Zero-temperature glassy transition in the θ-GLV model

Grigor Adamyan[1]
[1]*Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA*

The generalized Lotka-Volterra (θ-GLV) model has been recently proposed as an effective framework to describe diversity in large systems with random interactions. In this work, we investigate the thermodynamics of a θ-GLV model subjected to demographic noise. By mapping the stochastic dynamics to an equilibrium statistical mechanics model, we employ the replica method to analyze its zero-temperature phase diagram. Using the de Almeida-Thouless criterion, we determine the transition from a stable, replica-symmetric coexistence phase to a replica symmetry-breaking phase characterized by species extinctions. For the standard GLV case ($\theta = 1$), we recover the critical interaction variance $\sigma_c = 1/\sqrt{2}$ previously obtained via the cavity method. For $\theta < 1$, we show that the system loses stability, resulting in unbounded population growth. For $\theta > 1$, we derive the self-consistent integral equations that determine the phase boundary. Furthermore, we find that in the $\theta > 1$ regime, the species abundance remains finite as the system approaches the critical threshold, undergoing a discontinuous transition to the extinction state. This is in contrast to the continuous transition observed in the standard GLV model.

I. INTRODUCTION

Classical ecological models, particularly generalized Lotka–Volterra (GLV) systems with random interspecies interactions, predict that sufficiently large and complex communities tend to become unstable, leading to a loss of diversity [1]. This theoretical prediction contrasts with the widespread coexistence of many species observed in natural ecosystems [2]—an inconsistency often referred to as May's diversity–stability paradox [3].

Recent work [4] has explored this paradox by showing that introducing migration and environmental fluctuations can help stabilize highly diverse communities. Specifically, it was noted that these spatiotemporal noises effectively renormalize the standard GLV equation into a θ-GLV one. This suggests that the coarse-grained, effective dynamics of such ecosystems might be described by the θ-GLV model.

Motivated by this, we investigate the thermodynamic properties of the θ-GLV model subjected to demographic noise. Because such a system satisfies detailed balance, the stochastic dynamics map into an equilibrium system, which can be analyzed using the tools of statistical mechanics. By generalizing the replica approach used for standard GLV models [5], we study the zero-temperature phase transition that separates the stable, replica-symmetric coexistence phase from a replica symmetry-breaking extinction phase.

Our analysis yields several behaviors depending on the stiffness of the saturation term, θ. For the standard GLV case ($\theta = 1$), we recover the critical interaction variance $\sigma_c = 1/\sqrt{2}$ obtained previously via the cavity method [6]. For the $\theta < 1$, we find that the system loses stability, resulting in unbounded population growth. For the stiffer regime, $\theta > 1$, we derive the integral equation governing the phase boundary of the replica symmetry-breaking transition. Furthermore, we observe that for $\theta > 1$, the species abundance remains finite as the system approaches the critical threshold, undergoing a discontinuous transition to the extinction state. This contrasts with the continuous transition found in the standard GLV model.

II. THE θ-GLV MODEL WITH DEMOGRAPHIC NOISE

A. Stochastic Dynamical Equations

We consider a system of S interacting species, indexed by $i = 1, \ldots, S$. The natural framework to describe their population dynamics is the generalized Lotka-Volterra (GLV) model [5]:

$$\dot{N}_i = -N_i \left(\frac{\partial V_i(N_i)}{\partial N_i} + \sum_{j \neq i} \alpha_{ij} N_j \right) + \eta_i(t) + \lambda, \qquad (1)$$

where N_i is the abundance of species i. The potential function $V_i(N_i)$ captures the internal single-species dynamics, while the matrix elements α_{ij} represent interspecies interactions. The noise term $\eta_i(t)$ (interpreted in the Itô sense) is a Gaussian process with zero mean and covariance:

$$\langle \eta_i(t) \eta_j(t') \rangle = 2T N_i(t) \delta_{ij} \delta(t - t'). \qquad (2)$$

This multiplicative variance is characteristic of demographic noise. To prevent the populations from reaching an absorbing boundary (finite-time extinction), we also introduce a small, constant immigration rate, $\lambda > 0$.

Traditionally, the potential $V_i(N_i)$ takes the standard Lotka-Volterra form:

$$V_i(N_i) = -\rho_i \left(K_i N_i - \frac{N_i^2}{2} \right), \qquad (3)$$

where the linear term corresponds to exponential growth and the quadratic term enforces a carrying capacity, preventing indefinite growth. Here, $\rho_i = r_i / K_i$ is the ratio of the intrinsic growth rate r_i to the carrying capacity K_i.

In ecology, however, there is no fundamental requirement for the saturation term to be strictly quadratic. Furthermore, recent studies suggest that the effective, coarse-grained dynamics could be described by θ-GLV model [4]. Motivated by this, we investigate how the shape of the confining potential influences the macroscopic phase transition. Replacing the standard potential in Eq. (3), we introduce the generalized θ-LV potential:

$$V_i^\theta(N_i) = -\rho_i \left(K_i^\theta N_i - \frac{N_i^{1+\theta}}{1+\theta} \right), \tag{4}$$

where the exponent $\theta > 0$ controls the stiffness of the saturation term. A smaller θ yields a softer potential, while larger values create a steeper boundary. For an isolated single species, the stable fixed-point abundance remains independent of θ, always saturating at the carrying capacity:

$$N_i^* = K_i. \tag{5}$$

To enable a statistical mechanics treatment, we assume the interaction matrix is symmetric ($\alpha_{ij} = \alpha_{ji}$). The elements α_{ij} are drawn from a Gaussian distribution with a mean and variance that scale with the system size S:

$$\overline{\alpha_{ij}} = \frac{\mu}{S}, \quad \mathrm{Var}(\alpha_{ij}) = \frac{\sigma^2}{S}. \tag{6}$$

Because the Langevin equation (1) with symmetric interactions and demographic noise satisfies detailed balance, the stochastic dynamics strictly relax to a steady-state equilibrium. We can therefore define the effective Hamiltonian of the system:

$$H = \sum_i V_i^\theta(N_i) + \sum_{i<j} \alpha_{ij} N_i N_j + (T-\lambda) \sum_i \ln N_i. \tag{7}$$

With this effective Hamiltonian established, we can proceed to analyze the thermodynamic properties of the ecosystem using the replica method [5].

III. THERMODYNAMICS OF THE θ-GLV MODEL

A. The Replica Trick

Given the effective Hamiltonian, we can analyze the thermodynamics of the system by computing the average free energy F using the standard replica trick:

$$-\beta F = \lim_{n\to 0} \frac{\ln \overline{Z^n}}{n}. \tag{8}$$

By applying the Hubbard-Stratonovich transformation to the replicated partition function $\overline{Z^n}$, we can decouple the interactions between different species. Instead, the resulting effective theory couples the abundances of the same species N_i^α across different replicas α:

$$Z_i = \int \left(\prod_{\alpha=1}^{n} dN_i^\alpha \right) \exp\left(-\beta H_{\mathrm{eff}}(\{N_i^\alpha\})\right), \tag{9}$$

where the effective single-species Hamiltonian is given by:

$$\begin{aligned} H_{\mathrm{eff}}(\{N_i^\alpha\}) = &-\beta\rho^2\sigma^2 \sum_{\alpha<\beta} N_i^\alpha N_i^\beta Q_{\alpha\beta} \\ &-\beta\rho^2\sigma^2 \sum_\alpha \frac{Q_{\alpha\alpha}}{2} (N_i^\alpha)^2 + \sum_\alpha \rho\mu H_\alpha N_i^\alpha \\ &+ \sum_\alpha \left[V_i^\theta(N_i^\alpha) + (T-\lambda)\ln N_i^\alpha \right]. \end{aligned} \tag{10}$$

Here, $Q_{\alpha\beta}$ represents the overlap matrix between replicas, and H_α acts as an effective external field.

B. Replica Symmetric Ansatz

To proceed, we evaluate the partition function under the replica symmetric (RS) ansatz, which assumes that all replicas are equivalent:

$$Q_{\alpha\beta} = q_0 \quad \text{if} \quad \alpha \neq \beta, \tag{11}$$

$$Q_{\alpha\alpha} = q_d \quad \text{if} \quad \alpha = \beta, \tag{12}$$

$$H_\alpha = h \quad \forall\alpha. \tag{13}$$

By performing an additional Hubbard-Stratonovich transformation to decouple the replicas, we introduce a standard Gaussian variable z_i, which yields:

$$Z_i = \int_{-\infty}^{+\infty} \frac{dz_i}{\sqrt{2\pi}} e^{-z_i^2/2} \int \prod_{\alpha=1}^{n} dN_i^\alpha \, e^{-\beta \sum_\alpha H_{\mathrm{RS}}(N_i^\alpha, z_i)}, \tag{14}$$

where the RS Hamiltonian is:

$$\begin{aligned} H_{\mathrm{RS}}(N_i, z_i) = &-\beta\rho^2\sigma^2(q_d - q_0)\frac{N_i^2}{2} + \rho(\mu h - \sigma\sqrt{q_0}\, z_i) N_i \\ &+ V_i^\theta(N_i) + (T-\lambda)\ln N_i. \end{aligned} \tag{15}$$

Evaluating the saddle-point yields the self-consistent equations for the RS order parameters:

$$q_d = \overline{\langle N^2 \rangle}, \tag{16}$$

$$q_0 = \overline{\langle N \rangle^2}, \tag{17}$$

$$h = \overline{\langle N \rangle}, \tag{18}$$

where the angular brackets $\langle \dots \rangle$ denote the thermal average weighted by the effective single-site Hamiltonian $H_{\mathrm{RS}}(N_i, z_i)$, and the overline $\overline{(\dots)}$ represents the quenched average over the Gaussian noise, $\int \mathcal{D}z \equiv \int \frac{dz}{\sqrt{2\pi}} e^{-z^2/2}$.

C. Stability of the Replica Symmetric Solution

The stability of the RS solution against replica symmetry breaking (RSB) is determined by the sign of the replicon eigenvalue:

$$\lambda_{\mathrm{R}} = (\beta\rho\sigma)^2 \left[1 - (\beta\rho\sigma)^2 \overline{(\langle N^2\rangle - \langle N\rangle^2)^2}\right]. \tag{19}$$

The replica symmetric (RS) solution is locally stable provided $\lambda_{\mathrm{R}} > 0$ and becomes unstable when $\lambda_{\mathrm{R}} < 0$. Consequently, the de Almeida-Thouless (AT) transition point is determined by the condition $\lambda_{\mathrm{R}} = 0$ [7]. In our context, the stable RS solution corresponds to a phase of steady species coexistence, whereas the onset of replica symmetry breaking (RSB) corresponds to an extinction phase.

IV. ZERO-TEMPERATURE GLV MODEL

We analyze the zero-temperature regime by taking the limit $\beta \to \infty$. In this case, the logarithmic term in the replica symmetric Hamiltonian (15) vanishes, and the thermal averages simplify significantly as they are dominated by the saddle-point. Because thermal fluctuations vanish, we have $q_d - q_0 \to 0$, allowing us to define a single overlap order parameter $q = q_d = q_0$. However, the zero-temperature susceptibility is finite:

$$\chi = \lim_{\beta\to\infty} \beta(q_d - q_0). \tag{20}$$

In the standard GLV model ($\theta = 1$), the RS Hamiltonian (15) reduces to quadratic and linear terms:

$$H_{\mathrm{RS}}(N, z) = \rho\left[(1-\alpha_2)\frac{N^2}{2} + \alpha_1(z)N\right], \tag{21}$$

where we have defined:

$$\alpha_1(z) = \mu h - K - \sigma\sqrt{q}\, z, \tag{22}$$

$$\alpha_2 = \rho\sigma^2\chi. \tag{23}$$

A. Self-Consistency Equations

Working in the zero-temperature limit, we apply the saddle-point approximation. The condition for the global minimum of the Hamiltonian, $\frac{\partial H_{\mathrm{RS}}}{\partial N} = 0$, yields:

$$N^*(z) = -\frac{\alpha_1(z)}{1-\alpha_2}. \tag{24}$$

Since species abundances must be strictly non-negative, the physical macroscopic solution is given by:

$$N(z) = \max\left(0, -\frac{\alpha_1(z)}{1-\alpha_2}\right). \tag{25}$$

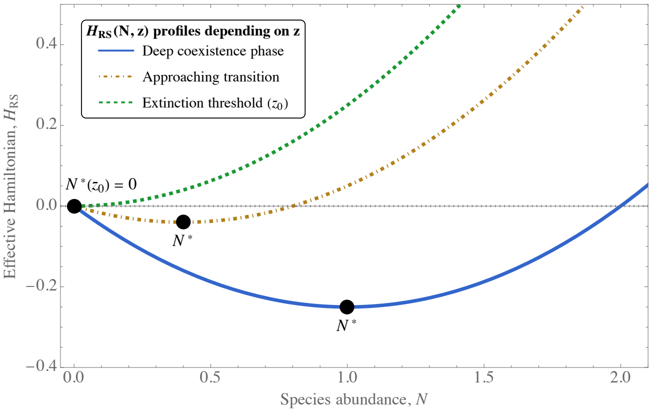

FIG. 1. Zero-temperature replica-symmetric Hamiltonian $H_{\mathrm{RS}}(N, z)$ profiles for the standard GLV model ($\theta = 1$). The system exhibits a single global minimum for any stable parameter regime ($\alpha_2 < 1$). As the z approaches z_0 (dashed green line), the saddle-point abundance N^* continuously shifts to zero. The black dots indicate the corresponding saddle-point abundance N^* for each value of z.

The critical threshold z_0, at which the species abundance vanishes, is determined by the condition $\alpha_1(z_0) = 0$, yielding:

$$z_0 = \frac{\mu h - K}{\sigma\sqrt{q}}. \tag{26}$$

As illustrated in Fig. 1, these equations show that as z approaches the z_0, the saddle-point abundance continuously shifts to zero.

In the zero-temperature limit, the thermal averages become simply $\langle N^k\rangle = N^k(z)$. Noting that $-\alpha_1(z) = \sigma\sqrt{q}(z - z_0)$, the self-consistent equations for h and q become:

$$h = \int_{z_0}^{\infty} \mathcal{D}z\, N(z) = \frac{\sigma\sqrt{q}}{1-\rho\sigma^2\chi}\phi_1(z_0), \tag{27}$$

$$q = \int_{z_0}^{\infty} \mathcal{D}z\, N^2(z) = \frac{\sigma^2 q}{(1-\rho\sigma^2\chi)^2}\phi_2(z_0), \tag{28}$$

where we have defined the family of integrals:

$$\phi_k(z_0) = \int_{z_0}^{\infty} \mathcal{D}z\, (z - z_0)^k. \tag{29}$$

The variance, which vanishes as $\beta \to \infty$, can be found to leading order by expanding the Hamiltonian around the saddle-point N^*:

$$H_{\mathrm{RS}}(N) = H_{\mathrm{RS}}(N^*) + \frac{1}{2}H''_{\mathrm{RS}}(N^*)\,(N - N^*)^2 + \ldots. \tag{30}$$

From this Gaussian fluctuations, we find:

$$\langle N^2\rangle - \langle N\rangle^2 \approx \frac{1}{\beta H''_{\mathrm{RS}}(N^*)} = \frac{1}{\beta\rho(1-\rho\sigma^2\chi)}. \tag{31}$$

Substituting this into the definition of the susceptibility, the self-consistency equation for χ (20) becomes:

$$\chi = \int_{z_0}^{\infty} \mathcal{D}z\, \beta(\langle N^2\rangle - \langle N\rangle^2) = \frac{1}{\rho(1-\rho\sigma^2\chi)}\phi_0(z_0). \tag{32}$$

B. Replica Symmetry Breaking

The replicon eigenvalue in Eq. (19) contains the quenched average of the squared variance of N. Evaluating this within the saddle-point approximation of Eq. (31), we obtain:

$$\lambda_{\rm R} = (\beta\rho\sigma)^2 \left[1 - \frac{\sigma^2}{(1-\rho\sigma^2\chi)^2}\phi_0(z_0)\right]. \tag{33}$$

The stability transition (AT line) occurs when the eigenvalue vanishes:

$$1 = \frac{\sigma^2}{(1-\rho\sigma^2\chi)^2}\phi_0(z_0). \tag{34}$$

Using the self-consistent equation for χ from Eq. (32), we can rewrite this stability condition as:

$$\chi = \frac{\sqrt{\phi_0(z_0)}}{\rho\sigma}. \tag{35}$$

Substituting this back into Eq. (34) yields:

$$\phi_0(z_0) = \frac{1}{4\sigma^2}. \tag{36}$$

Furthermore, comparing Eq. (34) with the q self-consistency equation (28), we arrive at the relation:

$$\phi_0(z_0) = \phi_2(z_0). \tag{37}$$

As detailed in the Appendix, the only solution satisfying this equality is $z_0 = 0$. Using this result in Eq. (36), we conclude that the critical interaction variance is given by:

$$\sigma_c = \frac{1}{\sqrt{2}}, \tag{38}$$

which recovers the previously obtained one via the cavity method [6].

Physically, for a small variance $\sigma < \sigma_c$, the system is in a stable replica symmetric coexistence phase, while for a large variance $\sigma > \sigma_c$, it transitions into a replica symmetry breaking phase characterized by extinctions.

V. ZERO-TEMPERATURE θ-GLV MODEL

We now generalize our analysis to the θ-GLV model. At zero temperature, the corresponding replica symmetric Hamiltonian takes the form:

$$H_{\rm RS}(N,z) = \rho\left[-\alpha_2\frac{N^2}{2} + \alpha_1(z)N + \frac{N^{1+\theta}}{1+\theta}\right], \tag{39}$$

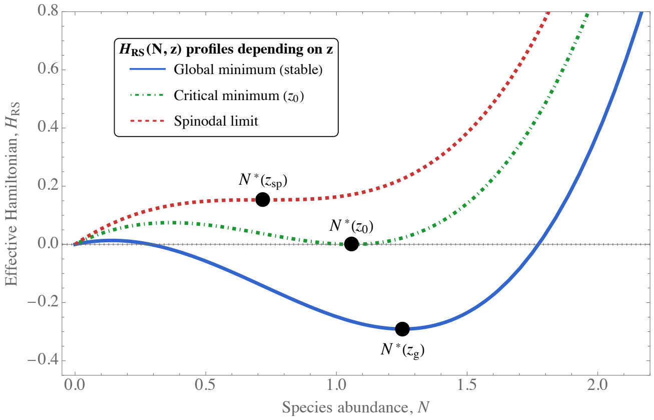

FIG. 2. Zero-temperature replica-symmetric Hamiltonian $H_{\rm RS}(N,z)$ profiles for the θ-GLV model (here, $\theta = 2.3$) at different values of z. The solid blue line illustrates a stable coexistence phase, where the system relaxes to a global minimum ($H_{\rm RS} < 0$). The dot-dashed green line represents the critical threshold z_0, at which the non-trivial minimum becomes strictly degenerate with the extinction state ($H_{\rm RS} = 0$). The dashed red line shows the spinodal limit, beyond which the metastable well is completely annihilated. The black dots indicate the corresponding saddle-point abundance N^* for each value of z.

where we have defined the effective field and interaction parameters:

$$\alpha_1(z) = \mu h - K^\theta - \sigma\sqrt{q}\, z, \tag{40}$$

$$\alpha_2 = \rho\sigma^2\chi. \tag{41}$$

A. Unbounded Growth for $\theta < 1$

For $\theta < 1$, the quadratic term dominates the Hamiltonian at large N. Since the susceptibility $\chi > 0$ by definition (20), it follows that $\alpha_2 > 0$. Consequently, the overall sign of the dominant quadratic term is negative, causing the Hamiltonian to be unbounded from below as $N \to \infty$. In this regime, no finite saddle-point solution exists.

Physically, this corresponds to unbounded population growth. Intuitively, when $\theta < 1$, the saturation term in the self-potential $V_i^\theta(N_i)$ lacks the sufficient stiffness required to suppress the effective variance-driven growth, leading to an explosion.

B. Saddle-Point Solution for $\theta > 1$

For $\theta > 1$, the system remains bounded. The condition for the minimum of the Hamiltonian, $\frac{\partial H_{\rm RS}}{\partial N} = 0$, yields the saddle-point equation:

$$-\alpha_2 N^* + (N^*)^\theta + \alpha_1(z) = 0. \tag{42}$$

Because $\theta \neq 1$, the profile of $H_{\rm RS}$ is no longer a simple parabola. Consequently, the Hamiltonian may possess a local non-zero minimum with $H_{\mathrm{RS}}(N^*) > 0$. When this occurs, this non-zero state has a higher free-energy cost than the extinction state, $H_{\mathrm{RS}}(0) = 0$, making it metastable (see Fig. 2). In such cases, the global minimum is exactly at $N = 0$, and the non-trivial saddle-point solution N^* becomes physically irrelevant.

To find the threshold for extinction, we must evaluate the critical point z_0 where the non-zero minimum becomes exactly degenerate with the extinction state, i.e., $H_{\mathrm{RS}}(N^*) = 0$:

$$-\alpha_2 \frac{(N^*)^2}{2} + \frac{(N^*)^{1+\theta}}{1+\theta} + \alpha_1(z) N^* = 0. \tag{43}$$

Substituting the relation $(N^*)^\theta = \alpha_2 N^* - \alpha_1(z)$ from Eq. (42) into this condition, we obtain:

$$\alpha_1(z_0) = \alpha_2 N^*(z_0) \frac{\theta - 1}{2\theta}. \tag{44}$$

Substituting this result back into Eq. (42) allows us to solve explicitly for the critical abundance $N^*(z_0)$ and the critical effective field $\alpha_1(z_0)$ at the transition:

$$N^*(z_0) = \alpha_2^{\frac{1}{\theta-1}} \left(\frac{\theta+1}{2\theta} \right)^{\frac{1}{\theta-1}}, \tag{45}$$

$$\alpha_1(z_0) = \alpha_2^{\frac{\theta}{\theta-1}} \frac{(\theta-1)(\theta+1)^{\frac{1}{\theta-1}}}{(2\theta)^{\frac{\theta}{\theta-1}}}. \tag{46}$$

The asymptotic behavior of these critical parameters in the limits $\theta \to 1^+$ (near standard GLV) and $\theta \to \infty$ (hard-wall limit) are detailed in the Appendix.

This result reveals a qualitative difference from the standard GLV model. In the standard GLV framework, as z decreases toward the critical threshold z_0, the species abundance vanishes continuously. In the θ-GLV model, however, the abundance remains finite as it approaches z_0. For $z < z_0$, saddle point solution become metastable and the population drops discontinuously to the extinction state, $N(z < z_0) = 0$.

C. Replicon Eigenvalue for $\theta > 1$

Using the saddle-point approximation from Eq. (30) and evaluating the second derivative $H''_{\mathrm{RS}}(N^*) = \rho\left(-\alpha_2 + \theta (N^*)^{\theta-1}\right)$, we can express the replicon eigenvalue in integral form:

$$\lambda_{\mathrm{R}} = (\beta\rho\sigma)^2 \left[1 - \sigma^2 \int_{z_0}^{\infty} \mathcal{D}z \, \frac{1}{\left[-\alpha_2 + \theta \left(N^*(z)\right)^{\theta-1}\right]^2} \right]. \tag{47}$$

Because we do not have an exact analytical expression for $N^*(z)$, this integral cannot be evaluated directly. However, we can rewrite it in a bit nicer form. To do so, we differentiate the saddle-point equation (42) with respect to z, which yields:

$$\left[-\alpha_2 + \theta \left(N^*(z)\right)^{\theta-1}\right]^{-1} = \frac{1}{\sigma\sqrt{q}} \frac{\partial N^*}{\partial z}. \tag{48}$$

Finally, the replicon eigenvalue becomes

$$\lambda_{\mathrm{R}} = (\beta\rho\sigma)^2 \left[1 - \frac{1}{q} \int_{z_0}^{\infty} \mathcal{D}z \left(\frac{\partial N^*}{\partial z} \right)^2 \right]. \tag{49}$$

The replica-symmetry-breaking transition occurs when the replicon eigenvalue vanishes, yielding:

$$q = \int_{z_0}^{\infty} \mathcal{D}z \left(\frac{\partial N^*}{\partial z} \right)^2. \tag{50}$$

We can cast this condition into a more intuitive form. Recalling the zero-temperature self-consistent equation for the overlap order parameter, $q = \int_{z_0}^{\infty} \mathcal{D}z \left(N^*(z)\right)^2$, the stability boundary can be rewritten as:

$$\int_{z_0}^{\infty} \mathcal{D}z \left(N^*(z)\right)^2 = \int_{z_0}^{\infty} \mathcal{D}z \left(\frac{\partial N^*}{\partial z} \right)^2. \tag{51}$$

This equality is a generalization of the condition (37) found in the standard GLV model.

VI. DISCUSSION

In this work, we explored the thermodynamic properties of the θ-GLV model subjected to demographic noise. We mapped stochastic population dynamics into an equilibrium statistical-mechanics problem. Following [5], we apply the replica method to evaluate the zero-temperature phase diagram, focusing specifically on the de Almeida-Thouless transition from a stable, replica-symmetric coexistence phase to a replica symmetry-breaking phase characterized by extinctions.

Our findings show how the stiffness of the saturation parameter, θ, affects the stability of the system. For the standard GLV model ($\theta = 1$), our replica method recovers the critical interaction variance $\sigma_c = 1/\sqrt{2}$ obtained previously via the cavity method [6]. In the regime $\theta < 1$, we find that the saturation term is insufficiently stiff to counteract the variance-driven fluctuations, leading to a loss of stability and unbounded population growth. For the stiffer regime, $\theta > 1$, we formulated the integral equation that determines the phase boundary of the replica symmetry-breaking transition.

A notable feature of the $\theta > 1$ regime is the qualitative difference of the extinction transition for different values of z. In the standard GLV framework, the species abundance vanishes continuously as z approaches the critical threshold z_0. In contrast, our analysis indicates that for $\theta > 1$, the abundance remains strictly finite right up to the critical threshold, eventually undergoing a discontinuous jump to the extinction state.

While these results provide theoretical insight into how saturation stiffness, θ, affects ecosystem stability, our approach relies on specific simplifying assumptions. The mapping to an equilibrium Hamiltonian strictly requires symmetric interspecies interactions ($\alpha_{ij} = \alpha_{ji}$). Natural ecosystems, however, often feature asymmetric interactions (such as predator-prey dynamics) that generate complex, non-equilibrium behaviors. Additionally, our analytical results are confined to the zero-temperature limit.

ACKNOWLEDGMENTS

I would like to thank Amer Al-Hiyasat and Mehran Kardar for suggesting this problem and for the useful discussions. I am also grateful to Mehran Kardar for introducing me to spin-glass physics.

APPENDIX

1. Critical σ at Zero Temperature

Here we solve the stability condition $\phi_0(z_0) = \phi_2(z_0)$ to determine the critical variance σ. Recall the definition of the integrals over the standard Gaussian measure $\mathcal{D}z \equiv \frac{dz}{\sqrt{2\pi}} e^{-z^2/2}$:

$$\phi_k(z_0) = \int_{z_0}^{\infty} \mathcal{D}z \, (z - z_0)^k. \tag{52}$$

Expanding the integrand for $k = 2$ and applying integration by parts to the z^2 term yields:

$$\begin{aligned} \phi_2(z_0) &= \int_{z_0}^{\infty} \mathcal{D}z \, z^2 - 2z_0 \int_{z_0}^{\infty} \mathcal{D}z \, z + z_0^2 \phi_0(z_0) \\ &= \big[z_0 \mathcal{N}(z_0) + \phi_0(z_0)\big] - 2z_0 \mathcal{N}(z_0) + z_0^2 \phi_0(z_0) \\ &= (1 + z_0^2)\phi_0(z_0) - z_0 \mathcal{N}(z_0), \end{aligned} \tag{53}$$

where $\mathcal{N}(z_0) = \frac{1}{\sqrt{2\pi}} e^{-z_0^2/2}$. Equating this to $\phi_0(z_0)$ and simplifying gives:

$$z_0 \big[z_0 \phi_0(z_0) - \mathcal{N}(z_0)\big] = 0. \tag{54}$$

Standard bounds on the Gaussian tail (Mills ratio) give that $z_0 \phi_0(z_0) < \mathcal{N}(z_0)$ for any finite z_0. Therefore, the bracketed term is never zero, leaving the trivial root as the unique solution:

$$z_0 = 0. \tag{55}$$

Evaluating the tail integral at this threshold gives

$$\phi_0(0) = \int_0^{\infty} \mathcal{D}z = \frac{1}{2}. \tag{56}$$

Substituting this back into the replicon stability condition

$$\phi_0(z_0) = \frac{1}{4\sigma^2} \tag{57}$$

derived in the main text, we find the critical interaction variance where the replica symmetric solution breaks down:

$$\sigma_c = \frac{1}{\sqrt{2}}. \tag{58}$$

2. Asymptotic Behavior of the Critical Parameters in θ-GLV

Here, we analyze the asymptotic behavior of the critical species abundance $N^*(z_0)$ and parameter $\alpha_1(z_0)$ in two extreme limits of the saturation stiffness: the near standard GLV limit ($\theta \to 1^+$) and the hard-wall limit ($\theta \to \infty$).

Recall the exact analytical expressions evaluated at the critical threshold z_0:

$$N^*(z_0) = \alpha_2^{\frac{1}{\theta-1}} \left(\frac{\theta+1}{2\theta} \right)^{\frac{1}{\theta-1}}, \tag{59}$$

$$\alpha_1(z_0) = \alpha_2 \left(\frac{\theta-1}{2\theta} \right) N^*(z_0). \tag{60}$$

a. *Limit $\theta \to 1^+$: The Near Standard GLV Limit*

To evaluate the limit as the system approaches the standard GLV model from above, we introduce a small positive parameter $\epsilon \ll 1$ such that $\theta = 1 + \epsilon$.

Expanding the following term to first order in ϵ, we get:

$$\frac{\theta+1}{2\theta} = \frac{2+\epsilon}{2(1+\epsilon)} = 1 - \frac{\epsilon}{2(1+\epsilon)} = 1 - \frac{\epsilon}{2} + \mathcal{O}(\epsilon^2). \tag{61}$$

Raising it to the power of $\frac{1}{\theta-1} = \frac{1}{\epsilon}$, we apply the standard limit definition of the exponential function:

$$\lim_{\epsilon \to 0^+} \left(1 - \frac{\epsilon}{2}\right)^{\frac{1}{\epsilon}} = e^{-1/2} = \frac{1}{\sqrt{e}}. \tag{62}$$

Substituting this back into Eq. (59), the critical abundance behaves asymptotically as:

$$N^*(z_0) \simeq \frac{1}{\sqrt{e}} \alpha_2^{\frac{1}{\theta-1}}. \tag{63}$$

This shows a singular dependence on the parameter α_2. As $\theta \to 1^+$, the critical abundance exponentially vanishes if $\alpha_2 < 1$, and exponentially diverges if $\alpha_2 > 1$.

Using Eq. (60), we substitute $\theta = 1 + \epsilon$ to find the prefactor:

$$\frac{\theta - 1}{2\theta} = \frac{\epsilon}{2(1+\epsilon)} \simeq \frac{\epsilon}{2} = \frac{\theta - 1}{2}. \tag{64}$$

Thus, the asymptotic expansion for the parameter is

$$\alpha_1(z_0) \simeq \frac{\alpha_2}{2\sqrt{e}}(\theta - 1)\alpha_2^{\frac{1}{\theta-1}}. \tag{65}$$

b. Limit $\theta \to \infty$: The Hard-Wall Limit

We now consider the opposite extreme, where the stiffness of the saturation term diverges ($\theta \to \infty$). In this limit, the parameter in the exponent approaches zero: $\frac{1}{\theta-1} \to 0$.

Taking the limit of Eq. (59):

$$\lim_{\theta\to\infty} N^*(z_0) = \lim_{\theta\to\infty} \alpha_2^{\frac{1}{\theta-1}} \left(\frac{1}{2} + \frac{1}{2\theta}\right)^{\frac{1}{\theta-1}} = 1. \tag{66}$$

Physically, the saturation term $\frac{N^{1+\theta}}{1+\theta}$ becomes an impenetrable energy barrier at $N = 1$. The critical metastable state is pushed exactly to this boundary.

To find the critical parameter α_1, we evaluate Eq. (60) using $N^*(z_0) \to 1$:

$$\lim_{\theta\to\infty} \alpha_1(z_0) = \lim_{\theta\to\infty} \alpha_2 \left(\frac{1}{2} - \frac{1}{2\theta}\right) N^*(z_0) = \frac{\alpha_2}{2}. \tag{67}$$

This result aligns with a simple argument: for a hard-wall potential at $N = 1$, the non-trivial free-energy minimum evaluates to $H_{\mathrm{RS}}(1) = -\frac{\alpha_2}{2} + \alpha_1$. Setting this equal to the extinction state energy $H_{\mathrm{RS}}(0) = 0$ yields $\alpha_1 = \alpha_2/2$.

[1] G. Hardin, The competitive exclusion principle: an idea that took a century to be born has implications in ecology, economics, and genetics., science **131**, 1292 (1960).

[2] E. O. Wilson, *The diversity of life* (WW Norton & Company, 1999).

[3] R. May and A. R. McLean, *Theoretical ecology: principles and applications* (OUP Oxford, 2007).

[4] A. Al-Hiyasat, D. W. Swartz, J. Gore, and M. Kardar, Spatiotemporal noise stabilizes unbounded diversity in strongly-competitive communities, arXiv preprint arXiv:2602.13423 (2026).

[5] A. Altieri, F. Roy, C. Cammarota, and G. Biroli, Properties of equilibria and glassy phases of the random lotka-volterra model with demographic noise, Phys. Rev. Lett. **126**, 258301 (2021).

[6] G. Bunin, Ecological communities with lotka-volterra dynamics, Phys. Rev. E **95**, 042414 (2017).

[7] J. R. de Almeida and D. J. Thouless, Stability of the sherrington-kirkpatrick solution of a spin glass model, Journal of Physics A: Mathematical and General **11**, 983 (1978).