RG Analysis of Microscopically Derived Active Field Theory Coupled to External Potential

Borna Perković
Massachusetts Institute of Technology
77 Massachusetts Ave., Cambridge, MA 02139-4307
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External potentials are known to induce long-range effects in non-equilibrium systems. This paper uses the Martin-Siggia-Rose-Janssen-De Dominicis (MSRJD) formalism to construct a functional for a microscopically derived field theoretic equation of motion for active Brownian particles (ABPs) in a low activity regime. The equations of motion include an external potential, resulting in both standard (equilibrium) and novel (non-equilibrium) terms, whose RG scaling we study up to the first loop correction. Flow equations for the Gaussian parameters are derived.

I. INTRODUCTION

External potentials, including disordered boundaries and quenched potentials, have been shown to affect long-range order in non-equilibrium systems, destroying both bulk phase separation and motility-induced phase separation [1][2]. These results beg the question of whether such an effect could be theoretically predicted. Active field theories have become widely used to probe and study the collective behaviours of active particles [3][4][5][6], thereby providing a natural framework for addressing this question. Given the destruction of order observed, it is interesting to examine how passive and active external-potential terms scale in the RG sense, as this scaling could provide a way to predict whether the large-scale behavior of a non-equilibrium system is disturbed by the potential. Recently, Tailleur and Kafri [7] derived a field-theoretic equation of motion for ABPs in an external potential directly from microscopic dynamics, rather than phenomenologically. Their approach applies in a dominantly passive regime where the persistence length ℓ_a is small, while also including additional subleading active terms. This paper builds on their work by using the equations of motion to construct a field theory and perform RG analysis. Because the starting equations are restricted to the low-ℓ_a regime, phases such as motility-induced phase separation cannot be studied within this theory, but low-activity, equilibrium-dominated phase transitions can.

II. CONSTRUCTING THE FIELD THEORY

Firstly, we study the system of non-interacting ABPs by starting with Eqn. (1.4) from Tailleur and Kafri [7]:

$$\partial_t \rho = -\nabla \cdot J_{\text{det}}, \tag{1}$$

$$J_{\text{det}} = -(D_p + D_a)\nabla\rho - \mu\rho\nabla V + \ell_a^2 \tilde{\gamma} D_a \nabla\nabla^2\rho - \frac{\mu\ell_a^2}{d}\sum_j \partial_j[(\nabla V)\partial_j\rho] + \frac{\mu\ell_a^2}{d}\nabla\big[\nabla\cdot(\rho\nabla V)\big], \tag{2}$$

where $\rho(x,t)$ represents the density field, D_p and D_a the passive and active diffusion rates, μ the mobility, V the external potential, d the dimension of space, and $\tilde{\gamma} \equiv \frac{4d-1}{d^2(d+2)}$ a positive number. As before, ℓ_a is the persistence length. In this entire paper, μ is assumed to be constant.

In order to create a functional to be used for RG, Dean-Kawasaki (DK) noise $J_{\text{noise}}(x,t) = \sqrt{2(D_p+D_a)\rho}\,\xi(x,t)$ is introduced into the system [8][9]. ξ is a d-dimensional Gaussian random variable defined by

$$\langle \xi(x,t)\rangle = 0, \tag{3}$$

$$\langle \xi_i(x,t)\xi_j(x',t')\rangle = \delta_{ij}\delta^{(d)}(x-x')\delta(t-t'). \tag{4}$$

The noisy equation of motion now reads

$$\partial_t\rho = -\nabla\cdot J_{\text{det}} - \nabla\cdot J_{\text{noise}}. \tag{5}$$

DK noise is a noise that preserves particles on the level of fluctuations, allowing the particle number to remain fixed throughout the motion. The partition function can be constructed from the equation of motion using the MSRJD formalism [10][11]:

$$\begin{aligned} Z &= \int D\rho D\xi \exp\Big(-\frac{\xi^2}{2}\Big)\delta\Big(\partial_t\rho + \nabla\cdot J_{\text{det}} + \nabla\cdot J_{\text{noise}}\Big) \\ &= \int D\rho D\xi D\tilde{\rho}\exp\Big(-\frac{\xi^2}{2}\Big) \\ &\quad \cdot \exp\Big[-\int d^dx dt\,\tilde{\rho}\Big(\partial_t\rho + \nabla\cdot J_{\text{det}} + \nabla\cdot J_{\text{noise}}\Big)\Big] \\ &= \int D\rho D\tilde{\rho}\exp\Big[-\int d^dx dt\,\tilde{\rho}\Big(\partial_t\rho + \nabla\cdot J_{\text{det}}\Big)\Big] \\ &\quad \cdot \int D\xi \exp\Big(-\frac{\xi^2}{2} - \int d^dx dt\,\tilde{\rho}\nabla\cdot J_{\text{noise}}\Big) \\ &= \int D\rho D\tilde{\rho}\exp\Big[-\int d^dx dt\, S[\rho,\tilde{\rho}]\Big], \end{aligned} \tag{6}$$

$$S[\rho,\tilde{\rho}] = \tilde{\rho}\Big(\partial_t\rho + \nabla\cdot J_{\text{det}}\Big) - (D_p+D_a)\rho(\nabla\tilde{\rho})^2 \tag{7}$$

where $\tilde{\rho}$ represents an auxiliary field that removes the random variable ξ from the formalism, and S is the dynamical action - the substitute for an energy functional. Going from line 3 to line 4, J_{noise} was expressed in terms of ξ, the term was integrated by parts, and the Gaussian integral over ξ performed. The external potential V is often chosen such that the force $F = -\nabla V$ is quenched [kardar paper]. In other words,

$$\langle F(x,t)\rangle = 0, \tag{8}$$

$$\langle F_i(x,t)F_j(x',t')\rangle = \Delta_F \delta_{ij}\delta^{(d)}(x-x'). \tag{9}$$

The derivation of Eqn. 2 assumes that V defines a length-scale $\ell_V \gg \ell_a$. To avoid breaking this assumption, one can instead assume a smoothed-out version of Eqn. (9),

$$\langle F_i(x,t)F_j(x',t')\rangle = \Delta_F \delta_{ij} C(x-x'), \tag{10}$$

$$C(x-x') \equiv \frac{1}{(\ell_V\sqrt{2\pi})^d}\exp\left(-\frac{(x-x')^2}{2\ell_V^2}\right), \tag{11}$$

and then take the limit $\frac{\ell_a}{\ell_V} \to 0$ and $\ell_V \to 0$, yielding the same result.

The quenched potential can be addressed using the replica method [12][13]. Writing $S = S_0 + S_V$, where S_V assumes the terms in S involving V, and S_0 being the rest, it follows that

$$\overline{Z^n} = \int \prod_{\alpha=1}^{n} D\rho^\alpha D\tilde{\rho}^\alpha \exp\left[-\sum_{\alpha=1}^{n} S_0^\alpha - \frac{1}{2}\left\langle\left(\sum_{\alpha=1}^{n} S_V^\alpha\right)^2\right\rangle_c\right], \tag{12}$$

where α is the index of the replica, S_0^α and S_V^α are S_0 and S_V of the α-th replica, and $\langle(\cdots)^2\rangle_c$ denotes a connected second moment.

$$S_0^\alpha = \int dt\, d^dx\, \{\tilde{\rho}^\alpha[\partial_t\rho^\alpha - (D_p+D_a)\nabla^2\rho^\alpha + \ell_a^2\tilde{\gamma}D_a\nabla^4\rho^\alpha] - (D_p+D_a)\rho^\alpha\ (\nabla\tilde{\rho}^\alpha)^2\}. \tag{13}$$

Through a series of integrations by parts, one can rewrite

$$S_V^\alpha = \frac{\mu\ell_a^2}{d}\int dtd^dx B_i^\alpha(x,t)\partial_i V, \tag{14}$$

$$B_i^\alpha(x,t) \equiv -\frac{d}{\ell_a^2}\rho^\alpha\partial_i\tilde{\rho}^\alpha + \sum_j(\partial_i\partial_j\tilde{\rho}^\alpha)(\partial_j\rho^\alpha) + \rho^\alpha\partial_i\nabla^2\tilde{\rho}^\alpha. \tag{15}$$

Hence

$$\left\langle\left(\sum_\alpha S_V^\alpha\right)^2\right\rangle_c = \\ = \frac{\mu^2\ell_a^4}{d^2}\int dtd^dxdt'd^dx' \sum_{i,\alpha,\beta} B_i^\alpha(x,t)B_i^\beta(x',t')\Delta_F C(x-x') \tag{16}$$

If $C(x-x')$ is replaced with $\delta(x-x')$, the equation becomes:

$$\left\langle\left(\sum_\alpha S_V^\alpha\right)^2\right\rangle_c = \\ = \frac{\mu^2\ell_a^4}{d^2}\int dtd^dxdt' \sum_{i,\alpha,\beta} B_i^\alpha(x,t)B_i^\beta(x,t')\Delta_F \tag{17}$$

Defining S_{eff} as

$$\overline{Z^n} = \int \prod_{\alpha=1}^{n} D\rho^\alpha D\tilde{\rho}^\alpha e^{-S_{\text{eff}}}, \tag{18}$$

it is obtained that

$$S_{\text{eff}} = \sum_\alpha S_0^\alpha - \\ -\frac{\Delta_F}{2}\frac{\mu^2\ell_a^4}{d^2}\int dtd^dxdt'd^dx' \sum_{i,\alpha,\beta} B_i^\alpha(x,t)B_i^\beta(x',t')C(x-x') \tag{19}$$

The quantity S_{eff} is the dynamical action to be used for RG. Assuming that the average density is constant and equal to ρ_0, the field can be expanded in terms of fluctuations $\phi(x,t)$ around the mean. In other words, the field is expanded to be $\rho(x,t) = \rho_0 + \phi(x,t)$. S_0^α is then

$$S_0^\alpha = \int dt\, d^dx\, \{\tilde{\rho}^\alpha[\partial_t\phi^\alpha - (D_p+D_a)\nabla^2\phi^\alpha + \ell_a^2\tilde{\gamma}D_a\nabla^4\phi^\alpha] - (D_p+D_a)\rho_0^\alpha\ (\nabla\tilde{\rho}^\alpha)^2 - (D_p+D_a)\phi^\alpha\ (\nabla\tilde{\rho}^\alpha)^2\}. \tag{20}$$

Expanding the external potential terms of the dynamical action around ρ_0, one obtains

$$B_i^\alpha = \underbrace{\rho_0\left[-\frac{d}{\ell_a^2}\partial_i\tilde{\rho}^\alpha + \partial_i\nabla^2\tilde{\rho}^\alpha\right]}_{\text{linear}} + \\ \underbrace{\left[-\frac{d}{\ell_a^2}\phi^\alpha\partial_i\tilde{\rho}^\alpha + \sum_j(\partial_i\partial_j\tilde{\rho}^\alpha)(\partial_j\phi^\alpha) + \phi^\alpha\partial_i\nabla^2\tilde{\rho}^\alpha.\right]}_{\text{bilinear}} \tag{21}$$

Naming the linear term $B_i^{(1)\alpha}$ and the bilinear term $B_i^{(2)\alpha}$, one can write out the product

$$B_i^\alpha(x,t)B_i^\beta(x',t') = \sum_{a,b\in\{1,2\}} B_i^{(a)\alpha}(x,t)B_i^{(b)\beta}(x',t') \tag{22}$$

S_{eff} remains to be defined by Eqn. (19), but S_0^α and B_i^α are then given by Eqns. (20) and (21).

Having solved the non-interacting ABP case, one can solve for pairwise-interacting ABPs in an analogous manner. Assuming $D_p = 0$ and the interaction potential $\sum_{i<j} w(x_i - x_j)$, the equation of motion becomes Eqn. (5.29) from Tailleur and Kafri [7]:

$$\partial_t\rho = -\nabla\cdot J_{\text{det, int}}, \\ J_{\text{det, int}} = -D_a\nabla\rho - \mu a_0\rho\nabla\rho - \mu a_2\rho\nabla\nabla^2\rho + \ell_a^2\tilde{\gamma}D_a\nabla\nabla^2\rho \\ -\frac{\mu a_0\ell_a^2}{d}\left[\frac{1}{2}\nabla[(\nabla\rho)^2] + \nabla^2\rho\,\nabla\rho\right] + \frac{\mu a_0\ell_a^2}{d}\nabla[\nabla\cdot(\rho\nabla\rho)], \tag{23}$$

where

$$a_0 \equiv \int d\mathbf{r}\, w(|\mathbf{r}|) \quad \text{and} \quad a_2 \equiv \frac{1}{2d}\int d\mathbf{r}\, w(|\mathbf{r}|)\mathbf{r}^2. \tag{24}$$

It follows that

$$S_{\text{eff, int}} = \sum_\alpha S^\alpha_{0,\text{int}} - \\ - \frac{\Delta_F}{2}\frac{\mu^2\ell_a^4}{d^2}\int dtd^dxdt'd^dx' \sum_{i,\alpha,\beta} B_i^\alpha(x,t)B_i^\beta(x',t')C(x-x'), \tag{25}$$

where

$$S^\alpha_{0,\text{int}} = \int dt\, d^dx \left\{\hat\rho^\alpha\left[\partial_t\rho^\alpha + (\nabla\cdot J^\alpha_{\text{det, int}})_0\right] - D_a\rho^\alpha\left(\nabla\hat\rho^\alpha\right)^2\right\}, \tag{26}$$

$$(\nabla\cdot J^\alpha_{\text{det, int}})_0 = -D_a\nabla^2\rho + \frac{\mu a_0\ell_a^2}{d}\nabla^2\left[\nabla\cdot(\rho\nabla\rho)\right] - \mu a_0\nabla\cdot(\rho\nabla\rho) - \mu a_2\nabla\cdot(\rho\nabla\nabla^2\rho) + \ell_a^2\tilde\gamma D_a\nabla^4\rho - \frac{\mu a_0\ell_a^2}{d}\nabla\cdot\left[\frac{1}{2}\nabla\big((\nabla\rho)^2\big) + (\nabla^2\rho)\nabla\rho\right] \tag{27}$$

and $B_i^\alpha(x,t)$ remains unchanged compared to the non-interacting case. The version of $S^\alpha_{0,\text{int}}$ expanded around ρ_0 is

$$S^\alpha_{0,\text{int}} = \int dt\, d^dx \left\{\hat\rho^\alpha\left[\partial_t\rho^\alpha + (\nabla\cdot J^\alpha_{\text{det, int}})_0\right] - D_a\rho_0\,(\nabla\hat\rho^\alpha)^2 - D_a\phi^\alpha\,(\nabla\hat\rho^\alpha)^2\right\}, \tag{28}$$

$$(\nabla\cdot J^\alpha_{\text{det, int}})_0 = -D_{\text{int}}\nabla^2\phi + \frac{\mu a_0\ell_a^2}{d}\nabla^2\left[\nabla\cdot(\phi\nabla\phi)\right] - \mu a_0\nabla\cdot(\phi\nabla\phi) - \mu a_2\nabla\cdot(\phi\nabla\nabla^2\phi) + \kappa_{\text{int}}\nabla^4\phi - \frac{\mu a_0\ell_a^2}{d}\nabla\cdot\left[\frac{1}{2}\nabla\big((\nabla\phi)^2\big) + (\nabla^2\phi)\nabla\phi\right], \tag{29}$$

where $D_{\text{int}} = D_a + \mu a_0\rho_0$ and $\kappa_{\text{int}} = \ell_a^2\tilde\gamma D_a - \mu a_2\rho_0 + \frac{\mu a_0\ell_a^2\rho_0}{d}$.

III. TREE-LEVEL RG SCALING

In the rest of the paper, only the non-interacting case is shown, as the pairwise interactions generate additional terms of subleading RG scaling at the diffusion fixed point that is being considered. To find the tree-level scaling, the following rescaling is used:

$$x\to bx, \quad t\to b^z t, \quad \rho\to b^\chi\rho, \quad \tilde\rho\to b^{\tilde\chi}\tilde\rho \tag{30}$$

Term in S_{eff}	Parameter	Coupling eigenvalue
$\tilde\rho\,\partial_t\phi$	1	$d+\chi+\tilde\chi$
$\tilde\rho\,\nabla^2\phi$	$-D\sim-(D_p+D_a)$	$d+z+\chi+\tilde\chi-2$
$\tilde\rho\,\nabla^4\phi$	$\kappa\sim\ell_a^2\tilde\gamma D_a$	$d+z+\chi+\tilde\chi-4$
$\phi(\nabla\tilde\rho)^2$	$-D$	$d+z+\chi+2\tilde\chi-2$
$\int B_{\text{eq}}B_{\text{eq}}$	$K_{e,e}\sim\Delta_F\mu^2$	$d+2z+2\chi+2\tilde\chi-2$
$\int B_{\text{eq}}B_{\text{non-eq}}$	$K_{e,n}\sim\Delta_F\mu^2\ell_a^2$	$d+2z+2\chi+2\tilde\chi-4$
$\int B_{\text{non-eq}}B_{\text{non-eq}}$	$K_{n,n}\sim\Delta_F\mu^2\ell_a^4$	$d+2z+2\chi+2\tilde\chi-6$
$(\nabla\tilde\rho)^2$	$\mathcal{D}\sim-D\rho_0$	$d+z+2\tilde\chi-2$

TABLE I: Scaling dimensions of the operators in S_{eff} under $x\to bx$, $t\to b^zt$, $\phi\to b^\chi\phi$, and $\tilde\rho\to b^{\tilde\chi}\tilde\rho$.

B_i^α breaks down into two terms under RG:

$$B^\alpha_{i,\text{eq}} = \rho^\alpha\partial_i\tilde\rho^\alpha \tag{31}$$

$$B^\alpha_{i,\text{non-eq}} = \sum_j(\partial_i\partial_j\tilde\rho^\alpha)(\partial_j\rho^\alpha) + \rho^\alpha\partial_i\nabla^2\tilde\rho^\alpha, \tag{32}$$

where $B^\alpha_{i,\text{eq}}$ scales as $b^{\chi+\tilde\chi-1}$ and $B^\alpha_{i,\text{non-eq}}$ scales as $b^{\chi+\tilde\chi-3}$. Note that all parameters such as μ,ℓ_a,Δ_F are combined into a single parameter. For example, if all parameters in front of $B_{\text{eq}}B_{\text{non-eq}}$ are combined into $K_{\text{e,n}}$ (e for equilibrium, n for non-equilibrium), then

$$S_{\text{eff}} \supseteq K_{\text{e,n}}\int B_{\text{eq}}B_{\text{non-eq}} \tag{33}$$

The scalings are shown in Table I. In order to determine the scalings of t and $\tilde\rho$, the diffusive point is fixed, such that the first two terms in Table I do not change under rescaling.

$$\chi+\tilde\chi+d=0 \tag{34}$$

$$z=2 \tag{35}$$

Noting that ϕ represents the density field fluctuations, $\chi=-d$, resulting in $\tilde\chi=0$. The scalings around this fixed point are shown in Table II. Immediately one can see that the equlibrium couplings with the potential dominate over non-equilibrium couplings. Around the diffusive fixed point, the non-equilibrium terms are irrelevant for $d>0$. The result is unsurprising, as the diffusive fixed point is equilibrium-dominated and perseveres in the low activity regime. The role of active diffusion is to simply rescale the passive diffusion coefficient.

Term in S_{eff}	Parameter	Coupling eigenvalue
$\tilde{\rho}\,\partial_t\phi$	1	0
$\tilde{\rho}\,\nabla^2\phi$	$-D \sim -(D_p + D_a)$	0
$\tilde{\rho}\,\nabla^4\phi$	$\kappa \sim \ell_a^2\tilde{\gamma}D_a$	-2
$\phi(\nabla\tilde{\rho})^2$	$-D$	0
$\int B_{\text{eq}}B_{\text{eq}}$	$K_{e,e} \sim \Delta_F\mu^2$	$2-d$
$\int B_{\text{eq}}B_{\text{non-eq}}$	$K_{e,n} \sim \Delta_F\mu^2\ell_a^2$	$-d$
$\int B_{\text{non-eq}}B_{\text{non-eq}}$	$K_{n,n} \sim \Delta_F\mu^2\ell_a^4$	$-d-2$
$(\nabla\tilde{\rho})^2$	$\mathcal{D} \sim -D\rho_0$	d

TABLE II: Scaling dimensions of the operators in S_{eff} under $x \to bx$, $t \to b^2t$, $\phi \to b^{-d}\phi$, and $\tilde{\rho} \to \tilde{\rho}$.

IV. ONE-LOOP CORRECTION

A. Propagators in k-space

In k-space, S_{eff} can be written as

$$S_{\text{eff}} = S_G + S_{\text{Dean}} + S_{V3} + S_{V4}, \tag{36}$$

$$\begin{aligned} S_G = \sum_{\alpha,\beta}\int_{k,\omega}\Big[&\delta_{\alpha,\beta}\tilde{\rho}^\alpha(-k,-\omega)\,A(k,\omega)\,\phi^\alpha(k,\omega) \\ &-\delta_{\alpha,\beta}D\,\rho_0\,k^2\,\tilde{\rho}^\alpha(-k,-\omega)\,\tilde{\rho}^\alpha(k,\omega) \\ &-\Lambda(k,\omega)\tilde{\rho}^\alpha(k,\omega)\tilde{\rho}^\beta(-k,-\omega)\Big] \end{aligned} \tag{37}$$

$$S_{\text{Dean}} = D\sum_\alpha\int_{q,k,t}(q\cdot k)\phi^\alpha(-q-k,t)\tilde{\rho}^\alpha(q,t)\tilde{\rho}^\alpha(k,t), \tag{38}$$

$$\begin{aligned} S_{V,3} = &\frac{\Delta_F\mu^2\ell_a^4}{2d^2}\rho_0\sum_{\alpha,\beta}\int_k\int_{q,\Omega}C(k)\left(\frac{d}{\ell_a^2}+k^2\right) \\ &k\cdot\Big[v(k,q)\tilde{\rho}^\alpha(k,0)\phi^\beta(q,\Omega)\tilde{\rho}^\beta(-k-q,-\Omega) \\ &+v(-k,q)\phi^\alpha(q,\Omega)\tilde{\rho}^\alpha(k-q,-\Omega)\tilde{\rho}^\beta(-k,0)\Big], \end{aligned} \tag{39}$$

$$\begin{aligned} S_{V,4} = &\frac{\Delta_F\mu^2\ell_a^4}{2d^2}\sum_{\alpha,\beta}\int_k\int_{q,\Omega}\int_{q',\Omega'}\Big[C(k)\,v(-k,q)\cdot v(k,q') \\ &\phi^\alpha(q,\Omega)\tilde{\rho}^\alpha(k-q,-\Omega)\phi^\beta(q',\Omega')\tilde{\rho}^\beta(-k-q',-\Omega')\Big]. \end{aligned} \tag{40}$$

where

$$A(k,\omega) \equiv -i\omega + Dk^2 + \kappa k^4, \tag{41}$$

$$\Lambda(k,\omega) \equiv 2\pi\delta(\omega)\frac{\Delta_F\mu^2\ell_a^4}{2d^2}\rho_0^2C(k)k^2\left(\frac{d}{\ell_a^2}+k^2\right)^2, \tag{42}$$

$$C(k) \equiv \exp\left[-\frac{\ell_V^2k^2}{2}\right], \tag{43}$$

$$v(k,q) \equiv -(k+q)\left(\frac{d}{\ell_a^2}+k^2+k\cdot q\right) \tag{44}$$

S_{Dean} denotes the three-field vertex produced by DK noise, $S_{V,3}$ and $S_{V,4}$ are is the three- and four-field vertices arising from V, and S_G is the Gaussian part. The term inside the integral in Eqn. 37 can be written as

$$\frac{1}{2}\begin{pmatrix}\phi & \tilde{\rho}\end{pmatrix}\mathbf{M}\begin{pmatrix}\phi \\ \tilde{\rho}\end{pmatrix}, \tag{45}$$

where

$$\mathbf{M} = \begin{pmatrix} 0 & A(-k,-\omega)\delta_{\alpha,\beta} \\ A(k,\omega)\delta_{\alpha,\beta} & -2N_{\alpha\beta}(k,\omega)\end{pmatrix}, \tag{46}$$

$$N_{\alpha\beta}(k,\omega) = D\rho_0k^2\delta_{\alpha,\beta} + \Lambda(k,\omega) \tag{47}$$

The inverse of this matrix is

$$\mathbf{M}^{-1} = \frac{1}{A(k,\omega)A(-k,-\omega)}\begin{pmatrix} 2N_{\alpha\beta}(k,\omega) & A(-k,-\omega)\delta_{\alpha,\beta} \\ A(k,\omega)\delta_{\alpha,\beta} & 0\end{pmatrix} \tag{48}$$

From here, the propagators can be readily read off:

$$G^{\alpha\beta}_{\phi\tilde{\rho}}(k,\omega;k',\omega') = \frac{(2\pi)^{d+1}\delta_{\alpha\beta}\delta^{(d)}(k+k')\delta(\omega+\omega')}{-i\omega+Dk^2+Kk^4}, \tag{49}$$

$$G^{\alpha\beta}_{\phi\phi}(k,\omega;k',\omega') = 2N_{\alpha\beta}(k,\omega)\frac{(2\pi)^{d+1}\delta_{\alpha\beta}\delta^{(d)}(k+k')\delta(\omega+\omega')}{\omega^2+(Dk^2+Kk^4)^2}, \tag{50}$$

$$G^{\alpha\beta}_{\tilde{\rho}\tilde{\rho}}(k,\omega;k',\omega') = 0, \tag{51}$$

where we defined

$$G^{\alpha\beta}_{\phi\phi}(k,\omega;k',\omega') \equiv \langle\phi^\alpha(k,\omega)\phi^\beta(k',\omega')\rangle, \tag{52}$$

$$G^{\alpha\beta}_{\phi\tilde{\rho}}(k,\omega;k',\omega') \equiv \langle\phi^\alpha(k,\omega)\tilde{\rho}^\beta(k',\omega')\rangle, \tag{53}$$

$$G^{\alpha\beta}_{\tilde{\rho}\tilde{\rho}}(k,\omega;k',\omega') \equiv \langle\tilde{\rho}^\alpha(k,\omega)\tilde{\rho}^\beta(k',\omega')\rangle \tag{54}$$

The goal is to perform a one-loop correction around the Gaussian point. As such, there are two terms contributing to the correction: the DK vertex, and the external potential contribution. For a one-loop correction, the DK vertex S_{Dean} and $S_{V,3}$ vanish because they have an odd number of fields, and contractions reduce the number of fields by multiples of two, so the result cannot have two fields like the Gaussian. Hence, only $S_{V,4}$ is important.

B. One-loop correction from $S_{V,4}$

We now compute the contribution to the Gaussian action coming from the quartic external-potential vertex $S_{V,4}$. The Wilsonian RG step is performed by splitting the fields into slow and fast modes,

$$\phi = \phi_< + \phi_>, \qquad \tilde{\rho} = \tilde{\rho}_< + \tilde{\rho}_>, \tag{55}$$

where the fast fields have momenta in the shell

$$\Lambda e^{-d\ell} < |k| < \Lambda. \tag{56}$$

To first order in $S_{V,4}$, the correction to the effective slow action is

$$\delta S_G^{(V4)} = \langle S_{V,4} \rangle_> , \tag{57}$$

where the average is taken with respect to the Gaussian action S_G.

The quartic vertex is given by Eqn. (40). Because $G_{\tilde{\rho}\tilde{\rho}} = 0$, $\langle S_{V,4} \rangle_>$ receives contributions only from $\phi\tilde{\rho}$ and $\phi\phi$ contractions. The first type renormalizes the response kernel $A(k,\omega)$, while the second type renormalizes the noise/disorder kernel $N_{\alpha\beta}(k,\omega)$.

1. *Correction to the response kernel*

First consider contractions of the form

$$\langle \phi_> \tilde{\rho}_> \rangle . \tag{58}$$

These leave one external $\tilde{\rho}_<$ field and one external $\phi_<$ field, and therefore correct the Gaussian response term

$$\sum_\alpha \int_{p,\omega} \tilde{\rho}^\alpha(-p,-\omega) A(p,\omega) \phi^\alpha(p,\omega). \tag{59}$$

The resulting correction can be written as

$$\delta S_{\tilde{\rho}\phi}^{(V4)} = \sum_\alpha \int_{p,\omega} \tilde{\rho}^\alpha(-p,-\omega) \delta A(p,\omega) \phi^\alpha(p,\omega), \tag{60}$$

where

$$\begin{aligned} \delta A(p,\omega) = \frac{\Delta_F \mu^2 \ell_a^4}{2d^2} \int_> \frac{d^d k}{(2\pi)^d} C(k) \\ \Bigg[\frac{v(-k,k+p)\cdot v(k,p)}{-i\omega + D|k+p|^2 + \kappa|k+p|^4} + \\ + \frac{v(-k,p)\cdot v(k,p-k)}{-i\omega + D|k-p|^2 + \kappa|k-p|^4} \Bigg]. \end{aligned} \tag{61}$$

This expression is then expanded for small external momentum p corresponding to long-range behavior. The leading correction is proportional to p^2:

$$\delta A(p,\omega) = p^2 \delta D + O(p^4, p^2\omega). \tag{62}$$

After angular averaging over the internal momentum k, one obtains

$$\begin{aligned} \delta D = \frac{\Delta_F \mu^2 \ell_a^4}{2d^2} \int_> \frac{d^d k}{(2\pi)^d} C(k) \Bigg[2 \frac{\frac{d}{\ell_a^2}\left(\frac{d}{\ell_a^2} + k^2\right) - \frac{k^4}{d}}{Dk^2 + \kappa k^4} \\ - \frac{4\frac{d}{\ell_a^2}\left(\frac{d}{\ell_a^2} + k^2\right)\left(D + 2\kappa k^2\right) k^2}{d\left(Dk^2 + \kappa k^4\right)^2} \Bigg]. \end{aligned} \tag{63}$$

The first term comes from the explicit p^2 dependence of the vertex, while the second term comes from expanding the internal response propagator in powers of the external momentum.

For a thin shell,

$$\int_> \frac{d^d k}{(2\pi)^d} f(k) = \frac{S_d}{(2\pi)^d} \Lambda^d f(\Lambda) d\ell, \tag{64}$$

where S_d is the area of the unit sphere in d dimensions. Therefore,

$$\begin{aligned} \left.\frac{dD}{d\ell}\right|_{V4} = \frac{\Delta_F \mu^2 \ell_a^4}{2d^2} \frac{S_d}{(2\pi)^d} \Lambda^d C(\Lambda) \Bigg[2 \frac{\frac{d}{\ell_a^2}\left(\frac{d}{\ell_a^2} + \Lambda^2\right) - \frac{\Lambda^4}{d}}{D\Lambda^2 + \kappa \Lambda^4} \\ - \frac{4\frac{d}{\ell_a^2}\left(\frac{d}{\ell_a^2} + \Lambda^2\right)\left(D + 2\kappa \Lambda^2\right) \Lambda^2}{d\left(D\Lambda^2 + \kappa \Lambda^4\right)^2} \Bigg]. \end{aligned} \tag{65}$$

Thus $S_{V,4}$ renormalizes the diffusion coefficient D. It does not generate a mass term because the correction begins at order p^2.

The same expansion also contains a term of order p^4, which renormalizes κ. We define this coefficient by

$$\delta A(p,\omega) = p^2 \delta D + p^4 \delta\kappa + \cdots . \tag{66}$$

Equivalently,

$$\delta\kappa = \frac{\Delta_F \mu^2 \ell_a^4}{2d^2} \int_> \frac{d^d k}{(2\pi)^d} C(k) \, \mathcal{C}_\kappa(k), \tag{67}$$

where $\mathcal{C}_\kappa(k)$ is the angularly averaged p^4 coefficient of

$$\left[\frac{v(-k,k+p)\cdot v(k,p)}{-i\omega + D|k+p|^2 + \kappa|k+p|^4} + \frac{v(-k,p)\cdot v(k,p-k)}{-i\omega + D|k-p|^2 + \kappa|k-p|^4} \right]. \tag{68}$$

Thus,

$$\left.\frac{d\kappa}{d\ell}\right|_{V4} = \frac{\Delta_F \mu^2 \ell_a^4}{2d^2} \frac{S_d}{(2\pi)^d} \Lambda^d C(\Lambda) \mathcal{C}_\kappa(\Lambda). \tag{69}$$

Since κ already has tree-level eigenvalue -2 at the diffusive fixed point, this correction is subleading compared with the correction to D.

2. Correction to the noise and disorder kernel

Next consider contractions of the two ϕ fields in $S_{V,4}$. These leave two external response fields and therefore correct the Gaussian noise/disorder term

$$-\sum_{\alpha,\beta}\int_{p,\omega} N_{\alpha\beta}(p,\omega)\tilde{\rho}^{\alpha}(p,\omega)\tilde{\rho}^{\beta}(-p,-\omega). \tag{70}$$

The relevant contraction is

$$\left\langle \phi^{\alpha}_{>}\phi^{\beta}_{>}\right\rangle . \tag{71}$$

This gives

$$\delta S^{(V4)}_{\tilde{\rho}\tilde{\rho}} = -\sum_{\alpha,\beta}\int_{p,\omega} \delta N_{\alpha\beta}(p,\omega)\tilde{\rho}^{\alpha}(p,\omega)\tilde{\rho}^{\beta}(-p,-\omega). \tag{72}$$

Using

$$v(-k,k-p)\cdot v(k,-k+p) = -p^2\left(\frac{d}{\ell_a^2}+k\cdot p\right)^2, \tag{73}$$

we find, to leading order in the external momentum,

$$v(-k,k-p)\cdot v(k,-k+p) = -\frac{d^2}{\ell_a^4}p^2+O(p^4). \tag{74}$$

Because the Gaussian action contains $-N_{\alpha\beta}\tilde{\rho}^{\alpha}\tilde{\rho}^{\beta}$, the induced correction to $N_{\alpha\beta}$ is positive:

$$\delta N_{\alpha\beta}(p,\omega) = \frac{\Delta_F\mu^2}{2}p^2\int_{>}\frac{d^dk}{(2\pi)^d}C(k)G^{\alpha\beta}_{\phi\phi}(k,\omega)+O(p^4). \tag{75}$$

Using

$$G^{\alpha\beta}_{\phi\phi}(k,\omega) = \frac{2N_{\alpha\beta}(k,\omega)}{\omega^2+\left(Dk^2+\kappa k^4\right)^2}, \tag{76}$$

we obtain

$$\delta N_{\alpha\beta}(p,\omega) = \Delta_F\mu^2p^2\int_{>}\frac{d^dk}{(2\pi)^d}C(k)\frac{N_{\alpha\beta}(k,\omega)}{\omega^2+\left(Dk^2+\kappa k^4\right)^2}. \tag{77}$$

Now decompose the Gaussian noise kernel into the DK noise part and the quenched-potential part:

$$N_{\alpha\beta}(k,\omega) = D\rho_0k^2\delta_{\alpha\beta}+\Lambda(k,\omega), \tag{78}$$

where

$$\Lambda(k,\omega) = 2\pi\delta(\omega)\frac{\Delta_F\mu^2\ell_a^4\rho_0^2}{2d^2}C(k)k^2\left(\frac{d}{\ell_a^2}+k^2\right)^2. \tag{79}$$

The diagonal noise amplitude is

$$\mathcal{D}\equiv D\rho_0. \tag{80}$$

The DK noise part therefore receives the correction coming exclusively from the term in $N_{\alpha\beta}(k,\omega)$ in front of $\delta_{\alpha\beta}$:

$$\delta\mathcal{D} = \Delta_F\mu^2\mathcal{D}\int_{>}\frac{d^dk}{(2\pi)^d}C(k)\frac{k^2}{\left(Dk^2+\kappa k^4\right)^2}. \tag{81}$$

Because we are interested in long time scales, we project to the $\omega\to 0$ space, thereby also ensuring that $\mathcal{D}$ remains white noise.

Hence,

$$\left.\frac{d\mathcal{D}}{d\ell}\right|_{V4} = \Delta_F\mu^2\mathcal{D}\frac{S_d}{(2\pi)^d}\Lambda^dC(\Lambda)\frac{\Lambda^2}{\left(D\Lambda^2+\kappa\Lambda^4\right)^2}. \tag{82}$$

The quenched-disorder Gaussian amplitude receives

$$\delta\left(\frac{\Delta_F\mu^2\ell_a^4\rho_0^2}{2d^2}\right) = \Delta_F\mu^2\left(\frac{\Delta_F\mu^2\ell_a^4\rho_0^2}{2d^2}\right)\cdot\int_{>}\frac{d^dk}{(2\pi)^d}C(k)^2\frac{k^2\left(\frac{d}{\ell_a^2}+k^2\right)^2}{\left(Dk^2+\kappa k^4\right)^2}. \tag{83}$$

Therefore,

$$\left.\frac{d}{d\ell}\left(\frac{\Delta_F\mu^2\ell_a^4\rho_0^2}{2d^2}\right)\right|_{V4} = \Delta_F\mu^2\left(\frac{\Delta_F\mu^2\ell_a^4\rho_0^2}{2d^2}\right)\frac{S_d}{(2\pi)^d}\Lambda^dC(\Lambda)^2\cdot\frac{\Lambda^2\left(\frac{d}{\ell_a^2}+\Lambda^2\right)^2}{\left(D\Lambda^2+\kappa\Lambda^4\right)^2}. \tag{84}$$

V. GAUSSIAN FLOW EQUATIONS

Combining these loop corrections with the tree-level scaling near the diffusive fixed point gives the Gaussian flow equations. Under

$$x\to bx,\qquad t\to b^zt,\qquad \phi\to b^{-d}\phi,\qquad \tilde{\rho}\to\tilde{\rho}, \tag{85}$$

the tree-level flows are

$$\left.\frac{dD}{d\ell}\right|_{\text{tree}} = (z-2)D, \tag{86}$$

$$\left.\frac{d\kappa}{d\ell}\right|_{\text{tree}} = (z-4)\kappa, \tag{87}$$

and

$$\left.\frac{d}{d\ell}\left(\frac{\Delta_F\mu^2\ell_a^4\rho_0^2}{2d^2}\right)\right|_{\text{tree}} = (2-d)\left(\frac{\Delta_F\mu^2\ell_a^4\rho_0^2}{2d^2}\right). \tag{88}$$

Including the one-loop corrections from $S_{V,4}$, one obtains

$$\frac{dD}{d\ell} = (z-2)D+\frac{\Delta_F\mu^2\ell_a^4}{2d^2}\frac{S_d}{(2\pi)^d}\Lambda^dC(\Lambda)\left[2\frac{\frac{d}{\ell_a^2}\left(\frac{d}{\ell_a^2}+\Lambda^2\right)-\frac{\Lambda^4}{d}}{D\Lambda^2+\kappa\Lambda^4}-\frac{4\frac{d}{\ell_a^2}\left(\frac{d}{\ell_a^2}+\Lambda^2\right)\left(D+2\kappa\Lambda^2\right)\Lambda^2}{d\left(D\Lambda^2+\kappa\Lambda^4\right)^2}\right], \tag{89}$$

$$\frac{d\kappa}{d\ell} = (z-4)\kappa + \frac{\Delta_F \mu^2 \ell_a^4}{2d^2}\frac{S_d}{(2\pi)^d}\Lambda^d C(\Lambda)\mathcal{C}_\kappa(\Lambda), \tag{90}$$

$$\frac{d\mathcal{D}}{d\ell} = (d+z-2)\mathcal{D} + \Delta_F \mu^2 \mathcal{D}\frac{S_d}{(2\pi)^d}\Lambda^d C(\Lambda)\frac{\Lambda^2}{(D\Lambda^2+\kappa\Lambda^4)^2}, \tag{91}$$

and

$$\frac{d}{d\ell}\left(\frac{\Delta_F \mu^2 \ell_a^4 \rho_0^2}{2d^2}\right) = (2-d)\left(\frac{\Delta_F \mu^2 \ell_a^4 \rho_0^2}{2d^2}\right) + \Delta_F \mu^2 \left(\frac{\Delta_F \mu^2 \ell_a^4 \rho_0^2}{2d^2}\right)\frac{S_d}{(2\pi)^d}\Lambda^d C(\Lambda)^2 \cdot \frac{\Lambda^2\left(\frac{d}{\ell_a^2}+\Lambda^2\right)^2}{(D\Lambda^2+\kappa\Lambda^4)^2}. \tag{92}$$

At the diffusive fixed point one chooses $z = 2$. Then the flow equations reduce to

$$\frac{dD}{d\ell} = \frac{\Delta_F \mu^2 \ell_a^4}{2d^2}\frac{S_d}{(2\pi)^d}\Lambda^d C(\Lambda)\left[2\frac{\frac{d}{\ell_a^2}\left(\frac{d}{\ell_a^2}+\Lambda^2\right)-\frac{\Lambda^4}{d}}{D\Lambda^2+\kappa\Lambda^4} - \frac{4\frac{d}{\ell_a^2}\left(\frac{d}{\ell_a^2}+\Lambda^2\right)\left(D+2\kappa\Lambda^2\right)\Lambda^2}{d\left(D\Lambda^2+\kappa\Lambda^4\right)^2}\right], \tag{93}$$

$$\frac{d\kappa}{d\ell} = -2\kappa + \frac{\Delta_F \mu^2 \ell_a^4}{2d^2}\frac{S_d}{(2\pi)^d}\Lambda^d C(\Lambda)\mathcal{C}_\kappa(\Lambda), \tag{94}$$

$$\frac{d\mathcal{D}}{d\ell} = d\mathcal{D} + \Delta_F \mu^2 \mathcal{D}\frac{S_d}{(2\pi)^d}\Lambda^d C(\Lambda)\frac{\Lambda^2}{(D\Lambda^2+\kappa\Lambda^4)^2}, \tag{95}$$

and

$$\frac{d}{d\ell}\left(\frac{\Delta_F \mu^2 \ell_a^4 \rho_0^2}{2d^2}\right) = (2-d)\left(\frac{\Delta_F \mu^2 \ell_a^4 \rho_0^2}{2d^2}\right) + \Delta_F \mu^2 \left(\frac{\Delta_F \mu^2 \ell_a^4 \rho_0^2}{2d^2}\right)\frac{S_d}{(2\pi)^d}\Lambda^d C(\Lambda)^2 \frac{\Lambda^2\left(\frac{d}{\ell_a^2}+\Lambda^2\right)^2}{(D\Lambda^2+\kappa\Lambda^4)^2}. \tag{96}$$

The quartic external-potential vertex therefore directly renormalizes the Gaussian sector. It shifts the diffusion coefficient D, the higher-gradient coefficient κ, the local DK noise amplitude $\mathcal{D} = D\rho_0$, and the Gaussian quenched-disorder amplitude

$$\frac{\Delta_F \mu^2 \ell_a^4 \rho_0^2}{2d^2}. \tag{97}$$

No mass term is generated, since the response correction vanishes at zero external momentum. Thus the diffusive structure of the Gaussian fixed point is preserved, while the external potential changes the Gaussian transport and noise parameters at one loop.

VI. DISCUSSION

This work performs an RG treatment of a microscopically derived active field theory for active Brownian particles in an external quenched potential. Starting from the low-persistence-length equation of motion derived by Tailleur and Kafri [7], the paper constructs an MSRJD dynamical action, introduces Dean-Kawasaki noise to preserve density conservation at the fluctuating level, and uses replicas to average over the quenched potential. This produces an effective dynamical action in which the external potential generates both equilibrium-like and genuinely nonequilibrium operators. The resulting theory provides a controlled way to ask whether disorder or externally imposed potentials can affect the large-scale dynamics of weakly active systems.

The main tree-level result is that, around the diffusion-dominated fixed point, the equilibrium potential coupling is the most relevant external-potential contribution. Its scaling eigenvalue is $2-d$, making it relevant below two dimensions, marginal in two dimensions, and irrelevant above two dimensions. Inspection of the one-loop correction reveals the term to be marginally relevant, as the correction is positive. By contrast, the nonequilibrium corrections generated by activity are irrelevant for all positive spatial dimensions at this fixed point. This is physically consistent with the low-activity regime assumed in the microscopic derivation: at large scales, the system remains controlled primarily by diffusive, equilibrium-like dynamics, while the active corrections appear only as higher-gradient perturbations. Thus, within the validity of the theory, activity does not destabilize the diffusive fixed point at tree level.

The one-loop calculation refines this conclusion by showing how the quartic external-potential vertex renormalizes the Gaussian sector. In particular, the external potential shifts the diffusion coefficient, the higher-gradient coefficient κ, the local Dean-Kawasaki noise amplitude, and the Gaussian quenched-disorder amplitude. The most relevant of these couplings has the same scaling structure as a random-force contribution in an equilibrium diffusive system, while the explicitly active pieces are suppressed by additional gradients and powers of the persistence length. Therefore, in the low-ℓ_a regime considered here, the dominant large-scale effect of the potential is essentially equilibrium-like, with activity entering perturbatively through irrelevant corrections.

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