Renormalization Group Analysis of the Six-Vertex Model

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We review the rigorous Renormalization Group (RG) analysis of the six-vertex model near the free fermion point ($\Delta = 0$) developed by P. Falco, enriched with modern mathematical viewpoints from integrable probability. By mapping the lattice model to a Grassmann integral, we outline a multi-scale RG procedure to derive the large-distance correlations. This synthesized presentation bridges the exact, determinantal framework of the non-interacting limit with the field-theoretic machinery required to rigorously analyze the interacting critical phase.

I. INTRODUCTION

The six-vertex model was first proposed by Pauling [1] in 1935 to study the residual entropy of ice. It can be seen as a model for "two-dimensional ice" as it can be defined as allowed configurations of water molecules on the square lattice. The six-vertex model became an archetypical lattice model of statistical mechanics and integrable probability with Lieb's solution of the model [2–5] using the Bethe ansatz, and there has been intense research activity on this model. We refer to [6, 7] for detailed expositions and reviews.

The characteristic parameter Δ determines various aspects of the model. The disordered regime $|\Delta| < 1$ is of particular interest, since the translation matrix is gapless and the correlation of the bulk observables are predicted to have a power-law decaying correlation functions. The continuum scaling limit is expected to be described by the $c = 1$ conformal field theory, specifically the massless sine–Gordon model, the 2D Coulomb gas, or the Luttinger liquid. However, direct validation of these predictions is very problematic in general.

The free fermion point $\Delta = 0$ case is much simpler and relatively well studied. At this point, the model becomes bijectively equivalent to a *dimer model*, or a domino tiling model, the partition functions collapse into explicit determinantal formulas and admits powerful exact methods that relates to various other fields of mathematics. There have been extensive mathematical results and frameworks on this regime, starting from the early work of Kasteleyn [8] and Temperley–Fisher [9] in 1960s, and including the groundbreaking works of [10] and [11] on the characterization of the ergodic translation-invariant Gibbs measures. The general critical regime $|\Delta| < 1$ is often understood as a small perturbation of $\Delta = 0$ with similar phenomenonology.

The Renormalization Group (RG) technique, developed by Wilson in the early 1970s, is a powerful framework for studying critical phenomena. Falco [12] addressed the six-vertex model from the RG perspective, and derived the arrow-arrow correlation asymptotics with an anomalous critical exponent for small Δ.

In this paper, we aim to reproduce the main mechanism of Falco's RG analysis adapted to the framework of this course, while incorporating a modern mathematical viewpoint rooted in integrable probability. The remainder of this paper is organized as follows. First, we reformulate the six-vertex model near the free fermion point into a system of interacting fermions, translating the lattice weights into a Grassmann integral representation. Next, we set up the multi-scale RG framework to systematically integrate out short-distance fluctuations. Finally, we analyze the resulting RG flow to derive the large-distance asymptotics of the correlation functions.

II. MODEL DEFINITION

Let Γ' be a finite square lattice with periodic boundary condition. We denote the orthogonal basis vector generating Γ' as $\mathbf{v}_0$ and $\mathbf{v}_1$. A configuration σ of the six-vertex model is a collection of orientation of the edges of Γ such that every vertex has indegree and outdegree of two. We assign them weights $p_1, \dots, p_6$ as in FIG. 1. For sake of simplicity, we only consider the *isotropic, zero-field* case, where

$$p_1 = p_2 = p_3 = p_4 = 1.$$

Without loss of generality, we can parametrize the remaining weights to $p_5 = 1$ and $p_6 = 2e^{\lambda}$ for a real λ. The characteristic parameter of the six-vertex model is

$$\Delta = \frac{p_1 p_2 + p_3 p_4 - p_5 p_6}{2\sqrt{p_1 p_2 p_3 p_4}} = 1 - e^{\lambda}.$$

The point $\Delta = 0$ ($\lambda = 0$) corresponds to the free fermion case. We will focus on the small Δ case and perform RG analysis.

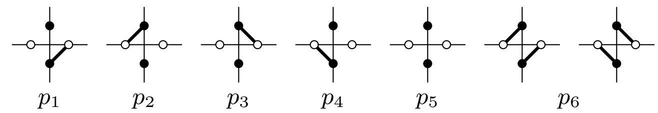

FIG. 1. The six possible configurations for vertices in the six-vertex model, and their corresponding dimer configurations. The vertex configuration 6 corresponds to two dimer configurations of the same weights.

Following the construction of Baxter [6], the six vertex model on Γ' is bijectively equivalent to a dimer model on a modified square lattice Γ generated by $\mathbf{e}_0$ and $\mathbf{e}_1$ consisting of black and white vertices, where $\mathbf{v}_0 = \mathbf{e}_0 - \mathbf{e}_1$ and $\mathbf{v}_1 = \mathbf{e}_0 + \mathbf{e}_1$. As depicted in FIG. 1, the first five vertex configurations of the six-vertex model map to dimer configurations on Γ of weight 1, while vertex configuration 6 maps to two possible dimer configurations of weight e^{λ} each. Hence, the partition function of our dimer model is

$$Z_\lambda = \sum_\sigma \exp\Big[\lambda \sum_{d,d'\in\sigma} v(d,d')\Big], \tag{1}$$

where the first sum runs over all dimer configurations, and we set $v(d,d') = 1$ when d and d' are two parallel dimers on adjacent edges, and 0 otherwise.

Let $\mathcal{B}$ and $\mathcal{W}$ be the sublattices of Γ of black and white vertices, respectively. For $\mathbf{w} \in \mathcal{W}$, let the *horizontal arrow orientation* $\sigma_0(\mathbf{w})$ be equal to 1 if the arrow along $\mathbf{w}$ points to the right, and -1 otherwise. Similarly, for $\mathbf{b} \in \mathcal{B}$, let the *vertical arrow orientation* $\sigma_1(\mathbf{b})$ be equal to 1 if the arrow along $\mathbf{b}$ points up, and -1 otherwise. For $\mathbf{x} \in \Gamma$, let the *dimer occupancy* $\nu_j(\mathbf{x})$ be equal to 1 if the edge $\{\mathbf{x}, \mathbf{x}+\mathbf{e}_j\}$ is occupied, and 0 otherwise. Then, we have the relations

$$\begin{aligned}\sigma_0(\mathbf{w}) &= \nu_0(\mathbf{w}-\mathbf{e}_0) + \nu_1(\mathbf{w}) - \nu_0(\mathbf{w}) - \nu_1(\mathbf{w}-\mathbf{e}_1),\\ \sigma_1(\mathbf{b}) &= \nu_0(\mathbf{b}) + \nu_1(\mathbf{b}) - \nu_0(\mathbf{b}-\mathbf{e}_0) - \nu_1(\mathbf{b}-\mathbf{e}_1).\end{aligned}$$

III. FREE FERMION POINT

The Kasteleyn matrix $K : \mathbb{C}^{\mathcal{B}} \to \mathbb{C}^{\mathcal{W}}$ is defined by

$$K_{\mathbf{w},\mathbf{b}} = \delta_{\mathbf{w}-\mathbf{e}_0,\mathbf{b}} - \delta_{\mathbf{w}+\mathbf{e}_0,\mathbf{b}} - \delta_{\mathbf{w}-\mathbf{e}_1,\mathbf{b}} - \delta_{\mathbf{w}+\mathbf{e}_0,\mathbf{b}}.$$

Then the unperturbed partition function at $\lambda = 0$ is $Z_0 = \det K$. Form an exterior algebra over the vector space $V = \bigoplus_{\mathbf{b}\in\mathcal{B}} \mathbb{C}\psi^+_{\mathbf{b}} \oplus \bigoplus_{\mathbf{w}\in\mathcal{W}} \mathbb{C}\psi^-_{\mathbf{w}}$. Then we can equivalently write

$$Z_0 = \int D\psi \exp\Big[-\sum_{\mathbf{b}\in\mathcal{B},\mathbf{w}\in\mathcal{W}} K_{\mathbf{w},\mathbf{b}}\psi^+_{\mathbf{b}}\psi^-_{\mathbf{w}}\Big]. \tag{2}$$

Hence, the lattice fermion field ψ is a Gaussian process with covariance given by the inverse Kasteleyn matrix K^{-1}.

Following the approach of Kenyon–Okounkov–Sheffield [11], to evaluate $[K^{-1}]_{\mathbf{b},\mathbf{w}}$ for large $|\mathbf{b}-\mathbf{w}|$, we project the translation-invariant operator K onto a fundamental domain. The magnetically altered Kasteleyn matrix $K_1(z,w)$, $z,w \in \mathbb{C}^*$, is the Kasteleyn matrix of a single fundamental domain, where the edge weights are multiplied by z^{-1} and w when the edges cross the two fundamental cycles of the torus. Note that the inverse Kasteleyn matrix is completely determined by the inverse of the magnetically altered Kasteleyn matrix:

$$\begin{aligned}[K^{-1}]_{\mathbf{b},\mathbf{w}} = \frac{1}{(2\pi i)^2}\oint_{|z|=1}\frac{dz}{z}\oint_{|w|=1}\frac{dw}{w}\\ \times [K_1(z,w)^{-1}]_{\mathbf{b},\mathbf{w}} z^{b_0-w_0} w^{b_1-w_1}.\end{aligned} \tag{3}$$

In our case, the magnetically altered Kastelyn matrix becomes a scalar function, yielding the characteristic polynomial of

$$P(z,w) = \det K_1(z,w) = \frac{1}{2}(z-z^{-1}) - \frac{1}{2i}(w+w^{-1}). \tag{4}$$

The spectral curve $\mathcal{R}$ is the curve defined by $P(z,w) = 0$. As $|\mathbf{b}-\mathbf{w}| \to \infty$, the value of the integral (3) is completely determined by the poles of the integrand, the points where the spectral curve $\mathcal{R}$ intersects the unit torus. Parametrizing $z = e^{ik_0}$ and $w = e^{ik_1}$, the characteristic polynomial becomes

$$P(e^{ik_0}, e^{ik_1}) = i\sin k_0 - \cos k_1, \tag{5}$$

so there are exactly two roots $\mathbf{p}_F^{\pm} = (0, \pm\frac{\pi}{2})$ within the first Brillouin zone $k_j \in (-\pi, \pi]$. These two points are the *Fermi points* in physics terminology.

To set up the RG, we restrict our attention to Fourier modes lying strictly within disks of radius $\Lambda \ll 1$, the UV cutoff, centered at the two roots $p_F^{\pm}$. More precisely, we change our basis to the Fourier space generators given by

$$\begin{aligned}\psi^+_{\mathbf{b}} &= \int \frac{d\mathbf{q}}{(2\pi)^2}\sum_{\omega=\pm} e^{i(\omega\mathbf{p}_F+\mathbf{q})\cdot\mathbf{b}}\psi^+_{\mathbf{q},\omega},\\ \psi^-_{\mathbf{w}} &= \int \frac{d\mathbf{q}}{(2\pi)^2}\sum_{\omega=\pm} e^{-i(\omega\mathbf{p}_F+\mathbf{q})\cdot\mathbf{w}}\psi^-_{\mathbf{q},\omega},\end{aligned} \tag{6}$$

where $\mathbf{k} = \mathbf{p}_F^{\omega} + \mathbf{q}$, and we only consider the small deviations $|\mathbf{q}| < \Lambda$. Linearizing the characteristic polynomial, we get

$$P(e^{i(0+q_0)}, e^{i(\pm\pi/2+q_1)}) \approx iq_0 \mp q_1,$$

yielding the covariance between the Fourier generators

$$\langle \psi^+_{\mathbf{q},\omega}\psi^-_{\mathbf{q}',\omega'}\rangle_0 = (2\pi)^2\delta^2(\mathbf{q}-\mathbf{q}')\delta_{\omega,\omega'}\frac{1}{iq_0 - \omega q_1}. \tag{7}$$

IV. PERTURBATIVE INTERACTION

For a pair p of adjacent parallel edges $\{\mathbf{b},\mathbf{w}\}$ and $\{\mathbf{b}',\mathbf{w}'\}$, consider the indicator $I_p(\sigma)$ which yields a value of 1 if both edges are occupied by dimers, and 0 otherwise. Then, the partition function in (1) can be expressed as

$$Z_\lambda = \sum_\sigma \prod_p e^{\lambda I_p(\sigma)} = \sum_\sigma \prod_p \big(1 - \Delta I_p(\sigma)\big),$$

where $\Delta = 1 - e^{\lambda}$ is the characteristic parameter. Hence, for the algebraic indicator

$$\nu_p = \pm\psi^+_{\mathbf{b}}\psi^+_{\mathbf{b}'}\psi^-_{\mathbf{w}}\psi^-_{\mathbf{w}'},$$

where the sign only depends on the slope of the edges in p, the sum can be rewritten as

$$Z_\lambda = \int D\psi \prod_p (1 + \Delta\nu_p) \exp\Big[- \sum_{\mathbf{b}\in\mathcal{B},\mathbf{w}\in\mathcal{W}} K_{\mathbf{w},\mathbf{b}}\psi^+_{\mathbf{b}}\psi^-_{\mathbf{w}}\Big].$$

Since the indicators ν_p are nilpotent and commuting each other, we can conclude that

$$Z_\lambda = Z_0\langle \exp(\Delta V(\psi))\rangle_0, \tag{8}$$

where $V(\psi)$ is the global fermion interaction is given by

$$V(\psi) = \sum_p \nu_p = \sum_p \pm\psi^+_{\mathbf{b}}\psi^+_{\mathbf{b}'}\psi^-_{\mathbf{w}}\psi^-_{\mathbf{w}'}. \tag{9}$$

Passing to the Fourier space, the global interaction can be written as

$$\begin{aligned} V(\psi) = \sum_{\underline{\omega}} \int \frac{d\mathbf{q}_1\cdots d\mathbf{q}_4}{(2\pi)^6}\psi^+_{\mathbf{q}_1,\omega_1}\psi^+_{\mathbf{q}_2,\omega_2}\psi^-_{\mathbf{q}_3,\omega_3}\psi^-_{\mathbf{q}_4,\omega_4} \\ \times\, \delta^2(-\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 + \mathbf{k}_4) v(\underline{\mathbf{q}};\underline{\omega}), \end{aligned} \tag{10}$$

where $\mathbf{k}_j = \mathbf{p}_F^{\omega_j} + \mathbf{q}_j$, with vertex weights

$$\begin{aligned} v(\underline{\mathbf{q}};\underline{\omega}) = 2\sin\Big[(\mathbf{q}_1 - \mathbf{q}_2)\cdot\frac{\mathbf{v}_1}{2} + (\omega_1 - \omega_2)\frac{\pi}{4}\Big] \\ \times \sin\Big[(\mathbf{q}_3 - \mathbf{q}_4)\cdot\frac{\mathbf{v}_0}{2} + (\omega_3 - \omega_4)\frac{\pi}{4}\Big]. \end{aligned} \tag{11}$$

Since the deviations $\mathbf{q}_j$ are small, we may replace the delta function in (10) by $\delta^2(-\mathbf{q}_1+\mathbf{q}_2-\mathbf{q}_3+\mathbf{q}_4)$ by leaving only the ω_j's such that $\sum_j \mathbf{p}_F^{\omega_j} \equiv \mathbf{0} \mod 2\pi$, which are precisely:

$$\underline{\omega} = -+-+, +-+-, ++++, ----, ++--, -++-.$$

V. RENORMALIZATION GROUP ANALYSIS

Split the field into $\psi = \psi^< + \psi^>$, where $\psi^>$ contains the fast modes in the shell $\Lambda/b < |\mathbf{q}| < \Lambda$. Averaging the action in (8) over the fast modes, we obtain the cumulant expansion of the effective action

$$\log\langle e^{\Delta V}\rangle_{0,>} = \Delta\langle V\rangle_{0,>} + \frac{\Delta^2}{2}\langle V\rangle^c_{0,>} + \mathcal{O}(\Delta^3). \tag{12}$$

Note that expanding the vertex weights $v(\underline{\mathbf{q}};\underline{\omega})$ as Taylor series around the Fermi points, all terms except the constant term are irrelevant under the RG rescaling $\mathbf{q}' = b\mathbf{q}$.

A. The First Cumulant

Let us evaluate the first cumulant $\langle V\rangle_{0,>}$. Up to additive constants, there are only two types of terms that survive: the quartic and the quadratic terms in the slow modes. Since the quartic term simply returns the interaction $V(\psi^<)$ restricted to the slow modes, we only need to evaluate the quadratic terms.

Suppose we contract $\psi^+_{\mathbf{q}_2,\omega_2}$ with $\psi^-_{\mathbf{q}_3,\omega_3}$. Using the two-point correlation in (7), we obtain a mass term with coefficient

$$\int_{\Lambda/b}^{\Lambda} \frac{d\mathbf{k}}{(2\pi)^2}\langle\psi^+_{\mathbf{k},\omega}\psi^-_{\mathbf{k},\omega}\rangle_0 = \int_{\Lambda/b}^{\Lambda}\frac{d\mathbf{k}}{ik_0 - \omega k_1},$$

which vanishes by radial symmetry. Similarly, all the other contractions vanish, so there is no mass term.

Since $\langle V\rangle_{0,>} = V(\psi^<)$, the effective action is just the original action defined on a slightly smaller disk $|\mathbf{q}| < \Lambda/b$. Now we rescale by $\mathbf{q}' = b\mathbf{q}$ and renormalize $\psi'(\mathbf{q}') = \psi^<(\mathbf{q})/z$. This yields the recursion relation $\Delta' = b^{-3}z^2\Delta$ in the first order, and gives the the fixed point $\Delta' = \Delta$ by setting $z = b^{3/2}$.

B. The Second Cumulant

Now we evaluate the second cumulant $\langle V^2\rangle_{0,>}$. Note that under the rescaling $\mathbf{q}' = b\mathbf{q}$ and $\psi'(\mathbf{q}') = \psi^<(\mathbf{q})/z$ with $z = b^{3/2}$ from above, the only relevant terms are the quadratic and the quartic terms in the slow modes.

We account the quadratic terms first. Note that the possible quadratic terms are the mass term $\psi^+\psi^-$ and the field renormalization term $K\psi^+\psi^-$. The latter term contributes by $\mathcal{O}(\Delta^2)$ to the renormalization factor z as it is generated by the second cumulant, so is negligible in the computation up to $\mathcal{O}(\Delta)$. For the mass term, it vanishes as like in the first cumulant; indeed, it vanishes to all orders as the mass term violates the lattice symmetries.

Now we address the quartic terms. Performing a casework with Feynmann diagrams, one can show that the quartic terms cancel out each other by lattice symmetries, yielding no contribution to the effective action.

Summing up, up to second order, the model remains massless, that is, it remains critical. Hence, the model possesses a continuous line of fixed points ranging over the small values of Δ.

VI. CORRELATION DECAY

We compute the dimer-dimer correlation function

$$\langle\nu_0(\mathbf{b})\nu_0(\mathbf{b}')\rangle_c = -\langle\psi^+_{\mathbf{b}}\psi^-_{\mathbf{b}'+\mathbf{e}_0}\rangle\langle\psi^+_{\mathbf{b}}\psi^-_{\mathbf{b}'+\mathbf{e}_0}\rangle. \tag{13}$$

For the first step, we address the correlation $\langle\psi^+_{\mathbf{b}}\psi^-_{\mathbf{w}}\rangle$.

We can perturbatively write

$$\langle \psi^+_{\mathbf{b}} \psi^-_{\mathbf{w}} \rangle = \langle \psi^+_{\mathbf{b}} \psi^-_{\mathbf{w}} e^{\Delta V} \rangle_0 = \sum_{n\geq 0} \frac{\Delta^n}{n!} \langle \psi^+_{\mathbf{b}} \psi^-_{\mathbf{w}} V^n \rangle_0.$$

Note that performing the Fourier transform as in (10) for each term, we can check that in the Fourier space, only the pairings with the same poles survive and the cross-terms vanish. In other words, the above correlation can be written as

$$\sum_{\omega=\pm} \int \frac{d\mathbf{q}_1 d\mathbf{q}_2}{(2\pi)^4} e^{i(\omega \mathbf{p}_F + \mathbf{q}_1)\cdot \mathbf{b} - i(\omega \mathbf{p}_F + \mathbf{q}_2)\cdot \mathbf{w})} \langle \psi^+_{\mathbf{q}_1,\omega} \psi^-_{\mathbf{q}_2,\omega} \rangle.$$

Plugging into (13), the correlation $\langle \nu_0(\mathbf{b}) \nu_0(\mathbf{b}') \rangle_c$ can be expressed as

$$-\sum_{\omega,\omega'} \int \frac{d\mathbf{q}_1 \dots d\mathbf{q}_4}{(2\pi)^8} e^{i((\mathbf{q}_1 - \mathbf{q}_4)\cdot \mathbf{b} - (\mathbf{q}_2 - \mathbf{q}_3)\cdot \mathbf{b}' - (\mathbf{q}_2 + \mathbf{q}_4)\cdot \mathbf{e}_0)} \times e^{i(\omega-\omega')\mathbf{p}_F \cdot (\mathbf{b}-\mathbf{b}')} \langle \psi^+_{\mathbf{q}_1,\omega} \psi^-_{\mathbf{q}_2,\omega} \rangle \langle \psi^+_{\mathbf{q}_3,\omega'} \psi^-_{\mathbf{q}_4,\omega'} \rangle \tag{14}$$

We split the expression (14) into two parts: $\omega = \omega'$ and $\omega \neq \omega'$. For the case $\omega = \omega'$, the oscillatory exponential phases cancel out, yielding an unstaggered part. Performing a perturbative computation, the $\mathcal{O}(\Delta)$ correction vanishes as in the previous section. Hence, the unstaggered part evaluates as

$$\int \frac{d\mathbf{q} d\mathbf{q}'}{(2\pi)^4} e^{i(\mathbf{q}-\mathbf{q}')\cdot(\mathbf{b}-\mathbf{b}')} \frac{2(q_0 q_0' - q_1 q_1')}{(q_0^2 + q_1^2)(q_0'^2 + q_1'^2)} \sim \frac{1}{|\mathbf{b}-\mathbf{b}'|^2}.$$

For the case $\omega \neq \omega'$, the oscillatory exponential phases add up, yielding a staggered part. Now the $\mathcal{O}(\Delta)$ correction does not vanish, yielding an additional logarithmic correction

$$\approx \frac{4}{\pi} \frac{\Delta \log |\mathbf{b}-\mathbf{b}'|}{|\mathbf{b}-\mathbf{b}'|^2} \sim \frac{1}{|\mathbf{b}-\mathbf{b}'|^{2-\frac{4\Delta}{\pi}}}.$$

VII. CONCLUSION

We have reproduced the core mechanism of Falco's rigorous RG analysis for the six-vertex model in the weakly interacting regime. By mapping the lattice model to a Grassmann integral, we applied a multi-scale RG procedure to compute the large-distance dimer-dimer correlations, depicting how the parameter Δ acts as an interaction coupling that shifts the system away from the exact determinantal baseline. This approach successfully bridges the exactly solvable, determinantal framework of integrable probability at $\Delta = 0$ with the field-theoretic tools to control perturbations and capture the universal, scale-invariant physics of the interacting model.

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