The Potts Model on Non-Integer-Dimensional Network

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The critical behavior of phase transitions in non-integer spatial dimensions provides profound insights into the limits of universality classes. In this project, we investigate the phase transition of the 3-state Potts model on a 1-dimensional long-range random ring (1DLR3) network. The spectral dimension d_s of this network can be continuously tuned via the long-range connection probability parameter σ. Employing Monte Carlo simulations, we analyze the Binder cumulant and the equilibrium energy distribution. We numerically confirm that as the spectral dimension increases, the nature of the phase transition shifts from continuous to first-order. Based on our data, we estimate that the threshold spectral dimension $d_s^c(q=3)$ lies within the range of $(2.852, 3.396)$. Finally, we compare this threshold with the existing conformal bootstrap prediction.

I. INTRODUCTION

Universality is a cornerstone of statistical physics, dictating that the critical behavior of phase transitions depends only on a few fundamental properties, such as internal symmetry and spatial dimensionality, rather than microscopic details. [1] While traditional studies predominantly focus on regular Euclidean lattices with integer dimensions, extending these concepts to non-integer-dimensional spaces provides a powerful theoretical playground to explore the limits of universality classes. Complex networks, characterized by continuously tunable topological parameters such as the spectral dimension, serve as ideal platforms for realizing such non-integer geometries.

In this project, we investigate how the nature of phase transitions evolves with continuous dimensionality by simulating the 3-state Potts model on a 1-dimensional long-range random ring (1DLR3) network. Specifically, we utilize Monte Carlo simulations with the Wolff algorithm to trace the evolution of the system's phase transition. By analyzing the Binder cumulant and the equilibrium energy distribution, we confirm the change in the nature of phase transition, and provide a range of the threshold dimension.

A. The Potts model

The Potts model is a natural generalization of the Ising model. [2] Instead of two states (spin-up and spin-down), on each site there are q discrete possible states. The Hamiltonian of the system reads

$$\mathcal{H} = -\sum_{\langle i,j \rangle} J\delta_{\sigma_i,\sigma_j}, \tag{1}$$

where $\sigma_i = 1, 2, ..., q$ denotes the present state on site i, and $\langle i,j \rangle$ denotes pairs of nearest neighboring sites. When $q = 2$, the Potts model becomes the Ising model.

Like the Ising model, the Potts model undergoes an order-disorder phase transition as temperature changes. Given the total number of sites N and the number of sites in each state n_s, the order parameter is defined as

$$m^2 = \frac{1}{q-1}\left(q\sum_{s=1}^{q}\frac{n_s^2}{N^2} - 1\right). \tag{2}$$

As the system goes from fully disordered (n_s are all equal) to fully ordered (all sites are on the same state), the order parameter m^2 goes from 0 to 1.

An important feature of the Potts model is that, for different q and different spatial dimension d, the nature of phase transition is different. (See Fig. 1.) For a fixed d, when q exceeds a threshold value $q_c(d)$, the phase transition changes from continuous to first-order. [3] In other words, for a fixed q, when d exceeds $d_c(q)$, the phase transition changes from continuous to first-order.

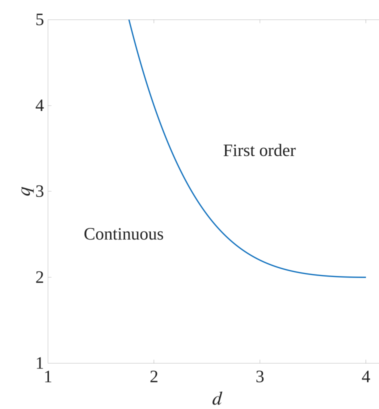

Figure 1: Schematic plot of $q_c(d)$, based on [3, 4].

The first order nature of the transition for $q > q_c(d)$ stems from the breaking of symmetry, which can be understood with simple mean field calculation.[3] Assume that in the ordered phase,

$$n_s = \begin{cases} Nx, & s = 1, \\ N\frac{1-x}{q-1}, & \text{otherwise}. \end{cases} \tag{3}$$

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The order parameter is $m = \frac{qx-1}{q-1}$ according to Eq. (2). By mean field approximation, the energy and entropy are

$$\begin{aligned}\mathcal{H}(m) =& -\frac{1}{2}Jz\sum_{s=1}^{q}\frac{n_s^2}{N^2}\\ =& -\frac{Jz}{2q}-\frac{Jz(q-1)}{2q}m^2, \end{aligned}\tag{4}$$

$$\begin{aligned}\mathcal{S}(m) =& -k_B\sum_{s=1}^{q}\frac{n_s}{N}\log\frac{n_s}{N}\\ =& k_B\frac{(q-1)(m-1)}{q}\log\frac{m-1}{q}\\ & -k_B\frac{qm-m+1}{q}\log\frac{qm-m+1}{q}. \end{aligned}\tag{5}$$

In the equation, z is the coordination number. Therefore, the free energy can be expanded into a series of m.

$$\begin{aligned}\mathcal{F}(m) =& \mathcal{H}(m) - T\mathcal{S}(m)\\ =& \mathcal{F}(0) + \frac{q-1}{2q}(k_BTq - Jz)m^2\\ & -\frac{k_BT}{6}(q-1)(q-2)m^3 + O(m^4). \end{aligned}\tag{6}$$

For $q > 2$, the existence of m^3 term in Eq. (6) indicates a first order transition. [1] Therefore, mean field approximation predicts a first order transition for $q > 2$.

However, the real system is more complicated than mean field approximation, as shown in Fig. 1. It is shown that $q_c(d = 2) = 4$ [5], $q_c(d \geq 4) = 2$ [6], and $d_c(q = 4) = 2$ [7, 8]. Through Fortuin-Kasteleyn cluster expansion, q can be extended to real values [9], and it is shown that $d_c(q = 3) = 2.45$ [10]. And numerical conformal booststrap shows $d_c(q = 3) \approx 2.5$.[11]

In this project, we focus on $q = 3$ case.

B. The complex network and spectral dimension

In this project, the spatial organization of sites in the Potts model is represented by complex networks. Here the complex network strictly means the simple graph, which can be described by a pair $G \equiv (V, E)$, where V is the set of all vertices, and E is the set of all neighboring vertex pairs (edges) in the network. (See Fig. 2) The structure of a network can also be described by its adjacency matrix $\mathbf{A}$, which is an $N \times N$ symmetric matrix. N is the number of vertices. $\mathbf{A}_{ij}$ is 1 if vertex i and j are connected by an edge, and is 0 if not. The Laplacian matrix $\mathbf{L}$ of a network is defined as [12]

$$\mathbf{L}_{ij} = \delta_{ij}\sum_k \mathbf{A}_{ik} - \mathbf{A}_{ij}. \tag{7}$$

It is an analog of the Laplacian operator ∇^2 on complex networks.

There are multiple ways to generalize dimensionality from Euclidean lattices to complex networks, each of which captures a specific feature of the network and gives a different value.[12, 13] One of them which is relevant to the phase transition and universality is the (average) spectral dimension. The spectral dimension is defined through the asymptotic behavior of a random walk on the network.[12] Given the adjacency matrix $\mathbf{A}$, a possible setup of random walk on a network can be described by the master equation

$$\partial_t p(i,t|i_0,t_0) = -\sum_j \mathbf{L}_{ij}p(j,t|i_0,t_0), \tag{8}$$

where $p(i,t|i_0,t_0)$ is the probability of a random walker being on vertex i at time t given it is on i_0 at t_0. In the limit $N \to \infty$, the spectral dimension d_s of a network is given by

$$d_s = -2\lim_{t\to\infty}\frac{\log\overline{p(i,t|i,0)}}{\log t}. \tag{9}$$

In Euclidean lattices, $p(i,t|i,0) \propto \frac{1}{t^{d/2}}$, hence spectral dimensions of Euclidean lattices coincide with spatial dimensions.

The spectral dimension also characterizes the distribution of eigenvalues of $\mathbf{L}$. The general solution of Eq. (8) is

$$\boldsymbol{p}(t) = e^{-\mathbf{L}t}\boldsymbol{p}(0) = \mathbf{U}e^{-\boldsymbol{\Lambda}t}\mathbf{U}^\dagger\boldsymbol{p}(0), \tag{10}$$

where $\mathbf{L} = \mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^\dagger$ is the eigenvalue decomposition of $\mathbf{L}$. Therefore, $\overline{p(i,t|i,0)}$ can always be written as a linear combination of $e^{-\lambda t}$:

$$\overline{p(i,t|i,0)} = \sum_\lambda C_\lambda e^{-\lambda t}. \tag{11}$$

In the thermodynamic limit $N \to \infty$, eigenvalues become dense and can be described by a density distribution $\rho(\lambda)$. In the long time limit $t \to \infty$, only small eigenvalues near $\lambda_{\min} = 0$ contribute significantly to $\overline{p(i,t|i,0)}$. When the distribution of small eigenvalues satisfies the power law

$$\rho(\lambda) \propto \lambda^\epsilon, \tag{12}$$

from Eq. (11) one obtains

$$\overline{p(i,t|i,0)} \propto \int_0 \mathrm{d}\lambda\ \lambda^\epsilon C(\lambda)e^{-\lambda t}. \tag{13}$$

In general, $C(\lambda) \approx C(0)$ near $\lambda = 0$, and is not a power of λ, hence $\overline{p(i,t|i,0)} \propto t^{-\epsilon-1} \propto t^{-d_s/2}$.

In statistical systems, the eigenstates of $\mathbf{L}$ can be mapped to eigenmodes of Gaussian models, and d_s characterizes the distribution of slow modes. Intuitively, eigenmodes of Gaussian models in Euclidean lattices are nothing but eigenvectors of ∇^2. In complex networks, ∇^2 is replaced by $\mathbf{L}$, hence eigenmodes become eigenvectors of $\mathbf{L}$.[14, 15]

The spectral dimension is proved to be invariant under a large class of transformations.[12, 14, 16] Especially, different details of random walk and Gaussian model setup give the same d_s in many cases. Therefore, d_s is an important control parameter of universality class.

C. 1-dimensional long range random ring

In this project, we restrict our work to a specific class of random networks called 1-dimensional long range random ring (1DLR3).[17] 1DLR3 is generated through the following process. (See Fig. 2) N indexed vertices are placed on a ring, and nearest neighbors are connected by edges. The distance of two vertices i and j is defined as

$$r_{ij} = \min(|i-j|, N - |i-j|). \tag{14}$$

Then edges between non-neighboring vertices are added independently randomly. For vertex i and j ($r_{ij} > 1$), the probability of $(i,j) \in E$ is

$$p_{ij} = \frac{1}{r_{ij}^{\sigma+1}}, \tag{15}$$

where σ is a control parameter. It is easy to show that, in thermodynamic limit $N \to \infty$, the average coordination number $\bar{z} = 2\zeta(\sigma+1)$.

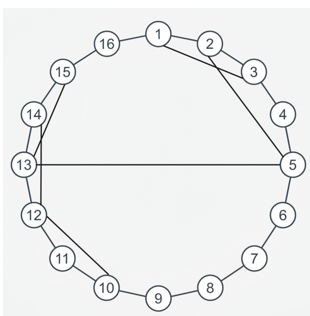

Figure 2: A schematic network generated with $N = 16$ and $\sigma = 1$.

By varying σ, the spectral dimension of 1DLR3 can be tuned within $[1, +\infty)$. In the limit $\sigma \to \infty$, 1DLR3 reduces to a 1D ring and has $d_s = 1$; In the limit $\sigma \to 0$, the average coordination number diverges and 1DLR3 has $d_s = +\infty$. We can use annealed approximation to estimate d_s.[16, 17] The quenched average of Laplacian matrix of 1DLR3, $\bar{\mathbf{L}}$, is an infinite Toeplitz matrix, with elements

$$\bar{\mathbf{L}}_{ij} = \begin{cases} 2\zeta(\sigma+1), & i = j, \\ -\frac{1}{|i-j|^{\sigma+1}}, & i \neq j. \end{cases} \tag{16}$$

The eigenvalue distribution of $\bar{\mathbf{L}}$ is determined by the first Szegö limit theorem.[18] Define function $f(\theta)$, $\theta \in [-\pi, \pi)$:

$$\begin{aligned} f(\theta) &= \sum_{j=-\infty}^{+\infty} \bar{\mathbf{L}}_{jk} e^{\mathrm{i}(j-k)\theta} \\ &= 2\zeta(\sigma+1) - 2\sum_{j=1}^{+\infty} \frac{1}{j^{\sigma+1}} \cos j\theta \\ &= 2\zeta(\sigma+1) - \left(\mathrm{Li}_{\sigma+1}\left(e^{-i\theta}\right) + \mathrm{Li}_{\sigma+1}\left(e^{i\theta}\right)\right) \end{aligned} \tag{17}$$

The first Szegö limit theorem states that, in the thermodynamic limit, the distribution of eigenvalues of $\bar{\mathbf{L}}$ converges to the distribution of $f(\theta)$ where θ is uniformly distributed on $[-\pi, \pi)$. Around the minimal point $\theta = 0$, the asymptotic form of $f(\theta)$ is

$$f(\theta) \propto \theta^{\sigma}. \tag{18}$$

Therefore, for small eigenvalues, $\rho(\lambda) \propto \frac{1}{\mathrm{d}\theta^{\sigma}/\mathrm{d}\theta} \propto \lambda^{-(\sigma-1)/\sigma}$, and $d_s = \frac{2}{\sigma}$. If $\sigma > 2$, Eq. (18) no longer holds true, giving unreasonable $d_s < 1$. After correction, the spectral dimension of $\bar{\mathbf{L}}$ is

$$d_s^{\mathrm{ann}} = \begin{cases} \frac{2}{\sigma}, & 0 < \sigma \leq 2, \\ 1, & \sigma > 2. \end{cases} \tag{19}$$

Theoretical analysis suggests that at $\sigma > 2$ and $\sigma < \frac{1}{3}$, 1DLR3's spectral dimension $d_s = d_s^{\mathrm{ann}}$.[17] In the range $\sigma \in (\frac{1}{3}, 2)$, d_s deviates from d_s^{ann}, as shown in Sec. II A.

1DLR3 has been used in the XY model and the Kuramoto model to investigate the influence of spectral dimension.[19]

II. RESULTS

A. Measurement of spectral dimension

We use the finite size scaling method to measure the spectral dimension d_s.[17] It is proved that small eigenvalue distribution of the symmetric normalized Laplacian matrix

$$\tilde{\mathbf{L}}_{ij} = \delta_{ij} - \frac{\mathbf{A}_{ij}}{\sqrt{\sum_k \mathbf{A}_{ik} \sum_l \mathbf{A}_{jl}}} \tag{20}$$

also satisfies Eq. (12). Eigenvalues of $\tilde{\mathbf{L}}$ are strictly confined in $[0, 2]$, and $\frac{1}{N}\rho(\lambda)$ converges to one curve in the limit $N \to \infty$. Therefore, when the system size changes, the ith eigenvalue scales as

$$\lambda_i(N) \propto N^{-2/d_s}. \tag{21}$$

We vary N from 2^{10} to 2^{16} and fit the value of 10th and 11th eigenvalue to measure d_s. Results are averaged over 20 replicates.

According to the measurement, d_s deviates from the annealed approximation d_s^{ann} in $\sigma \in (\frac{1}{3}, 2)$. At small enough σ, d_s also deviates from d_s^{ann}. This is a systematic error caused by limited size of the network. We are interested in the range $d_s \in [2, 4]$, which corresponds approximately to $\sigma \in [0.5, 0.9]$.

B. Phase transition of the Potts model

For simplicity, in this section we let

$$J = \frac{1}{\bar{z}} = \frac{1}{2\zeta(\sigma+1)}. \tag{22}$$

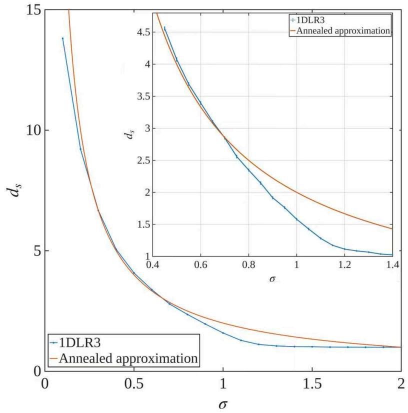

Figure 3: Measured d_s of 1DLR3 and d_s^{ann} in Eq. (19).

We use the Wolff algorithm to sample equilibrium configurations of the Potts model.[20] Our system size N ranges from 2^{13} to 2^{17}. For each pair (N, σ), at least 200 replicates of networks are created. On each replicate, at least 5000 samples are generated.

As discussed in Sec. I A, when spatial dimension d grows from 2 to 3, the phase transition of 3-states Potts model changes from continuous to first-order. We find a similar change on 1DLR3 when d_s grows from 2 to 4. This can be shown by Binder cumulant U_4 and the equilibrium distribution of energy.

Binder cumulant of the Potts model is defined as

$$U_4 = 1 - \frac{q-1}{q+1} \frac{\langle m^4 \rangle}{\langle m^2 \rangle^2}. \tag{23}$$

In ordered phase, U_4 converges to $1 - \frac{q-1}{q+1}$ in thermodynamic limit; in disordered phase, U_4 converges to 0. For different types of phase transitions, U_4 exhibits distinct behaviors around the critical point. When σ is small, i.e., d_s is big, we find a sharp dip near the critical point, see Fig. 4 far left. The sharp dip on U_4 curve indicates first-order transition. When σ is big, i.e., d_s is small, there is no obvious dip on U_4 curve, but curves of different N intersect at critical point (see Fig. 4 middle right), which suggests continuous transition.[22] A more direct evidence of first-order transition is the equilibrium energy distribution around the critical point. For cases with small σ, we find obvious bimodal structure in the probability density of energy E, which proves the first-order nature of the transition. (See Fig. 4 middle left.)

According to our data, at $\sigma < 0.8$ ($d_s > 3.396$), we can find obvious bimodal structure in $p(E)$. At $\sigma > 0.7$ ($d_s < 2.852$, we can find intersection of U_4 curves, see Fig. 5. Our data suggest a threshold dimension $d_s^c(q = 3)$ in the range $(2.852, 3.396)$.

III. DISCUSSION

In this project, we investigate the phase transition of 3-state Potts model on 1DLR3. We show that, as in the Euclidean lattices, when the spectral dimension d_s of 1DLR3 increases, the phase transition changes from continuous to first-order. We also provide a range of the threshold dimension, based on our data.

It is still unclear why the range we obtain cannot cover the result in [11], which is about 2.5. One possibility is other topological features of 1DLR3 influence the phase transition, causing 1DLR3 and Euclidean lattices being in different universal classes. Since the Potts model is not a Gaussian model, and the threshold range of d_s is much lower than the upper critical dimension, we do not expect d_s can capture all topological features of the network. Another possibility is that the quenched disorder of random network influences the phase transition. Also, due to the limit of N in our simulations, we may fail in distinguish weak first-order transition from continuous transition.

There are still many questions related to the Potts model and non-integer dimensionality. For example, what is the nature of the transition from continuous phase transition to first-order transition, are critical exponents obtained from non-integer-dimensional systems consistent with other theoretical results like ϵ-expansion, etc. Further simulations with higher precision and better numerical method can be done in the future. Moreover, the 1DLR3 network, serving as a tool for non-integer dimensions and universality, can be applied to study other statistical systems.

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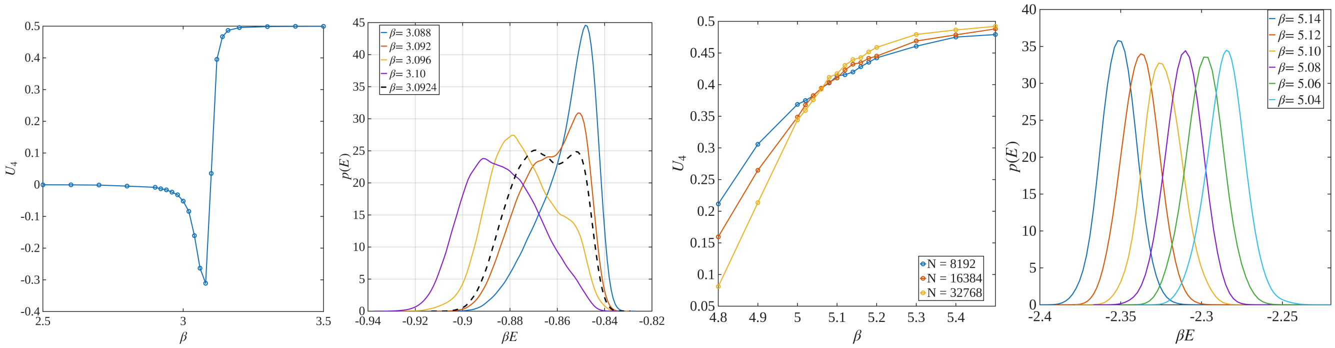

Figure 4: Far left: U_4 of different β. $N = 32768$. Middle left: probability density of energy in different β. $N = 131072$. Solid lines are results of simulations, and the dash line is generated by Ferrenberg-Swendsen method based on the data of $\beta = 3.092$.[21] Left two panels: $\sigma = 0.5, d_s = 4.075 \pm 0.026$. Middle right: U_4 of different β and N. Far right: probability density of energy in different β. $N = 32768$. Right two panels: $\sigma = 0.9, d_s = 1.916 \pm 0.015$.

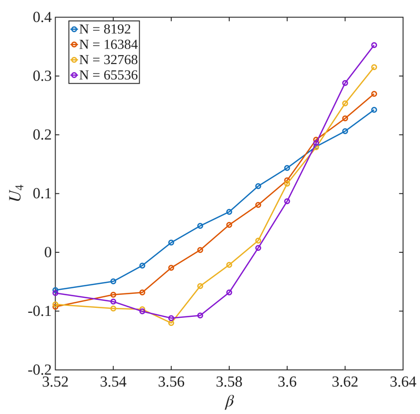

Figure 5: U_4 of different β and N. $\sigma = 0.7, d_s = 2.852 \pm 0.017$. U_4 curves intersect at one point, while a dip also appears.

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