rangle$ has maximum uncertainty for S_z, so the larger γ is, the stronger the quantum fluctuations of S_z.

For generic quantum-mechanical systems with a spatial extent, though, equation (1) is unsatisfactory because the exponentials involved do not preserve the short-rangedness of the interaction. In other words, a classical Hamiltonian with only nearest-neighbor interactions would nevertheless produce a (very complicated) quantum-mechanical H_q with nonzero interactions at arbitrarily long range, and vice-versa. This problem may be averted by working in a highly anisotropic region of the classical parameter space: if there is a very high energy cost for σ to differ from σ', then the transfer matrix will be close to the identity, and can thus be safely Taylor expanded. This is the procedure used to accomplish the previously mentioned map between the six-vertex model and the spin-1/2 XXZ model.

Let us examine what happens to the Mermin-Wagner theorem under this correspondence. In classical statistical physics, in the form presented in 8.334, the Mermin-Wagner theorem rules out the possibility of symmetry-breaking order for a continuous symmetry in 2 spatial dimensions. The order is destroyed by thermal excitation of the Goldstone modes. One might thus suspect that, as a result of the previously discussed correspondence, quantum fluctuations would rule out the possibility of continuous symmetry breaking order in quantum chains. However, it is easy to devise a counterexample, like the Heisenberg ferromagnet:

$$H = -J\sum_i \vec{S}_i\cdot\vec{S}_{i+1}.$$

Any product state where all the spins have maximal projection along the same axis is a ground state of this system, which therefore spontaneously breaks rotation symmetry.

What has gone wrong? I wish to argue, elaborating on a brief comment to [8], that when this occurs, the corresponding classical system is at zero temperature (meaning all configurations with a nonzero weight have the same weight). However, unlike [8], I do not believe that the unentangled nature of the Heisenberg ground state is relevant, and in fact something similar can occur in $0+1$-dimensional quantum systems where entanglement isn't even possible.

Making a quantum-classical correspondence using a transfer matrix requires a choice of basis on the quantum side, and thermal fluctuations on the classical side result from the fact that the quantum-mechanical ground state is a superposition of various vectors in this basis. But if the ground state *belongs* to the basis, this superposition is trivial, and the classical system is at zero temperature. For instance, if we had used σ_z instead of σ_x in our earlier example of a single quantum spin, we would have found that the corresponding classical system assigns a finite weight only to the state where all spins are up.

There are more interesting gapless states than the aforementioned Heisenberg ferromagnet ground state, like the Heisenberg antiferromagnet ground state alluded to in the introduction. But before studying that, we must introduce the relevant piece of mathematical machinery, namely spin coherent states.

III. SPIN COHERENT STATES

In the previous section, we explained how to rewrite a classical partition function as a quantum-mechanical one by introducing a quantum Hamiltonian equal to the logarithm of the transfer matrix. How do we go in the other direction? If we insert many resolutions of the identity $\mathbf{1} = \sum_n |n\rangle\langle n|$ into the quantum-mechanical partition function, the sums in the resolution of the identity become a sum over classical configurations, and products of the matrix elements of $e^{-\beta_q H_q/N}$ become classical Boltzmann weights. Thus, any way of resolving the identity provides a way to go from a quantum Hamiltonian to a classical one.

A particularly convenient resolution of the identity for studying spin systems is provided by the spin coherent states. In this section, we establish the properties of the spin coherent states (loosely following [6]) and use them to study the classical analogue of a quantum particle in a magnetic field. We notice a complication: there is no guarantee that the Boltzmann weights obtained by this procedure are real. As also observed in [7], these **Berry phases** are physically very important.

Following [6], we define the spin coherent state associated with a particular direction $\vec{n}$ as the state obtained by rotating the highest-weight S_z eigenstate to point in the $\vec{n}$ direction. We accomplish this by rotating by an amount θ along the axis parallel to $\vec{n}\times\hat{z}$[9]:

$$|\vec{n}\rangle = \exp\left(i\theta(\widehat{\vec{n}\times\hat{z}})\cdot\vec{S}\right)|\hat{z}\rangle.$$

In the same way that coherent states for the harmonic oscillator may be pictured as a sharply peaked (area $\hbar$) Gaussian in the (x, p) plane, the spin coherent states point in a particular direction to the greatest extent allowed by the angular momentum commutation relations. In particular, as the spin of the particle gets larger, the angular uncertainty decreases.

Let us evaluate the overlap of two spin coherent states. This will help clarify the preceding remark, and it will be useful in later calculations. The result is:

$$|\langle\vec{n}_1|\vec{n}_2\rangle| = \left(\frac{1+\vec{n}_1\cdot\vec{n}_2}{2}\right)^s.$$

As s gets big, the overlap falls off more rapidly as $\vec{n}_1$ and $\vec{n}_2$ get father apart. We sketch a proof, following [10, 11]. First, we re-express the coherent state in terms of the angular momentum raising and lowering operators $S_\pm = S_x \pm iS_y$:

$$|\vec{n}\rangle = \exp\left(\frac{\theta}{2}(S_- e^{i\phi} - S_+ e^{-i\phi})\right)|\hat{z}\rangle.$$

Since S_+ annihilates $|\hat{z}\rangle$, we want to use a Baker-Campbell-Hausdorff style argument to get it out of the exponential. This is done in [11] by proving the following fact in the spin-1/2 representation, which thus holds in all representations since the spin-1/2 rep is faithful:

$$\exp\left(\frac{\theta}{2}(S_- e^{i\phi} - S_+ e^{-i\phi})\right) = e^{\mu S_-} e^{-\log(1+|\mu|^2)S_z} e^{-\mu^* S_+}.$$

Using that $|\hat{z}\rangle$ is an eigenstate of S_z gives:

$$|\vec{n}\rangle = \left(1+|\mu|^2\right)^{-s} e^{\mu S_-}|\hat{z}\rangle \tag{2}$$

where we have introduced:

$$\mu = e^{i\phi}\tan\left(\frac{\theta}{2}\right).$$

To compute the overlap:

$$\langle\vec{n}_1|\vec{n}_2\rangle = \left(1+|\lambda|^2\right)^{-s}\left(1+|\mu|^2\right)^{-s}\langle\hat{z}|e^{\lambda^* S_+}e^{\mu S_-}|\hat{z}\rangle,$$

we may perform a similar maneuver to obtain:

$$\langle\vec{n}_1|\vec{n}_2\rangle = \left(1+|\lambda|^2\right)^{-s}\left(1+|\mu|^2\right)^{-s}(1+\lambda^*\mu)^{2s}.$$

The modulus of this gives the result quoted above, while the phase is $e^{is\Phi(\vec{n}_1,\vec{n}_2,\hat{z})}$, where Φ gives the area of the spherical triangle defined by the 3 points.

The last property of the spin coherent states we need is that they resolve the identity. This appears to be done by using (2) to obtain the components of $|\vec{n}\rangle$ in the S_z eigenbasis, and then explicitly evaluating the integral over S^2. The result is:

$$\mathbf{1} = \frac{2s+1}{4\pi}\int d\Omega\,|\vec{n}\rangle\langle\vec{n}|.$$

Having reviewed some important properties of the spin coherent states, let us now apply them to some 0-dimensional quantum systems of interest. We begin with a particle in an external magnetic field. In [6], the author considers a continuous classical analogue, but I would like to consider a discrete one (i.e. a chain). This will allow us to avoid making potentially hard-to-justify approximations. We are thus interested in evaluating $\langle\vec{n}_1|e^{-\beta_q H_q}|\vec{n}_2\rangle$, where $\beta_q H_q = \gamma S_z$.

The trick here is to Wick rotate: if we let $\gamma = it$, the exponential becomes a rotation matrix, which acts on the coherent state to produce another coherent state (and a phase). The overlap can then be computed using the previously established formula. When this is done, we replace it by γ and we are done. In particular, we have:

$$e^{-itS_z}|\vec{n}_2\rangle = e^{-its}|R\vec{n}_2\rangle,$$

which allows us to compute:

$$\langle\vec{n}_1|e^{-\gamma S_z}|\vec{n}_2\rangle = \left[e^{-\gamma/2}\cos\left(\frac{\theta_1}{2}\right)\cos\left(\frac{\theta_2}{2}\right) + e^{i(\phi_2-\phi_1)+\gamma/2}\sin\left(\frac{\theta_1}{2}\right)\sin\left(\frac{\theta_2}{2}\right)\right]^{2s}$$

Taking the log of this gives our classical $\mathcal{H}$ analogue, which is complex. Nevertheless, it is amusing to take this model seriously and to evaluate e.g. correlation functions in it. We have:

$$\begin{aligned}\langle S_{0,i}S_{n,j}\rangle &= \frac{1}{Z}\sum_{\text{cfgs}} e^{-\beta H} S_{0,i}S_{n,j} = \frac{1}{Z}\int_{\{S_i\}} S_{0,i}\langle S_0|e^{-\gamma S_z}|S_1\rangle\langle S_1|e^{-\gamma S_z}\cdots|S_n\rangle S_{n,j}\langle S_n|\cdots|S_N\rangle\langle S_N|e^{-\gamma S_z}|S_0\rangle,\\ &= \frac{1}{Z}\int_{S_0,S_n} S_{0,i}S_{n,j}\langle S_0|e^{-n\gamma S_z}|S_n\rangle\langle S_n|e^{-(N-n)\gamma S_z}|S_0\rangle.\end{aligned}$$

In the limit $N\to\infty$, we have:

$$\langle S_{0,i}S_{n,j}\rangle = \begin{pmatrix} Ae^{-n\gamma} & iAe^{-n\gamma} & 0\\ -iAe^{-n\gamma} & Ae^{-n\gamma} & 0\\ 0 & 0 & B\end{pmatrix},$$

where:

$$A = \frac{2s}{(2s+2)^2},$$
$$B = \frac{(2s)^2}{(2s+2)^2}.$$

The B term is due to the disconnected expectation $\langle S_{0,i}\rangle\langle S_{n,j}\rangle$ which comes from the external potential, while the upper left block comes from correlations in the quantum fluctuations. Note that in the large s limit we recover classical behavior, as we would expect.

We may also consider two spins interacting by a Heisenberg term $\vec{S}^{(1)} \cdot \vec{S}^{(2)}$. I could not find a simple closed form for the classical Hamiltonian $\mathcal{H}$. For correlation functions, let us consider the case $s = 1/2$ and calculate:

$$\left\langle \vec{S}^{(1)}_{0,i} \vec{S}^{(1)}_{n,j} \right\rangle = \delta_{ij} \frac{2 + e^{(N-n)\gamma} + e^{n\gamma}}{27 + 9e^{N\gamma}}.$$

The qualitative behavior as we send N to ∞ depends on whether $\gamma > 0$ (i.e. antiferromagnetic) or $\gamma < 0$ (ferromagnetic). In the antiferromagnetic case, the terms multiplied by $e^{N\gamma}$ dominate, and we have:

$$\left\langle \vec{S}^{(1)}_{0,i} \vec{S}^{(1)}_{n,j} \right\rangle = \delta_{ij} \frac{e^{-n\gamma}}{9}.$$

In the ferromagnetic case, the terms multiplied by $e^{N\gamma}$ are suppressed, and we have:

$$\left\langle \vec{S}^{(1)}_{0,i} \vec{S}^{(1)}_{n,j} \right\rangle = \delta_{ij} \frac{2 + e^{-n|\gamma|}}{27}.$$

These are qualitatively very different behaviors, with the classical analogue of the ferromagnet exhibiting long range order and the classical analogue of the antiferromagnet being disordered. The reason for this is that a product spin coherent state is a ground state for the ferromagnet, but not the antiferromagnet—the only ground state of the antiferromagnet is in fact maximally entangled. Nevertheless, the classical ensemble corresponding to the ferromagnetic pair of spins is still at a finite temperature. This appears to contradict the argument I gave in the introduction that if the ground state of the system is one of the states in the basis used to construct the classical analogue, the classical analogue must be at zero temperature. But that argument assumed that the states used to resolve the identity were all orthogonal to each other, which the spin coherent states are not—the finite temperature comes from the nonzero overlap of the spin coherent states.

IV. SPIN CHAINS

Now that we have some experience with spin coherent states and their application to the quantum-classical correspondence, we consider the application of these techniques to the problem of 1d antiferromagnetic Heisenberg spin chains. We will show that these map onto an $O(3)$ nonlinear sigma model with a topological term. Then, we review an argument, first due to Affleck [3], that uses this language to explain why half-integer antiferromagnetic spin chains are gapless while integer-spin ones are gapped (the Haldane conjecture).

Instead of considering discrete imaginary time, as before, imagine inserting enough resolutions of the identity that the classical configuration may be thought of as a continuously evolving spin. Then, following Fradkin, it is a reasonable approximation that the classical Hamiltonian as a function of $\vec{n}(t)$ is just the expectation of the quantum Hamiltonian in the $|\vec{n}\rangle$ state, plus the log of $\langle \vec{n}(t)|\vec{n}(t+\delta t)\rangle$. From the previously evaluated overlap, this log will have a real part equal to:

$$\mathrm{re}\log\langle \vec{n}(t)|\vec{n}(t+\delta t)\rangle = s \log\left(\frac{1 + \vec{n}(t)\cdot\vec{n}(t+\delta t)}{2}\right),$$

which in the continuum limit will produce a kinetic term which penalizes dramatic changes in $\vec{n}$. The imaginary part, the so-called **Wess-Zumino term**, gives the area swept out by the line connecting the spin to the north pole:

$$\mathrm{im}\log\langle \vec{n}(t)|\vec{n}(t+\delta t)\rangle = \Phi(\vec{n}(t), \vec{n}(t+\delta t), \hat{z}).$$

To summarize, the spin coherent state formalism maps any quantum spin system onto a classical system in one higher dimension, with a term coming from the quantum Hamiltonian, a kinetic term for the imaginary time direction, and the imaginary Wess-Zumino term:

$$\mathcal{H} = \mathcal{H}_Q + \mathcal{H}_{\mathrm{kin}} + \mathcal{H}_{\mathrm{WZ}}.$$

Let us consider what happens to the Wess-Zumino term on a 1d antiferromagnetic chain. This argument closely follows [6], but I have attempted to be more wordy since I found that treatment quite telegraphic. For a single spin, the Wess-Zumino term gave us the area swept out by the trajectory of the spin in imaginary time [12]. For the chain, since the spins are staggered (i.e. it is favorable for the spins to be antialigned), the area swept out by one spin will nearly cancel the area swept out by its neighbor. To make this precise, let us characterize the state of our chain using unit vectors $\vec{n}_i$ which are staggered relative to the physical orientation of the spins. This means, for example, that a constant $\vec{n}_i$ configuration corresponds to the Néel state. Then the topological portion of the action may be written:

$$S_{\mathrm{top}} = S\sum_{j=1}^{N}(-1)^j \mathcal{S}_{\mathrm{WZ}}[\vec{n}(j)],$$

where $\mathcal{S}_{\mathrm{WZ}}$ is the area swept out by $\vec{n}$ as it evolves and the $(-1)^j$ comes from the fact that the n's are staggered. We now group this sum into pairs of terms to obtain:

$$S_{\mathrm{top}} = S\sum_{j\ \mathrm{odd}} \mathcal{S}_{\mathrm{WZ}}[\vec{n}(j+1)] - \mathcal{S}_{\mathrm{WZ}}[\vec{n}(j)],$$

If we let $\delta\vec{n}(j)$ be the (assumed small) difference between $\vec{n}(j)$ and $\vec{n}(j+1)$, we may evaluate this area difference by accumulating small quadrilaterals with sides $\delta\vec{n}(j)$ and $\partial_0\vec{n}(j)$, where x_0 is time. The area of each small quadrilateral may be obtained by dotting the sides after rotating one by 90° by crossing with $\vec{n}(j)$:

$$S_{\rm top} = S\sum_{j\ {\rm odd}}\int_0^T dx_0\, \delta\vec{n}(j)\cdot(\vec{n}(j)\times\partial_0\vec{n}(j)). \qquad (3)$$

At this point, following Affleck [13], Fradkin rewrites $\vec{n}$ as a sum of two terms:

$$\vec{n}(j) = \vec{m}(j) + (-1)^j a_0\vec{\ell}(j),$$

where $\vec{m}$ is normalized and $\vec{\ell}$ is small. The reason for this is that the staggering transformation we have performed, $\vec{n}(j)\mapsto(-1)^j\vec{n}(j)$, does not preserve the quantum mechanical angular momentum commutation relations: $[S_i, S_j] = i\hbar\varepsilon_{ijk}S_k$ is even under inversion on the L.H.S. but odd on the R.H.S. Thus, the "unstaggered" field ℓ must be used to capture quantum mechanical fluctuations in the angular momentum. Using this, we have:

$$\begin{aligned}\delta\vec{n}(j) &= \vec{n}(j+1) - \vec{n}(j)\\ &= (\vec{m}(j+1) + a_0\vec{\ell}(j+1)) - (\vec{m}(j) - a_0\vec{\ell}(j))\\ &\approx a_0(\partial_1\vec{m})(j) + 2a_0\vec{\ell}(j)\end{aligned}$$

Plugging this into (3) and replacing $\sum_{j\ {\rm odd}} \mapsto (2a_0)^{-1}\int dx_1$ gives:

$$\begin{aligned}S_{\rm top} &= S\int d^2x\left(\frac{1}{2}(\partial_1\vec{m}) + \vec{\ell}\right)\cdot(\vec{m}\times\partial_0\vec{m}),\\ \mathcal{L}_{\rm top} &= \frac{S}{2}(\partial_1\vec{m})\cdot(\vec{m}\times\partial_0\vec{m}) + S\vec{\ell}\cdot(\vec{m}\times\partial_0\vec{m}).\end{aligned}$$

Having expressed the Wess-Zumino contribution to the Lagrangian in terms of the $\vec{m}$ and $\vec{\ell}$ fields, we would now like to add the term coming from the antiferromagnetic exchange interaction, which on the lattice is:

$$\mathcal{L}_{\rm mag} = -JS^2\vec{n}(j)\cdot\vec{n}(j+1) = \frac{1}{2}JS^2\left(\delta\vec{n}(j)\right)^2.$$

Rewriting in terms of $\vec{m}$ and $\vec{\ell}$ gives:

$$\mathcal{L}_{\rm mag} = \frac{1}{2}JS^2\left(a_0^2(\partial_1\vec{m})^2 + 4a_0^2\vec{\ell}^2\right).$$

Putting these together (and using the cyclic property of the scalar triple product) gives:

$$\begin{aligned}\mathcal{L}(\vec{m},\vec{\ell}) &= -2a_0JS^2\vec{\ell}^2 + S\vec{\ell}\cdot(\vec{m}\times\partial_0\vec{m})\\ &\quad - \frac{a_0JS^2}{2}(\partial_1\vec{m})^2 + \frac{S}{2}\vec{m}\cdot(\partial_0\vec{m}\times\partial_1\vec{m}),\end{aligned}$$

compare (7.67) in Fradkin. Now, we integrate out the $\vec{\ell}$ modes:

$$\begin{aligned}Z_1[\vec{m}] &= \int\mathcal{D}\ell\, e^{i\int d^2x\, -\frac{1}{2}4a_0JS^2\vec{\ell}^2 + S\vec{\ell}\cdot(\vec{m}\times\partial_0\vec{m})},\\ &= e^{\frac{i}{2}\frac{1}{4a_0JS^2}S^2(\vec{m}\times\partial_0\vec{m})\cdot(\vec{m}\times\partial_0\vec{m})},\\ &= e^{\frac{i}{2}\frac{1}{4a_0J}(\partial_0\vec{m})^2},\end{aligned}$$

where we have used that $\vec{m}\cdot\vec{m} = 1$, which also implies $\vec{m}\cdot\partial_0\vec{m} = 0$. This gives:

$$\begin{aligned}\mathcal{L}(\vec{m}) &= \frac{1}{2}\frac{1}{4a_0J}(\partial_0\vec{m})^2 - \frac{a_0JS^2}{2}(\partial_1\vec{m})^2\\ &\quad + \frac{S}{2}\vec{m}\cdot(\partial_0\vec{m}\times\partial_1\vec{m}),\\ &= \frac{1}{2g}\left(\frac{1}{v_s}(\partial_0\vec{m})^2 - v_s(\partial_1\vec{m})^2\right)\\ &\quad + \frac{\theta}{4\pi}\vec{m}\cdot(\partial_0\vec{m}\times\partial_1\vec{m}).\end{aligned}$$

where we have introduced g, v_s, and the topological angle $\theta = 2\pi S$. Now, when we Wick rotate back to imaginary time, each time derivative ∂_0 picks up a factor of i, so that the kinetic term becomes negative and the topological term becomes imaginary. The topological term is a pullback of the area form on the target S^2, so if we compactify the domain spacetime of the nonlinear sigma model into a sphere, it counts the degree of the configuration seen as a map from S^2 to S^2. Since this is always an integer, the partition function seen as a function of θ is periodic with period 2π, which is why the symbol θ is conventionally used for this parameter. Since $\theta = 2\pi S$, this means we only care about whether S is an integer or half-integer. The behavior in each of these two cases is what concerns us next.

With our mapping between 1d quantum antiferromagnets and classical $O(3)$ nonlinear sigma models with a θ term in hand, we are in a position to discuss the Haldane conjecture, which says that half integer spin Heisenberg antiferromagnets are gapless and integer spin ones are gapped. Let us consider integer spin first. The topological term is zero in this case, and we just have a classical NLSM. While the 2d NLSM is scale invariant at the dimension counting level, its conformal symmetry is anomalous and the model acquires a correlation length (which on the quantum side translates to a gap).

To see what happens to half-integer spins, Fradkin notes that the topological term does not change under RG flow, while the coupling strength g increases (corresponding to decreasing S). Thus, we must only understand what happens to the spin-1/2 chain to understand what happens to all half-integer antiferromagnetic Heisenberg chains. The spin-1/2 chain is integrable by the Bethe ansatz, which reveals gapless spinon excitations.

Affleck [3] presents a less rigorous but perhaps more direct argument [14]. Imagine adding an anisotropy Δm_1^2 to the $O(3)$ sigma model energy. For very large Δ, spins would be heavily penalized for pointing out of the plane, and the model would be an XY model. We know the XY model has two phases: a quasi-long-range ordered phase consisting of bound vortices and a disordered phase which is a plasma of unbound vortices. If we examine the phase diagram in the (g, Δ) plane, the entire $\Delta = 0$ line belongs to the same phase as the vortex-plasma phase of the XY model. The gist of the argument is that for half integer spin, the topological term suppresses vortex proliferation, and thus the mechanism which gives the $O(3)$ model a finite correlation length.

To show this, note that at any non-infinite value of Δ, there are two kinds of vortices: vortices where the spin moves up and vortices where the spin moves down. Imagine two configurations that are identical except for which direction the spin at the vortex core points. Because each vortex configuration covers half of the target S^2, and $\theta = \pi$, these two configurations will have Boltzmann weights which differ by a factor of $i/(-i) = -1$, and will thus cancel.

CONCLUSION

In this note, we have developed the spin coherent state formalism, and applied it to obtain discrete-imaginary-time classical analogues for a single spin in a magnetic field and two spins with an isotropic exchange interaction. We computed some correlation functions of interest in those models and commented on their physical interpretation. Then, we noted that for continuous imaginary time, the spin coherent state formalism produces a classical analogue Hamiltonian which consists of a term coming from the quantum Hamiltonian, a kinetic term, and a WZ term. We explained how the WZ term gives rise to the topological term in the $O(3)$ description of antiferromagnetic spin chains, and argued how this topological term supresses the generation of a mass at $\theta = \pi$ (i.e. half integer spin).

There are some things which still puzzle me. One issue which was raised at several points was the origin of classical temperature in the quantum-mechanical model. The fact that the ferromagnetic spin pair exhibited long range order while still being at a finite temperature is surprising, and suggests that I haven't yet completely understood what is going on (or made an error in the calculation). Another thing I wish I understood better was where Affleck's parameterization of $\vec{n} = \vec{m} \pm a_0\vec{\ell}$ comes from in the spin coherent state language. If $\vec{n}$, $\vec{m}$, and $\vec{\ell}$ are operators, as they are in Affleck's derivation (he doesn't use spin coherent states), then this makes sense, because as mentioned in the main text one cannot negate all the components of $\vec{n}$ without violating the $\mathfrak{su}(2)$ algebra. But if $\vec{n}$ is just a label on spin coherent states, it seems like there should be nothing wrong with staggering them.

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[6] Eduardo Fradkin. *Field Theories of Condensed Matter Physics*. Cambridge University Press, February 2013. Google-Books-ID: x7_6MX4ye_wC.
[7] Timothy H Hsieh. From d-dimensional Quantum to d + 1-dimensional Classical Systems.
[8] octonion (https://physics.stackexchange.com/users/59281/octonion). Mermin-wagner and heisenberg spin chains. Physics Stack Exchange. URL:https://physics.stackexchange.com/q/404586 (version: 2018-05-07).
[9] This differs from the equation given in Fradkin slightly, but I think what I have written here is right.
[10] J M Radcliffe. Some properties of coherent spin states. *Journal of Physics A: General Physics*, 4(3):313–323, May 1971.
[11] Anna Vershynina. NOTES ON COHERENT STATES.
[12] Actually, for this portion of the argument, I work in real time to follow Fradkin, but this is not a major issue.
[13] I. Affleck. Quantum spin chains and the Haldane gap. *Journal of Physics: Condensed Matter*, 1(19):3047–3072, May 1989.
[14] It is very related to Problem 9 on Exam Review 3.