The Kondo problem: Boundaries, defects, and critical phenomena

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Boundary critical phenomena offer a rich playground for exploring universality classes and renormalization group (RG) flows that differ significantly from bulk behavior. A historical such example is the Kondo problem, in which a magnetic impurity of spin s coupled to a conduction electron bath. While the Bethe ansatz provides exact solutions for thermodynamic observables of this system, it remains limited in its ability to describe dynamical properties and spatial profiles. In this work, we review a recently developed analytical framework that treats the Kondo problem in the large-spin limit ($s \to \infty$). By parameterizing small impurity spin fluctuations around a frozen boundary condition, we construct an effective boundary action where $SU(2)$ symmetry is restored order-by-order through coupling to marginal "tilt" operators. We outline the perturbative RG analysis that captures the crossover from weak to strong coupling at order $1/s$ and discuss the role of the Berry phase contact term in maintaining spin quantization. Finally, we provide a roadmap for extending this semi-classical field-theoretic approach for studying the multi-defect problem.

Introduction.—The study of bulk critical behavior is, by now, relatively well understood, with universality classes, scaling hypotheses, and renormalization group (RG) flows providing a coherent framework for describing continuous phase transitions [1]. However, the introduction of boundaries, particularly defects, enriches this picture: boundary conditions can give rise to new universality classes, distinct scaling behavior, and lower-dimensional dynamics that differ from the bulk. Furthermore, boundaries can host their own critical degrees of freedom, leading to phenomena such as boundary phase transitions, modified operator spectra, and the interplay between bulk fluctuations and boundary ordering. In recent years, these questions have seen renewed interest from multiple directions, including boundary conformal field theory (BCFT) [2], the role of topological line and surface defects [3], and connections to quantum critical systems and edge physics.

Recent work on the $O(N)$ model in semi-infinite space has uncovered a novel "extraordinary-log" universality class in $d = 3$, in which boundary correlations receive inverse logarithmic corrections rather than undergoing simple power-law or exponential decay [4, 5]. This behavior arises from the marginal nature of Goldstone fluctuations at the boundary, leading to RG flows in which the effective stiffness runs logarithmically. This structure is well-suited for a field-theoretic description, where the computation and interpretation of the boundary stiffness is straightforward, allowing one to understand how the presence (or absence) of topological defects controls the stability of boundary phases.

Inspired by the semiclassical techniques developed to study critical fields coupled to nearly-ordered boundaries [6], these ideas have recently been adapted to the context of magnetic impurities in metals, famously known as the Kondo problem [7]. The Kondo problem consists of a single magnetic impurity of spin s in a 3D metal. In the low-frequency limit, the problem of scattering of electrons off the magnetic impurity can be reduced to a 1+1-dimensional semi-infinite spatial geometry, where the impurity spin couples to the conduction electrons at the boundary. In this picture one can diagnose the phase of the system based on the phase shift of the scattered wave. The physics of this model is notoriously rich: perturbative RG describes the weak-coupling regime [8] where the effective coupling grows as the temperature is lowered, dynamically generating the Kondo energy scale T_K. At strong coupling [9], the impurity spin is completely screened, forming a local Fermi liquid.

Historically, the full crossover between these regimes was resolved using Wilson's Numerical Renormalization Group (NRG) [10] and later exactly via the Bethe ansatz [11, 12]. The Bethe ansatz is an exact analytical method that solves the 1D model non-perturbatively by demonstrating that multi-particle states can be diagonalized as superpositions of plane waves. This allows for the formal computation of thermodynamic observables such as the impurity entropy and susceptibility for all energy scales. However, the Bethe ansatz struggles to describe dynamical and spatial properties, such as the impurity spectral function, the finite-temperature resistivity, and the spatial profile of the Kondo screening cloud.

To overcome these limitations, Ref. [13] considered the Kondo problem in the large spin limit $s \to \infty$ with the aim of an analytical understanding of the intermediate coupling scales. In this limit, the magnetic impurity is frozen and acts on the electrons as a static exchange field. This effectively freezes the boundary conditions for the fermions on the semi-infinite line. By expanding in small impurity spin fluctuations (powers of $1/s$) combined with RG, this method provides an analytically tractable flow along a line of fixed points. Notably, this framework can compute the dynamical observables that the Bethe ansatz cannot easily access, and it can be naturally extended to the multichannel Kondo problem, accessing even the "non-Fermi-liquid" overscreened fixed points.

In this work, we introduce the formal setup of the problem in the large-s limit and outline the perturbative boundary RG calculations. We then discuss the extension of this analysis to the multi-defect problem before concluding with a summary of the conceptual novelties of this framework.

Setup: The Single-Channel Kondo Problem in the Large-s Limit.— In the single-channel Kondo problem, a bulk 3D Fermi liquid is coupled to a single magnetic impurity of spin s at a point. Due to the point-like nature of the defect, only the s-wave channel for the scattering of bulk fermions off the defect is affected. Following Ref. [13], we reduce the problem to the scattering of bulk fermions off of a point-like magnetic impurity in 1D on the semi-infinite line. The starting point of our analysis is therefore the Euclidean action

$$S = S_B + \int d\tau \int_0^\infty dx \left[\psi^\dagger_{L\alpha}(\partial_\tau + i\partial_x)\psi_{L\alpha} + \psi^\dagger_{R\alpha}(\partial_\tau - i\partial_x)\psi_{R\alpha}\right] + J\int_{x=0} d\tau \psi^\dagger_R \frac{\sigma^a}{2}\psi_R S^a \tag{1}$$

where S_B is the Berry phase action for the impurity spin S^a, and $\psi_{R/L,\alpha}$ are the right- and left-moving electron fields (with Fermi velocity $v_F = 1$). The boundary condition in the absence of the defect is $\psi_{R\alpha}(0^+,\tau) = \psi_{L\alpha}(0^+,\tau)$. In the presence of the defect, the boundary conditions are, in general,

$$\psi_{L\uparrow}(0^+,\tau) = e^{i\rho}\psi_{R\uparrow}(0^+,\tau) \tag{2}$$

$$\psi_{L\downarrow}(0^+,\tau) = e^{-i\rho}\psi_{R\downarrow}(0^+,\tau). \tag{3}$$

where ρ is the Kondo phase shift.

We consider the limit $s \to \infty$ with sJ held fixed, so that the impurity spin direction is frozen. Without loss of generality, we choose the frozen impurity spin to point along the z-axis. The interaction therefore acts like a static delta-function potential. The resulting frozen action $S_{\rm frozen}(\rho)$ for the fermions is characterized by the boundary conditions above and parameterizes a line of fixed points.

To account for finite-s fluctuations, we parameterize the spin as $\vec{n} = (\pi_1, \pi_2, \sqrt{1-\vec{\pi}^2})$. The effective action is:

$$S_{\rm eff} = S_{\rm frozen}(\rho) + S_B[\vec{n}] + S_{\rm int} + S_{\rm cont} \tag{4}$$

where

$$S_B[\vec{n}] = \frac{is}{2}\int d\tau(\pi_1\partial_\tau\pi_2 - \pi_2\partial_\tau\pi_1) + \mathcal{O}(\pi^4) \tag{5}$$

is the Berry's phase action for the spin, and

$$S_{\rm cont} = \delta m_\pi \int d\tau dx \pi^2 + \frac{i\delta s_B}{2}\int d\tau(\pi_1\partial_\tau\pi_2 - \pi_2\partial_\tau\pi_1) \tag{6}$$

are the contact interactions allowed to quadratic order in π. The Berry phase contact term coefficient δs_B is fixed by continuity. By considering the impurity in a finite system and adiabatically following the ground state as ρ is tuned from the free-spin limit ($\rho = 0$), the total Berry phase must remain quantized as s. This fixes the coefficient to be:

$$\delta s_B = -\frac{1}{2\pi}\left(\rho - \frac{1}{2}\sin 2\rho\right). \tag{7}$$

This term accounts for the "sucking in" of spin from the bulk into the boundary as the Kondo interaction develops, effectively shifting the impurity spin magnitude from its bare value s to $s_{\rm eff} = s + \delta s_B$. At the strong-coupling fixed point ($\rho = \pi$), $\delta s_B = -1/2$. In this limit, the bulk fermions decouple from the boundary, leaving an underscreened free boundary spin of magnitude:

$$s_{\rm eff} = s + \delta s_B = s - \frac{1}{2}. \tag{8}$$

The final interaction term $S_{\rm int}$ is required to restore the global $SU(2)$ symmetry that was explicitly broken by freezing the impurity spin along the z-axis in $S_{\rm frozen}(\rho)$. Under an infinitesimal rotation, the variation of the frozen action is compensated by the "tilt" operators, which are marginal boundary operators derived from the bulk spin currents. To linear order in π, the interaction is:

$$S_{\rm int} = \int d\tau\, \epsilon^{ij}\pi^i(\tau) j^j_x(0^+,\tau) + O(\pi^2) \tag{9}$$

where the boundary currents are obtained from the bulk spin current,

$$j^a_x(x,\tau) = \frac{1}{2}\left(\psi^\dagger_R\sigma^a\psi_R - \psi^\dagger_L\sigma^a\psi_L\right), \tag{10}$$

yielding at the boundary for $i = 1,2$:

$$j^i_x(0^+,\tau) = \sin\rho\big(\sin\rho\, \psi^\dagger_R\sigma^i\psi_R(0^+,\tau) - \cos\rho\, \epsilon^{ij}\psi^\dagger_R\sigma^j\psi_R(0^+,\tau)\big). \tag{11}$$

By requiring that $SU(2)$ symmetry be restored order-by-order in π (treating a constant π as a rigid rotation of the spin direction), one can also fix the quadratic correction to the interaction:

$$\delta S_{\rm int} = r \int d\tau \vec{\pi}^2(\tau) \psi_R^\dagger \sigma^3 \psi_R(0^+, \tau), \quad r = -\frac{\sin 2\rho}{4}. \tag{12}$$

Notably, these interactions vanish whenever the phase shift ρ is a multiple of π, indicating that the spin fluctuations fully decouple from the bulk at these $SU(2)$-invariant fixed points.

Renormalization Group flow in the large spin limit.— A central result of Ref. [13] is that the Kondo crossover may be accessed analytically by combining a semiclassical expansion in $1/s$ with boundary RG. The crucial point is that the limit $s \to \infty$ is *not* a weak-coupling limit in the bare Kondo exchange J. Rather, one first solves the impurity problem exactly at $s = \infty$, where the impurity behaves as a static exchange field and the boundary condition is characterized by a phase shift ρ, and then studies the effect of impurity spin fluctuations perturbatively in $1/s$. In this formulation, the running coupling is the phase shift ρ, and the resulting beta function is non-perturbative in the microscopic exchange J.

Lets us now compute the beta function for ρ. The expansion parameter is $1/s$. Indeed, from the Berry phase term the π-field propagator is off-diagonal and scales as $1/s$:

$$\langle \pi^i(\tau) \pi^j(0) \rangle = \frac{i}{2s} \epsilon^{ij} \, \mathrm{sgn}(\tau). \tag{13}$$

Thus every π-contraction carries an explicit suppression by $1/s$, and to leading nontrivial order the RG is controlled.

The RG computation is performed in the fixed-ρ theory $S_{\rm frozen}(\rho)$, which is exactly solvable because it consists of free fermions with a nontrivial boundary condition (2). In particular, the boundary two-point functions in the frozen theory are exactly calculable. The boundary OPE is

$$\begin{aligned} &\left(\psi_R^\dagger \sigma^a \psi_R\right)(\tau) \left(\psi_R^\dagger \sigma^b \psi_R\right)(0) \sim \\ &\qquad \frac{\delta^{ab}}{2\pi^2 \tau^2} + \frac{i}{\pi \tau} \epsilon^{abc} \left(\psi_R^\dagger \sigma^c \psi_R\right)(0) + \cdots . \end{aligned} \tag{14}$$

Using Eq. (11), this implies the current OPE

$$j_x^i(\tau) j_x^j(0) \sim \sin^2 \rho \left[\frac{\delta^{ij}}{2\pi^2 \tau^2} + \frac{i}{\pi \tau} \epsilon^{ij} \psi_R^\dagger \sigma^3 \psi_R(0) + \cdots \right]. \tag{15}$$

Note the appearance of the operator $\psi_R^\dagger \sigma^3 \psi_R$: this is precisely the exactly marginal boundary operator that moves the theory along the line of frozen fixed points parameterized by ρ. In other words, a perturbation $\delta S = c \int d\tau \, \psi_R^\dagger \sigma^3 \psi_R(0^+, \tau)$ induces a shift $\rho \to \rho + c + O(c^2)$, and so the generation of this term can be interpreted as a renormalization of the phase shift.

We now integrate out the short-time modes of the π-fluctuations to obtain the leading β-function for ρ. The leading correction to the action arises at second order in the linear interaction $\epsilon^{ij} \pi^i j_x^j$,

$$\delta S = -\frac{1}{2} \int_{\mathcal{C}} d\tau_1 \, d\tau_2 \, \epsilon^{ij} \epsilon^{kl} \, \pi^i(\tau_1) \pi^k(\tau_2) \, j_x^j(\tau_1) j_x^l(\tau_2), \tag{16}$$

where the integration region is $\mathcal{C} = a < |\tau_1 - \tau_2| < a e^{d\ell}$, a is the short-time cutoff, and $d\ell$ the infinitesimal RG step. After contracting the π terms, replacing the product $j_x^j(\tau_1) j_x^l(\tau_2)$ by its OPE (15), and keeping only the logarithmically divergent contribution proportional to the marginal operator $\psi_R^\dagger \sigma^3 \psi_R$, one obtains

$$\begin{aligned} \delta S = -\frac{1}{2} \int_{a<|\tau|<a e^{d\ell}} d\tau \int d\tau_2 \, \epsilon^{ij} \epsilon^{kl} \left(\frac{i}{2s} \epsilon^{ik} \, \mathrm{sgn} \tau \right) \\ \times \sin^2 \rho \left(\frac{i}{\pi \tau} \epsilon^{jl} \psi_R^\dagger \sigma^3 \psi_R(\tau_2). \right). \end{aligned} \tag{17}$$

Simplifying the above expression and evaluating the τ integral for infinitesimal $d\ell$ we have

$$\delta S = \frac{\sin^2 \rho}{\pi s} \, d\ell \int d\tau \, \psi_R^\dagger \sigma^3 \psi_R(0^+, \tau), \tag{18}$$

and so the beta function for ρ is

$$\frac{d\rho}{d\ell} = -\beta(\rho) = \frac{1}{\pi s} \sin^2 \rho + O\!\left(\frac{1}{s^2}\right). \tag{19}$$

Equation (19) is non-perturbative in the bare Kondo coupling J: note that ρ itself is a nonlinear function of sJ, determined already in the frozen impurity problem. For small sJ, one has $\rho \approx sJ/2$, while for a particular δ-function regularization one may write $\rho = 2\arctan(sJ/4)$. Thus the beta function resums arbitrarily high powers of J into a simple trigonometric function of the phase shift. Expanding Eq. (19) for weak coupling yields $dJ/d\ell \approx J^2/(2\pi)$, reproducing the standard perturbative scaling result at leading order [8]. The large-s construction therefore interpolates between the conventional weak-coupling RG and the strong-coupling regime in a single analytic framework.

Solution of the Flow and the Kondo Scale.—The RG equation (19) is separable and integrates to:

$$\rho(\ell) = \frac{\pi}{2} - \tan^{-1}\left(\cot \rho_0 - \frac{\ell}{\pi s} \right). \tag{20}$$

For $0 < \rho_0 < \pi$, the flow is toward $\rho_{\rm IR} = \pi$. This identifies the strong-coupling fixed point with the screened/underscreened boundary condition. Near $\rho = \pi$, $\pi - \rho(\ell) \sim \pi s/\ell$, so the approach to the IR fixed point is only logarithmic. Physically, this is the familiar residual ferromagnetic coupling of the remnant spin $s - \frac{1}{2}$ to the bulk electrons in the underscreened single-channel problem.

As usual, dimensional transmutation introduces the Kondo scale. Writing $\ell = \log(\Lambda/\omega)$, one defines T_0 by the condition $\rho(\omega = T_0) = \pi/2$, which gives

$$T_0 = \Lambda \, \exp[-\pi s \cot \rho_0] \, . \tag{21}$$

Then Eq. (20) becomes

$$\rho(\omega) = \frac{\pi}{2} - \tan^{-1}\left(\frac{\log(\omega/T_0)}{\pi s}\right). \tag{22}$$

The dependence on the microscopic cutoff Λ and bare phase shift ρ_0 is thus traded for a dynamically generated IR scale.

RG Improvement and Observables.—Once the running phase shift $\rho(\omega)$ is known, observables may be RG-improved by evaluating their frozen-theory expressions at the appropriate scale. In this way one obtains explicit crossover formulae at large s. For example, the zero-temperature impurity magnetization in a field h is controlled by the running phase shift at $\omega \sim h$, $M(h) = s - \rho(h)/(2\pi) + O(1/s)$, so that

$$M(h) = s - \frac{1}{4} + \frac{1}{2\pi} \tan^{-1}\left(\frac{\log(h/T_0)}{\pi s}\right) + O(1/s). \tag{23}$$

Likewise, finite-temperature thermodynamics follow from $\rho(T)$, and one obtains closed-form expressions for the impurity entropy, specific heat, and susceptibility along the full crossover.

The multi-defect problem.— In this section, we consider an extension of the Kondo problem to include multiple magnetic impurities in a 3D metal. For simplicity, we assume each impurity has identical coupling to the metal and identical impurity-impurity coupling. We will again consider the large spin limit $s \to \infty$ so that we can expand in $1/s$. The action for the system is

$$S = \sum_i S_B^i + \int d\tau \int_0^\infty dx \left[\psi^\dagger_{L\alpha}(\partial_\tau + i\partial_x)\psi_{L\alpha} + \psi^\dagger_{R\alpha}(\partial_\tau - i\partial_x)\psi_{R\alpha}\right] + J \int_{x=0} d\tau \psi_R^\dagger \frac{\sigma^a}{2}\psi_R \sum_i S_i^a + J_1 \sum_{\langle i,j\rangle} \vec{S}_i \cdot \vec{S}_j, \tag{24}$$

where S_B^i is the Berry phase action for the i-th spin, and J_1 is the spin-spin coupling. The boundary conditions for the fermions are the same as before (2), but ρ will have a different renormalization that depends on J_1.

It is reasonable to expect that the RG flow depends on the configuration of the spins. If $J_1 > 0$, the classical ground state of the spins is ferromagnetic, whereas if $J_1 < 0$, the classical ground state is antiferromagnetic. Further, for the case of three (3) spins, there can be frustration between the spins, akin to the 120° collinear order of the Heisenberg antiferromagnet on the triangular lattice.

For simplicity, we will assume $J_1 > 0$ and consider the case of two (2) defect spins. We take the semi-classical limit where the two spins are aligned and expand the action in terms of impurity fluctuations. Without loss of generality we take the spins to be aligned along the z-axis, and we expand each spin in terms of fluctuations according to $\vec{n_i} = (\pi_1^i, \pi_2^i, \sqrt{1 - \vec{\pi}^{i2}})$. The Berry phase action for each spin reduces as before to (5). The interaction term between the spins is

$$J_1 \vec{S}_1 \cdot \vec{S}_2 = J_1 \vec{\pi}_1 \cdot \vec{\pi}_2. \tag{25}$$

Once again, the freezing of the spins breaks the $SU(2)$ rotation symmetry of (24). We restore the symmetry order-by-order by coupling the marginal boundary operators to the π^i. The j_x^i boundary OPE is the same as before, except now the defect interaction is

$$\delta S_{\rm int} = -\frac{\sin 2\rho}{4} \sum_i \int d\tau \vec{\pi}^{i2} \psi_R^\dagger \sigma^3 \psi_R(0^+, \tau). \tag{26}$$

The contact terms for the Berry phase term for each spin is similarly fixed as before.

The leading-order interaction obtained from integrating out the high-energy modes is

$$\delta S = -\frac{1}{2} \sum_m \int_{\mathcal{C}} d\tau_1 \, d\tau_2 \, \epsilon^{ij} \epsilon^{kl} \, \pi_i^m(\tau_1) \pi_k^m(\tau_2) \, j_x^j(\tau_1) j_x^l(\tau_2). \tag{27}$$

Using the boundary OPE and contracting the π, we obtain

$$\delta S = -\frac{1}{2} \frac{\sin^2 \rho}{\pi} \frac{1}{s(1 + \mathcal{O}(J_1))} d\ell \int d\tau \psi_R^\dagger \sigma^3 \psi_R(0^+, \tau), \tag{28}$$

and so we see that the beta function will be altered by the coupling J_1. Since we have considered a ferromagnetic coupling, this has little effect on the RG flow (indeed, from the point of view of RG, this just looks like an effective coupling $J \to 2J$ for a single spin).

Summary and future work.— In this work, we have reviewed and extended a recently developed analytical framework for the Kondo problem based on the large-spin ($s \to \infty$) limit [13]. A major advantage of this large-s renormalization group (RG) formalism is that it provides a systematic way to calculate observables that have historically been difficult to extract from the exact Bethe ansatz solution. Since the effective theory is a boundary field theory with explicit fermionic correlators, one can compute dynamical and spatial properties—such as the impurity spectral function $A_f(\omega)$, the finite-temperature resistivity, and the real-space profile of the Kondo screening cloud throughout the entire strong-to-weak coupling crossover.

There are also conceptual novelties due to this approach. First, the starting point is not weak coupling in the bare exchange J, but the exactly solvable $s = \infty$ limit, which defines a continuous line of boundary fixed points parameterized by the phase shift ρ. The RG flow describes motion along this line induced by finite-s spin fluctuations. Second, the couplings of the Goldstone-like impurity fluctuations π^i to the boundary fermions are constrained by symmetry: the linear coupling to the tilt operator j_x^i follows from the broken $SU(2)$ Ward identity, while the quadratic interactions are fixed by requiring that rigid rotations of the frozen spin be reproduced order-by-order. Third, the beta function obtained in this way is non-perturbative in the microscopic Kondo exchange, but perturbative in $1/s$, allowing for the capture of crossover physics at intermediate and strong coupling where ordinary perturbation theory in J fails.

Building on these insights, we have provided a roadmap for extending this semiclassical field-theoretic approach to multi-impurity defects. By considering a system of two spin-s impurities with a ferromagnetic coupling ($J_1 < 0$), we demonstrated that the frozen-spin limit and subsequent $1/s$ expansion can be generalized to characterize the interplay between inter-impurity interactions and the bulk electron bath. We showed that the boundary OPE structure and the symmetry-restoration mechanism are preserved, though the resulting beta function for the phase shift ρ is modified by the inter-impurity coupling J_1.

This framework offers a promising analytical tool for future studies. A primary direction is the extension of this analysis to frustrated spin configurations, such as triangular clusters of antiferromagnetically coupled impurities, or the study of extended boundary conditions like line and surface defects. Yet another interesting direction would be the analysis of the multi-impurity problem with translational symmetry reduced to the line. Here, the impurities can be modeled (after course-graining) by an $O(N)$-model in coupled to the Fermi sea in 1+1D. We leave these directions for future work.

[1] K. G. Wilson and J. Kogut, Physics Reports **12**, 75 (1974).
[2] J. L. Cardy, Nuclear Physics B **240**, 514 (1984).
[3] M. Billò, M. Caselle, D. Gaiotto, F. Papini, S. Pellegrini, and E. Vescovi, Journal of High Energy Physics **2013**, 1 (2013).
[4] M. A. Metlitski, SciPost Physics **12**, 131 (2022).
[5] F. Parisen Toldin and M. A. Metlitski, Physical Review Letters **128**, 215701 (2022).
[6] G. Cuomo, Z. Komargodski, and M. Meineri, Communications in Mathematical Physics **396**, 229 (2022).
[7] J. Kondo, Progress of theoretical physics **32**, 37 (1964).
[8] P. W. Anderson, G. Yuval, and D. R. Hamann, Physical Review B **1**, 4464 (1970).
[9] P. Nozières, Journal of Low Temperature Physics **17**, 31 (1974).
[10] K. G. Wilson, Reviews of modern physics **47**, 773 (1975).
[11] N. Andrei, Physical Review Letters **45**, 379 (1980).
[12] P. B. Wiegmann, Journal of Physics C: Solid State Physics **14**, 1463 (1981).
[13] A. Krishnan and M. A. Metlitski, arXiv preprint arXiv:2408.12650 (2024).