Directed Percolation Transition in Random Media

David DeCoursy
Physics Department, Massachusetts Institute of Technology

In this paper, we discuss a 1+1d 45° rotated square lattice with quenched, missing random bonds, occuring with probability p. This model contains a percolation transition at a critical value p_c, linking both the KPZ and DP universality classes through this transition. We analyze the system through the lens of the largest eigenvalue of the fully-weighted transfer matrix, and discuss whether this approach can be extended towards similar models.

I. Introduction

The subject of directed paths in random media, or DPRM for short, is a very well-studied class of systems that fall under the umbrella of the larger Kardar-Parisi-Zhang (KPZ) universality class, which governs a wide range of phenomena, ranging from turbulence and magnetic domain walls, to semiconductors and interface growth[1]. In this discussion, we link the study of DPRM to the related, yet distinct, subject of percolation. More specifically, we consider a model of directed paths on a 1+1d 45° rotated square lattice in which bonds are randomly removed with probability p. In the usual DPRM problem, the randomness changes the relative weights of different paths, but they are still generically able to propagate through the lattice. In the present model, however, the disorder can remove bonds entirely, meaning that directed paths may become blocked. This introduces a natural percolation transition: for small enough p, paths can survive to long times and one expects the usual KPZ/DPRM scaling behavior, whereas above a critical probability p_c, the missing bonds prevent directed paths from propagating indefinitely, signaling percolating behavior corresponding to a different, directed percolation (DP) universality class.

The goal of this discussion is to characterize this transition by using the transfer matrix of a single directed path, which can be written as a time-ordered product of random, single-step transfer matrices. We focus on the largest eigenvalue of this matrix, since at long times it should contain the dominant behavior of the system. The central question of this discussion, and its overall goal, is to evaluate whether this approach provides a practical way of characterizing both DPRM and DP when in the same framework, and whether or not it could benefit later analyses for related models.

II. The System

A. The Model

We consider a 1+1d 45° rotated square lattice, with random bonds removed with probability p, and directed paths trickling down the lattice. Eventually, at some p_c, we expect all directed paths to eventually get stopped, signaling a transition to percolating behavior. To be more exact, the system is

$$Z[\{K\}] = \sum_{\{\sigma_i\}} \prod_{\langle i,j\rangle} e^{K_{ij}\sigma_i\sigma_j} \tag{1}$$

$$P(K_{ij}) = p\delta_{K_{ij},0} + (1-p)\delta_{K_{ij},K} \tag{2}$$

Since the bonds themselves are now quenched random variables, the fully-weighted transfer matrix will now be a *time-ordered* product of single-step transfer matrices, which themselves are random[2, 3]. We can see this by using the Markovian property of the system. Letting $W(x,t) \equiv \langle x|W(t)|0\rangle$ denote the total weight of going from $(0,0)$ to (x,t), we can see that

$$W(i,t+1) = \tau_{i+1\to i}W(i+1,t) + \tau_{i-1\to i}W(i-1,t)$$

where we have introduced the bond weight $\tau_{ij} \equiv \tanh(K_{ij})$. Matching this to the form $W(x,t+1) = \sum_{x'}\langle x|\mathrm{T}(t)|x'\rangle W(x',t)$, we obtain the an expression for the single step transfer matrix

$$\langle i|\mathrm{T(t)}|i\pm 1\rangle = \tau_{i\pm 1\to i} \tag{3}$$

To simplify our results for the numerical analysis later, from here on we will consider our system in the $T=0$ limit, so that $K\to\infty$ and the fully weighted transfer matrix will only take integer value bond weights. Relabeling the bond weights as just $\tau_j = \tau_{j\to i}$, and $i' \equiv -i$, we can write the transfer matrices as [2]

$$\mathrm{T(t)} = \begin{pmatrix} 0 & \tau_1 & 0 & 0 & \cdots \\ \tau_{1'} & 0 & \tau_2 & 0 & \cdots \\ 0 & \tau_{2'} & 0 & \tau_3 & \cdots \\ 0 & 0 & \tau_{3'} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{4}$$

$$W(t) = \prod_{t'=1}^{t} \mathrm{T(t')} \tag{5}$$

$$P(\tau) = p\delta_{\tau,0} + (1-p)\delta_{\tau,1} \tag{6}$$

By Perron-Frobenius theorem, we know that the largest eigenvalue of this matrix will be both unique and nonnegative, justifying our approach to studying this system.

B. Scaling and Relevant Variables

We will take the standard approach to analyzing this system as is commonly done for the nonrandom case, albeit we have to keep in mind our relevant variables are

now random, and depend on the specific quenched landscape that is realized. On intuitive grounds, we expect that for a small enough p, the presence of missing bonds in the lattice aren't really felt by the directed paths, and so the system should roughly display the standard KPZ behavior. However, at a large enough p one would expect the presence of missing bonds to start preventing paths from propagating, resulting in percolating behavior. We will let the transition between these two behaviors be denoted by a critical probability p_c.

For $p \ll p_c$, we expect the standard KPZ behavior, and thus the free energy scales as [2]

$$F(t) \simeq \bar{f}t + \sigma t^{1/3}\chi \tag{7}$$

with the standard KPZ $t^{1/3}$ fluctuations and Tracy-Widom random variable χ. We will let this exponent stay general and denote it as β, which by the argument above we postulate to be 1/3. As the system approaches p_c^-, we need to account for the presence of missing bonds in the lattice that start to kill directed paths. We can correct for this by introduce a scaling function g and an appropriate correlation length[4], $\xi_\parallel$, so that

$$F(t) \simeq \bar{f}t + \sigma t^{\beta} g(t/\xi_\parallel)\chi, \tag{8}$$

$$\text{with } \xi_\parallel \sim |p - p_c|^{-\nu_\parallel} \tag{9}$$

This works because for $p \ll p_c$, $t/\xi_\parallel \sim 0 \to g \sim 1$ and the standard KPZ behavior is achieved, while closer to p_c the missing bonds become more and more relevant.

Introducing $\lambda(t)$ as the largest eigenvalue of $W(t)$, we can thus write

$$F(t) = \log Z(t) \simeq \log \lambda(t) \tag{10}$$

to leading order and for $t \gg 1$. Matching (11) to (9), we can thus infer the properties of the system by characterizing the behavior of $\log \lambda(t)$.

However, for $p > p_c$, since the possibility arises of no directed paths of length t being possible, we would get $\lambda = 0$ and thus $\log \lambda(t)$ would no longer be well defined. We can get around this by introducing instead a survival probability $P_s(t) \equiv P(\lambda(t) > 0)$ to characterize instead. We can then expect on intuitive grounds something like exponential decay for $p > p_c$, i.e. $P_s \sim e^{-t/\xi_\parallel}$, and something like a power law at criticality, i.e. $P_s \sim t^{-\delta}$ for some corresponding critical exponent δ [5]. This, in theory, gives us a method for extracting the behavior of the system for $p > p_c$ in terms of $\lambda(t)$.

III. Numerics

A. Method

We first obtain an estimate for p_c by looking at the behavior of the survival probability P_s near criticality. Since $P_s(t; p_c, L) \sim t^{-\delta}$, we can use the fact that the ratio

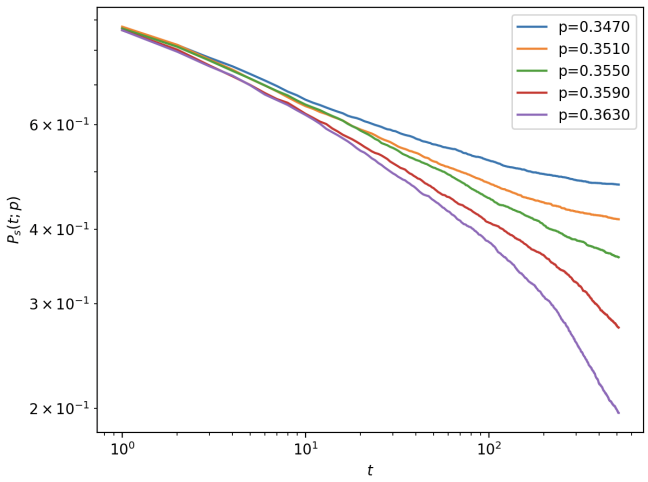

FIG. 1: $P_s(t; p, L = 256)$ vs t for different p. Note the change in behavior around $p_c \simeq 0.355$, the inflection signifying a transition to percolating behavior.

$D(t, p_c) \equiv P_s(2t; p_c, L)/P_s(t; p_c, L) = 2^{-\delta}$ is independent of t to get an estimate for p_c. Since each $D(t, p = p_c)$ will have the same value for each t, we can sweep over the possible t and identify the critical point as the shared intersection. This is shown in Figure 3 and 4. To obtain an estimate for the exponent $\nu_\parallel$, we can use the scaling form of P_s near criticality,

$$P_s(t, p) \simeq t^{-\delta} g((p - p_c)t^{1/\nu_\parallel}) \tag{11}$$

$$\Rightarrow \frac{\partial}{\partial p} \log P_s(t, p) \simeq t^{1/\nu_\parallel} \frac{g'((p - p_c)t^{1/\nu_\parallel})}{g((p - p_c)t^{1/\nu_\parallel})} \tag{12}$$

$$\Rightarrow |\frac{\partial}{\partial p} \log P_s(t, p)|_{p=p_c} \sim t^{1/\nu_\parallel} \tag{13}$$

since the scaling form vanishes to a constant at the criticality. Plotting this on a log-log plot thus gives a line with slope $1/\nu_\parallel$, as shown in figure 2. Finally, to obtain δ, we can rather simply plot $P_s(t, p_c)$ vs t directly in a log-log plot to obtain the exponent. It is also worth noting that due to finite size limitations, we probe the behavior of the system in the standard KPZ range $1 \ll t \ll L^{3/2}$.

B. Analysis and Results

Carrying out this procedure for $L = 256$, the quantities were measured to be:

$$p_c \simeq 0.355413 \tag{14}$$

$$\delta \simeq 0.161287 \tag{15}$$

$$\nu_\parallel \simeq 1.672986 \tag{16}$$

Given the roughness of the method and analysis, we have managed to get a fairly good estimate of the behavior of the critical point. Equations 15-17 can be compared to the better known values for directed percolation

in 1+1d of $p_c \simeq .3553$, $\delta \simeq 0.159464(6)$, and $\nu_\parallel \simeq 1.7338$ [5]. The results, while not exactly the same as the more well-documented estimations of the percolation quantities, serve as a good enough estimate considering the system size and the roughness of the method used. This analysis can be performed more carefully to improve the estimates of the relevant quantities of interest, though the practicality of using the eigenvalue $\lambda(t)$ is clearly evident as a possible method of analyzing this system.

IV. Conclusion

We have thus shown the viability of characterizing the largest eigenvalue of the fully weighted transfer matrix as a possible avenue for studying this particular directed percolation model. Its strengths lie in its direct calculation for $p < p_c$ and its relation to the survival probability for $p > p_c$, providing a straightforward way of analyzing the system. Even with the rather rough analysis above, the extracted estimates for p_c, δ, and $\nu_\parallel$ agree reasonably well with the known values for directed percolation in $1 + 1$d, while the subcritical behavior of $\log \lambda(t)$ is consistent with the expected KPZ scaling. This suggests that $\lambda(t)$ contains both the KPZ behavior below the transition and the directed percolation behavior at and above the transition. A more careful finite-size scaling analysis, along with a larger system size, would likely improve the numerical estimates and clarify the crossover governed by $g(t/\xi_\parallel)$.

Additionally, the success of this method also suggests a possible approach for studying a system of multiple directed paths propagating through the same random landscape. This would require constructing the corresponding transfer matrix, and applying a similar analysis as the one above, serving as a natural extension to this project.

Acknowledgments

The author would like to thank the 334 staff for an amazing semester, and Professor Kardar for introducing them to the subject.

V. Figures

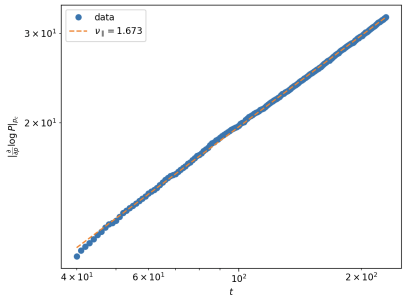

FIG. 2: Calculation of $\nu_\parallel$ using the survival probability at criticality.

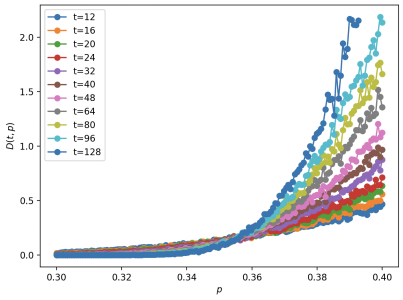

FIG. 3: $D(t, p)$ as a function of p over different t slices.

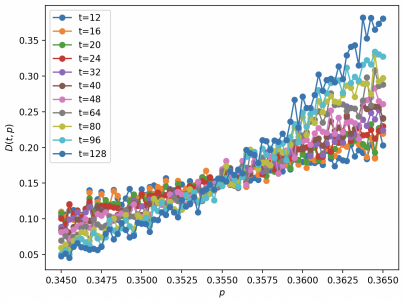

FIG. 4: Zoom in near $p_c \simeq 0.3554$.

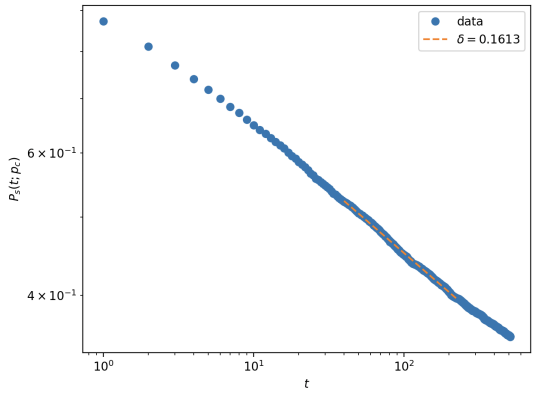

FIG. 5: Survival probability P_s vs t at criticality.

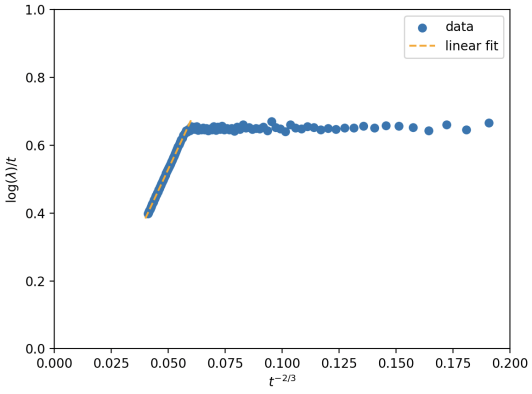

FIG. 6: Confirmation of the $\beta = 1/3$ KPZ scaling for $p \ll p_c$.

[1] T. Halpin-Healy and K. A. Takeuchi, A kpz cocktail–shaken, not stirred: Toasting 30 years of kinetically roughened surfaces, Journal of Statistical Physics **160**, 794 (2015), arXiv:1505.01910 [cond-mat.stat-mech].

[2] S. Y. S. Chu, *Fluctuating Interfaces and Paths in Disordered and Non-Equilibrium Systems*, Ph.d. thesis, Massachusetts Institute of Technology (2019).

[3] S. Mu, A. A. Saberi, R. Moessner, and M. Kardar, Directed polymer transfer matrices as a unified generator of distinct one-point fluctuation laws (2026), arXiv:2603.14477 [cond-mat.soft].

[4] E. Perlsman and S. Havlin, The directed-polymer–directed-percolation transition, EPL **46**, 13 (1999), arXiv:cond-mat/9904106 [cond-mat.stat-mech].

[5] I. Jensen, Low-density series expansions for directed percolation i: A new efficient algorithm with applications to the square lattice, Journal of Physics A: Mathematical and General **32**, 5233 (1999), arXiv:cond-mat/9906036 [cond-mat.stat-mech].

[6] M. Kardar and Y.-C. Zhang, Scaling of directed polymers in random media, Physical Review Letters **58**, 2087 (1987).

[7] M. Kardar, *Statistical Physics of Fields* (Cambridge University Press, Cambridge, 2007).

[8] S. Chu and M. Kardar, Probability distributions for directed polymers in random media with correlated noise, Physical Review E **94**, 010101 (2016), arXiv:1605.04298 [cond-mat.stat-mech].