# Conformal Field Theory characterization on the Fuzzy Sphere

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Conformal field theories (CFTs) describe various second-order transitions, where emergent scale and conformal invariance enable various theoretical calculations. It is known that the State-Operator correspondence maps the scaling dimensions of 3D CFT operators to energy eigenvalues on the 2D sphere. Recently, numerics using the Fuzzy sphere regularization produced remarkably accurate operator spectra within a small number of particles. In this paper, we review recent developments on fuzzy sphere regularization and its application to 3D Ising CFT.

## I. 3D CFT BASICS

We introduce the minimum foundation of (3D) CFT before getting into the Fuzzy Sphere (FS) regularization. CFT assumes that the critical point exhibits symmetries of the conformal group enhanced from isometry. The conformal group in $d$ dimensions $SO(d+1,1)$ is generated by four sets of generators: translations $P_\mu = i\partial_\mu$, rotations $M_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu)$, dilatation $D = ix^\mu\partial_\mu$, and special conformal transformation $K_\mu = i[2x_\mu(x^\nu\partial_\nu) - x^2\partial_\mu]$. These operators have commutation relations

$$[D, P_\mu] = -iP_\mu,\ [D, K_\mu] = iK_\mu, \tag{1}$$
$$[P_\mu, K_\nu] = 2i(\delta_{\mu\nu}D - M_{\mu\nu}). \tag{2}$$

Commutation relations of isometries or those of commuting operators are not written above. These operators act on the field operator $\mathcal{O}(x)$

$$[P_\mu, \mathcal{O}(x)] = -i\partial_\mu\mathcal{O}(x), \tag{3}$$
$$[D, \mathcal{O}(x)] = -i(\Delta + x^\mu\partial_\mu)\mathcal{O}(x), \tag{4}$$
$$[M_{\mu\nu}, \mathcal{O}(x)] = -i(\Sigma_{\mu\nu} + x_\mu\partial_\nu - x_\nu\partial_\mu)\mathcal{O}(x), \tag{5}$$
$$[K_\mu, \mathcal{O}(x)] = \tag{6}$$
$$-i(2x_\mu\Delta + 2x^\lambda\Sigma_{\lambda\mu} + 2x_\mu(x^\rho\partial_\rho) - x^2\partial_\mu)\mathcal{O}(x). \tag{7}$$

Relations (1) to (7) can be obtained from calculus and infinitesimal transformation. The scaling dimension $\Delta$ and the finite dimensional spin matrix $\Sigma_{\mu\nu}$ characterize the operator $\mathcal{O}(x)$. Especially, $\Delta$ governs the correlation function. For example, operator $\phi(x)$ with the scaling dimension $\Delta$ has correlation function

$$\langle\phi(x)\phi(y)\rangle \propto \frac{1}{|x-y|^{2\Delta}}. \tag{8}$$

In the 3D Ising critical point (or any others), we can relate the critical exponent $\eta$ with the scaling dimension of the spin field $\Delta_\sigma$:

$$\eta = 2\Delta_\sigma - d + 2 = 2\Delta_\sigma - 1. \tag{9}$$

Thus, studying $\Delta_\sigma$, or generally $\Delta$ and $\Sigma_{\mu\nu}$, helps understanding critical exponents beyond perturbative expansion.

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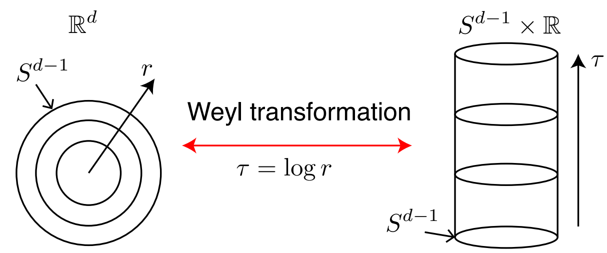


FIG. 1. Illustration of radial quantization via the Weyl transformation. Adapted from [1].

Now we use Wick rotation to treat 3D CFT in (2+1)D. Conventionally, we choose one of the spacetime axes as $t$, and the translation generator along that axis becomes a Hamiltonian operator. However, since we have enriched symmetries and we don't have an "arrow of time," we can choose an arbitrary direction to define the Hamiltonian. Moreover, we can exploit some coordinates to choose different operators as the Hamiltonian. As an example, we use the Weyl transformation $\tau = \log r$ and choose $\tau$ as the time axis. In this coordinate system, the dilatation generator is simply

$$D = ix^\mu\partial_\mu = i\partial_\tau. \tag{10}$$

Thus, $D$ becomes Hamiltonian along $\tau$ axis. Taking time slices (also called foliations) along $\tau$ and thinking of the respective generator as Hamiltonian is called *radial quantization*. We illustrate this in Fig. 1. We assume the vacuum state $|0\rangle$ with zero energy. We can act a dilatation operator on the center of the sphere $\mathcal{O}(0)$ to produce a new state $\mathcal{O}(0)|0\rangle$. Energy of this state is obtained,

$$D(\mathcal{O}(0)|0\rangle) = \mathcal{O}(0)D|0\rangle + [D, \mathcal{O}(0)]|0\rangle \tag{11}$$
$$= -i\Delta(\mathcal{O}(0)|0\rangle). \tag{12}$$

Physically, this means that insertion of operator $\mathcal{O}$ at the origin generates a state with eigenvalue $-i\Delta$. This is called the *state-operator correspondence*. Acting momentum operator on the state $\mathcal{O}(0)|0\rangle \equiv |\Delta\rangle$ generates states with higher scaling dimension:

$$DP_\mu|\Delta\rangle = P_\mu D|\Delta\rangle + [D, P_\mu]|0\rangle \tag{13}$$
$$= -i(\Delta+1)P_\mu|\Delta\rangle. \tag{14}$$

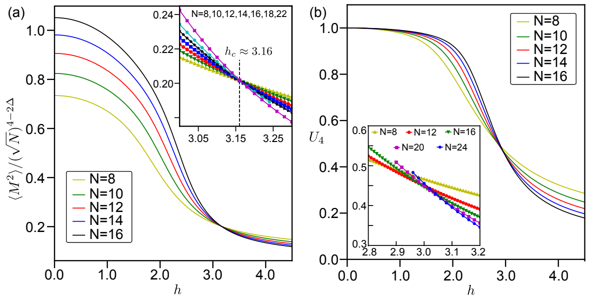


FIG. 2. Finite size scaling of an order parameter $\langle M^2\rangle$. Critical field $h_c$ is obtained from crossings of the curves. Adapted from [1].

On the other hand, acting on special conformal transformation operators produces states with lower scaling dimension:

$$DK_\mu|\Delta\rangle = K_\mu D|\Delta\rangle + [D, K_\mu]|0\rangle \tag{15}$$
$$= -i(\Delta-1)K_\mu|\Delta\rangle. \tag{16}$$

Therefore, towers of states arise from momentum and special conformal transformation operators, which are similar to creation and annihilation operators in quantum harmonic oscillators. Analogous to the ground state of a harmonic oscillator that is annihilated by the annihilation operator $a$, we define *primary operator* $\mathcal{O}$, where the corresponding state is annihilated by $K_\mu$:

$$K_\mu \mathcal{O}(0)|0\rangle = 0. \tag{17}$$

Thus, the operator spectrum of the CFT is fully characterized by its primary operators. Studying CFT requires uncovering how many primary operators we have and their scaling dimension. Given the state-operator correspondence, we aim to realize the conformal tower spectrum from the energy spectrum of a quantum mechanical model on a 2D sphere. However, it is hard to think of a numerically calculable discrete model (like lattices), which conserves the whole symmetry of the 2D sphere. Alternatively, some studies used polyhedra with rich symmetries to mimic a 2D sphere. Still, finite-size effects and explicit breaking of continuous rotational symmetry remain, making accurate extraction of scaling dimensions impossible. This is where the Fuzzy sphere comes in. The Fuzzy sphere is a set of spherical Landau levels: solutions of single particle eigenstates of a charged particle embedded on a 2D sphere, where a magnetic monopole (Wu-Yang monopole) is inserted at the center. This basis conserves full rotational symmetry of the sphere. By writing the many-body Hamiltonian and diagonalizing it, it is shown that a spectrum matching the CFT's operator spectrum is produced.

## II. MICROSCOPIC MODEL

In this section, we write a microscopic model that realizes a phase transition belongs to (2+1)D Ising universality. We first elaborate on the details of the single particle basis. Single particle wavefunctions

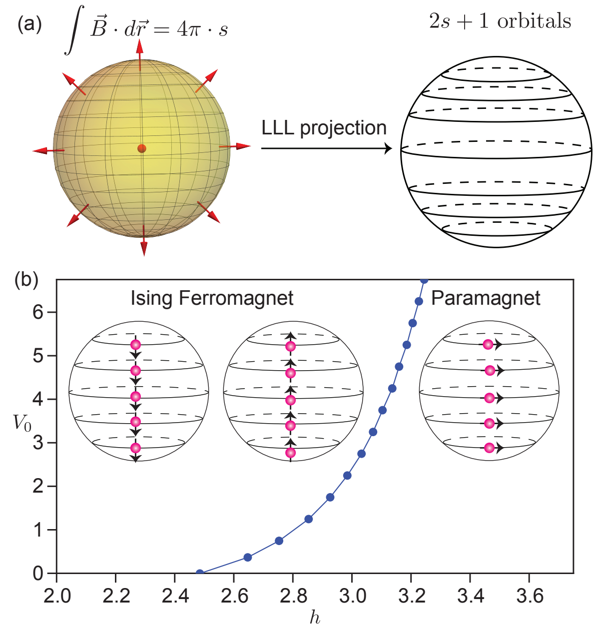


FIG. 3. (a) Illustration of spherical Landau levels. Phase diagram on $(V_0, h)$ space from finite size scaling. Adapted from [1].

$$Y^{(s)}_{lm}(\hat{\mathbf{n}}) \quad (l = s, s+1, \ldots, m = -l, \ldots, l) \tag{18}$$

are solutions of the Wu-Yang monopole problem

$$H_0 = \frac{1}{2MR^2}\left(\partial_i + iA_i\right)^2, \tag{19}$$

under a particular gauge

$$A_\theta = 0, A_\phi = -\frac{s}{R}\cot\theta. \tag{20}$$

These are also called the monopole spherical harmonics. Contrast to usual spherical harmonics $Y_{lm}(\hat{\mathbf{n}})$, these functions' $l$ starts from $s$, resulting from existence of monopole. Energy eigenvalues are

$$E_l = \frac{1}{2MR^2}\left(l(l+1) - s^2\right). \tag{21}$$

In the Landau level picture, $l = s$ wavefunctions are the lowest Landau level, with the lowest energy. To consider only low-energy physics with few degrees of freedom, we consider the lowest Landau level-projected model. We can write the lowest Landau level basis explicitly

$$Y^{(s)}_{sm}(\hat{\mathbf{n}}) = \sqrt{\frac{(2s+1)!}{4\pi(s-m)!(s+m)!}}\,u^{s+m}v^{s-m} \tag{22}$$

using spinor variables

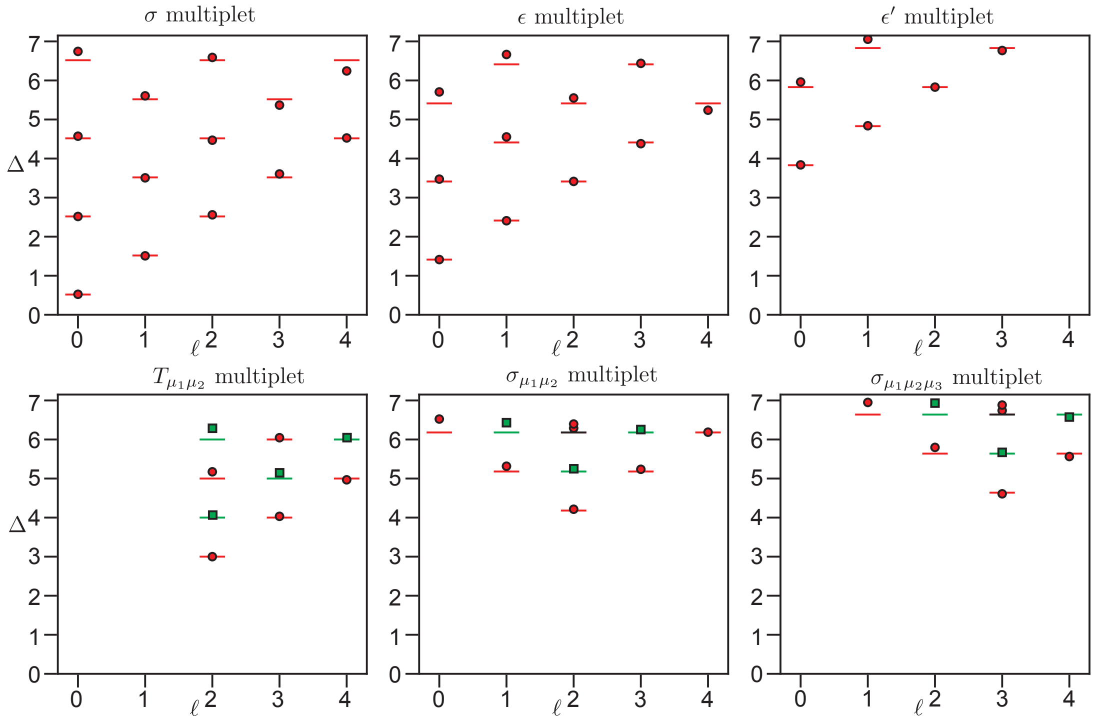


FIG. 4. Conformal towers of 3D Ising CFT. Each plot corresponds to spectra of dilatation operator $D$, for each primary operator. Lines are results from the conformal bootstrap method. Adapted from [1]

$$(u, v) = \left(\cos(\theta/2)\, e^{i\phi/2}, \sin(\theta/2)\, e^{-i\phi/2}\right). \tag{23}$$

We consider a multi-fermion system, with creation and annihilation operators

$$\begin{aligned} \hat{\psi}^\dagger_f(\hat{\mathbf{n}}) &= \sum_{m=-s}^{+s} \bar{Y}^{(s)}_{sm}(\hat{\mathbf{n}}) \hat{c}^\dagger_{mf}, \\ \hat{\psi}_f(\hat{\mathbf{n}}) &= \sum_{m=-s}^{+s} Y^{(s)}_{sm}(\hat{\mathbf{n}}) \hat{c}_{mf}. \end{aligned} \tag{24}$$

In particular, we consider the phase transition from a quantum Hall ferromagnet (FM) to a paramagnet (PM), which resembles the quantum transverse field Ising model. The model consists of three terms

$$H_{00} = \frac{1}{2}\sum_{m_i} V_{m_1,m_2,m_3,m_4} : \left(\mathbf{c}^\dagger_{m_1}\mathbf{c}_{m_4}\right)\left(\mathbf{c}^\dagger_{m_2}\mathbf{c}_{m_3}\right) :, \tag{25}$$

$$H_{zz} = -\frac{1}{2}\sum_{m_i} V_{m_1,m_2,m_3,m_4} : \left(\mathbf{c}^\dagger_{m_1}\sigma^z\mathbf{c}_{m_4}\right)\left(\mathbf{c}^\dagger_{m_2}\sigma^z\mathbf{c}_{m_3}\right) :, \tag{26}$$

$$H_t = -h\sum_{m} \mathbf{c}^\dagger_{m}\sigma^x\mathbf{c}_{m}. \tag{27}$$

In the absence of a transverse magnetic field $H_t$, the fermions are fully polarized in a certain direction, forming FM. As we increase $h$, in the high magnetic field regime, all of the fermions direct positive $x$ direction (PM). By exactly diagonalizing the Hamiltonian, we can find $h$ at which the phase transition occurs. In detail, we set $V_1 = 1$ and scan theoretical phase space $(V_0, h)$, where $V_l$ encodes interaction coefficients:

$$V_{m_1,m_2,m_3,m_4} = \sum_l V_l(4s-2l+1)\begin{pmatrix} s & s & 2s-l \\ m_1 & m_2 & -m_1-m_2 \end{pmatrix} \tag{28}$$

$$\times \begin{pmatrix} s & s & 2s-l \\ m_4 & m_3 & -m_3-m_4 \end{pmatrix}. \tag{29}$$

The critical line ($h_c$ for each $V_0$) is given by analyzing the order parameter $\langle M^2 \rangle$, shown in Fig. 2. The phase diagram is shown in Fig. 3. Scanning crtical line, we find a point with minimal finite size effects. Finally, we get critical point $(V_0, h) = (4.75, 3.16)$.

## III. RESULTS

By analyzing the low-energy spectrum of the Hamiltonian at $(V_0, h) = (4.75, 3.16)$, we study 3D Ising CFT. Since every CFT carries an energy-momentum tensor $T_{\mu\nu}$, which has $\Delta = 3$ and $l = 2$, we can normalize data using the lowest energy state in $l = 2$ sector. In Fig. 4, we show conformal data obtained from the rescaled energy spectrum. We compare the results with those of the conformal bootstrap approach, which shows the state-of-the-art accuracy on the conformal data. Surprisingly, with in small size simulation, the Fuzzy sphere regularization already produce accurate conformal data, benchmarking the conformal bootstrap method.

## IV. DISCUSSION

In this paper, we review recent developments on the Fuzzy sphere regularization. We introduce the foundations of CFT and the microscopic model that realizes 3D Ising CFT. By analyzing the Hamiltonian, we show that the Fuzzy sphere approach produces remarkably high quality conformal data.

We discuss the current status of this idea. Until now, the FS regularization has been applied to numerous CFTs, including 3D Ising [1], free scalar CFT [2], O(3) Wilson-Fisher [3], SO(5) deconfined critical point [4], Sp(N) symmetric [5], Chern-Simons matter theory [6], and non-unitary Yang-Lee CFT [7–9], as well as conformal defects [10] and boundary CFTs [11, 12]. In addition to scalar CFTs, free and interacting fermionic CFTs are realized using a mixed Hilbert space of fermions and bosons. Also, the dimension of the target CFT has not been limited to 3D. Though the Landau level is defined only in even-dimensional space, we can use thin torus geometry where we freeze one direction to study 2D CFT. In principle, the Landau level basis in higher dimensions can be utilized to study exotic CFTs in higher dimensions.

For future study, it will be interesting to target more CFTs. Additionally, quantitative reasoning beyond qualitative argument (like preservation of rotational symmetry) is needed to explain the usefulness of this approach. Also, the current Fuzzy sphere technique requires prior conformal data to define the loss function and find the optimized critical point. It will be remarkable if the technique itself can find a critical point or improve conformal data.


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