Problems & Solutions

1. *Spin waves:* In the XY model of $n = 2$ magnetism, a unit vector $\vec{s} = (s_x, s_y)$ (with $s_x^2 + s_y^2 = 1$) is placed on each site of a d–dimensional lattice. There is an interaction that tends to keep nearest–neighbors parallel, i.e. a Hamiltonian

$$-\beta\mathcal{H} = K \sum_{<ij>} \vec{s}_i \cdot \vec{s}_j \qquad .$$

The notation $< ij >$ is conventionally used to indicate summing over all *nearest–neighbor* pairs (i, j).

(a) Rewrite the partition function $Z = \int \prod_i d\vec{s}_i \exp(-\beta\mathcal{H})$, as an integral over the set of angles $\{\theta_i\}$ between the spins $\{\vec{s}_i\}$ and some arbitrary axis.

• The partition function is

$$Z = \int \prod_i d^2\vec{s}_i \exp\left(K \sum_{\langle ij\rangle} \vec{s}_i \cdot \vec{s}_j\right) \delta\left(\vec{s}_i^{\ 2} - 1\right).$$

Since $\vec{s}_i \cdot \vec{s}_j = \cos(\theta_i - \theta_j)$, and $d^2\vec{s}_i = ds_i d\theta_i s_i = d\theta_i$, we obtain

$$Z = \int \prod_i d\theta_i \exp\left(K \sum_{\langle ij\rangle} \cos(\theta_i - \theta_j)\right).$$

(b) At low temperatures ($K \gg 1$), the angles $\{\theta_i\}$ vary slowly from site to site. In this case expand $-\beta\mathcal{H}$ to get a quadratic form in $\{\theta_i\}$.

• Expanding the cosines to quadratic order gives

$$Z = e^{N_b K} \int \prod_i d\theta_i \exp\left(-\frac{K}{2} \sum_{\langle ij\rangle} (\theta_i - \theta_j)^2\right),$$

where N_b is the total number of bonds. Higher order terms in the expansion may be neglected for large K, since the integral is dominated by $|\theta_i - \theta_j| \approx \sqrt{2/K}$.

(c) For $d = 1$, consider L sites with periodic boundary conditions (i.e. forming a closed chain). Find the normal modes θ_q that diagonalize the quadratic form (by Fourier transformation), and the corresponding eigenvalues $K(q)$. Pay careful attention to whether the modes are real or complex, and to the allowed values of q.

• For a chain of L sites, we can change to Fourier modes by setting

$$\theta_j = \sum_q \theta(q) \frac{e^{iqj}}{\sqrt{L}}.$$

Since θ_j are real numbers, we must have

$$\theta(-q) = \theta(q)^*,$$

and the allowed q values are restricted, for periodic boundary conditions, by the requirement of

$$\theta_{j+L} = \theta_j, \quad \Rightarrow \quad qL = 2\pi n, \quad \text{with} \quad n = 0, \pm 1, \pm 2, \ldots, \pm\frac{L}{2}.$$

Using

$$\theta_j - \theta_{j-1} = \sum_q \theta(q) \frac{e^{iqj}}{\sqrt{L}} \left(1 - e^{-iq}\right),$$

the one dimensional Hamiltonian, $\beta\mathcal{H} = \frac{K}{2}\sum_j \left(\theta_j - \theta_{j-1}\right)^2$, can be rewritten in terms of Fourier components as

$$\beta\mathcal{H} = \frac{K}{2} \sum_{q,q'} \theta(q)\,\theta(q') \sum_j \frac{e^{i\left(q+q'\right)j}}{L} \left(1 - e^{-iq}\right)\left(1 - e^{-iq'}\right).$$

Using the identity $\sum_j e^{i\left(q+q'\right)j} = L\delta_{q,-q'}$, we obtain

$$\beta\mathcal{H} = K \sum_q |\theta(q)|^2 \left[1 - \cos(q)\right].$$

(d) Generalize the results from the previous part to a d–dimensional simple cubic lattice with periodic boundary conditions.

• In the case of a d dimensional system, the index j is replaced by a vector

$$j \mapsto \mathbf{j} = (j_1, \ldots, j_d),$$

which describes the lattice. We can then write

$$\beta\mathcal{H} = \frac{K}{2} \sum_{\mathbf{j}} \sum_\alpha \left(\theta_{\mathbf{j}} - \theta_{\mathbf{j}+\mathbf{e}_\alpha}\right)^2,$$

where $\mathbf{e}_\alpha$'s are unit vectors $\{\mathbf{e}_1 = (1, 0, \cdots, 0), \cdots, \mathbf{e}_d = (0, \cdots, 0, 1)\}$, generalizing the one dimensional result to

$$\beta\mathcal{H} = \frac{K}{2} \sum_{\mathbf{q},\mathbf{q}'} \theta(\mathbf{q})\,\theta(\mathbf{q}') \sum_\alpha \sum_{\mathbf{j}} \frac{e^{i\left(\mathbf{q}+\mathbf{q}'\right)\cdot\mathbf{j}}}{L^d} \left(1 - e^{-i\mathbf{q}\cdot\mathbf{e}_\alpha}\right)\left(1 - e^{-i\mathbf{q}'\cdot\mathbf{e}_\alpha}\right).$$

Again, summation over $\mathbf{j}$ constrains $\mathbf{q}$ and $-\mathbf{q}'$ to be equal, and

$$\beta\mathcal{H} = K\sum_{\mathbf{q}} |\theta(\mathbf{q})|^2 \sum_{\alpha} [1 - \cos(q_\alpha)] .$$

(e) Calculate the contribution of these modes to the free energy and heat capacity. (Evaluate the *classical* partition function, i.e. do not quantize the modes.)

• With $K(\mathbf{q}) \equiv 2K\sum_\alpha [1 - \cos(q_\alpha)]$,

$$Z = \int \prod_{\mathbf{q}} d\theta(\mathbf{q}) \exp\left[-\frac{1}{2}K(\mathbf{q})|\theta(\mathbf{q})|^2\right] = \prod_{\mathbf{q}} \sqrt{\frac{2\pi}{K(\mathbf{q})}},$$

and the corresponding free energy is

$$F = -k_B T \ln Z = -k_B T \left[\text{constant} - \frac{1}{2}\sum_{\mathbf{q}} \ln K(\mathbf{q})\right],$$

or, in the continuum limit (using the fact that the density of states in $\mathbf{q}$ space is $(L/2\pi)^d$),

$$F = -k_B T \left[\text{constant} + \frac{1}{2}L^d \int \frac{d^d q}{(2\pi)^d} \ln K(\mathbf{q})\right].$$

As $K \sim 1/T$, we can write

$$F = -k_B T \left[\text{constant}' - \frac{1}{2}L^d \ln T\right],$$

and the heat capacity per site is given by

$$C = -T\frac{\partial^2 F}{\partial T^2} \cdot \frac{1}{L^d} = \frac{k_B}{2}.$$

This is because there is one degree of freedom (the angle) per site that can store potential energy. Of course, at sufficiently high temperatures the replacement of cosines by the quadratic form is no longer valid.

(f) Find an expression for $\langle \vec{s}_0 \cdot \vec{s}_{\mathbf{x}} \rangle = \Re\langle \exp[i\theta_{\mathbf{x}} - i\theta_0] \rangle$ by adding contributions from different Fourier modes. Convince yourself that for $|\mathbf{x}| \to \infty$, only $\mathbf{q} \to \mathbf{0}$ modes contribute appreciably to this expression, and hence calculate the asymptotic limit.

• We have

$$\theta_{\mathbf{x}} - \theta_{\mathbf{0}} = \sum_{\mathbf{q}} \theta(\mathbf{q}) \frac{e^{i\mathbf{q}\cdot\mathbf{x}} - 1}{L^{d/2}},$$

and by completing the square for the argument of the exponential in $\left\langle e^{i(\theta_{\mathbf{x}}-\theta_{\mathbf{0}})}\right\rangle$, *i.e.* for

$$-\frac{1}{2}K\left(\mathbf{q}\right)\left|\theta\left(\mathbf{q}\right)\right|^{2}+i\theta\left(\mathbf{q}\right)\frac{e^{i\mathbf{q}\cdot\mathbf{x}}-1}{L^{d/2}},$$

it follows immediately that

$$\left\langle e^{i(\theta_{\mathbf{x}}-\theta_{\mathbf{0}})}\right\rangle=\exp\left\{-\frac{1}{L^{d}}\sum_{\mathbf{q}}\frac{\left|e^{i\mathbf{q}\cdot\mathbf{x}}-1\right|^{2}}{2K\left(\mathbf{q}\right)}\right\}=\exp\left\{-\int\frac{d^{d}q}{\left(2\pi\right)^{d}}\frac{1-\cos\left(\mathbf{q}\cdot\mathbf{x}\right)}{K\left(\mathbf{q}\right)}\right\}.$$

For x larger than 1, the integrand has a peak of height $\sim x^{2}/2K$ at $q=0$ (as it is seen by expanding the cosines for small argument). Furthermore, the integrand has a first node, as q increases, at $q\sim 1/x$. From these considerations, we can obtain the leading behavior for large x:

• In $d=1$, we have to integrate $\sim x^{2}/2K$ over a length $\sim 1/x$, and thus

$$\left\langle e^{i(\theta_{\mathbf{x}}-\theta_{\mathbf{0}})}\right\rangle\sim\exp\left(-\frac{|x|}{2K}\right).$$

• In $d=2$, we have to integrate $\sim x^{2}/2K$ over an area $\sim(1/x)^{2}$. A better approximation, at large x, than merely taking the height of the peak, is given by

$$\begin{aligned}\int\frac{d^{d}q}{\left(2\pi\right)^{d}}\frac{1-\cos\left(\mathbf{q}\cdot\mathbf{x}\right)}{K\left(\mathbf{q}\right)}&\approx\int\frac{dqd\varphi q}{\left(2\pi\right)^{2}}\frac{1-\cos\left(qx\cos\varphi\right)}{Kq^{2}}\\&=\int\frac{dqd\varphi}{\left(2\pi\right)^{2}}\frac{1}{Kq}-\int\frac{dqd\varphi}{\left(2\pi\right)^{2}}\frac{\cos\left(qx\cos\varphi\right)}{Kq},\end{aligned}$$

or, doing the angular integration in the first term,

$$\int\frac{d^{d}q}{\left(2\pi\right)^{d}}\frac{1-\cos\left(\mathbf{q}\cdot\mathbf{x}\right)}{K\left(\mathbf{q}\right)}\approx\int^{1/|x|}\frac{dq}{2\pi}\frac{1}{Kq}+\text{subleading in }x,$$

resulting in

$$\left\langle e^{i(\theta_{\mathbf{x}}-\theta_{\mathbf{0}})}\right\rangle\sim\exp\left(-\frac{\ln|x|}{2\pi K}\right)=|x|^{-\frac{1}{2\pi K}},\quad\text{as }x\to\infty.$$

• In $d\geq 3$, we have to integrate $\sim x^{2}/2K$ over a volume $\sim(1/x)^{3}$. Thus, as $x\to\infty$, the x dependence of the integral is removed, and

$$\left\langle e^{i(\theta_{\mathbf{x}}-\theta_{\mathbf{0}})}\right\rangle\to\text{constant},$$

implying that correlations don't disappear at large x.

The results can also be obtained by noting that the fluctuations are important only for small q. Using the expansion of $K(\mathbf{q}) \approx Kq^2/2$, then reduces the problem to calculation of the Coulomb Kernel $\int d^d\mathbf{q} e^{i\mathbf{q}\cdot\mathbf{x}}/q^2$, as described in the preceding chapter.

(g) Calculate the transverse susceptibility from $\chi_t \propto \int d^d\mathbf{x} \langle \vec{s}_0 \cdot \vec{s}_\mathbf{x} \rangle_c$. How does it depend on the system size L?

- We have

$$\left\langle e^{i(\theta_\mathbf{x}-\theta_\mathbf{0})} \right\rangle = \exp\left\{ -\int \frac{d^dq}{(2\pi)^d} \frac{1-\cos(\mathbf{q}\cdot\mathbf{x})}{K(\mathbf{q})} \right\},$$

and, similarly,

$$\left\langle e^{i\theta_\mathbf{x}} \right\rangle = \exp\left\{ -\int \frac{d^dq}{(2\pi)^d} \frac{1}{2K(\mathbf{q})} \right\}.$$

Hence the connected correlation function

$$\langle \vec{s}_\mathbf{x} \cdot \vec{s}_\mathbf{0} \rangle_c = \left\langle e^{i(\theta_\mathbf{x}-\theta_\mathbf{0})} \right\rangle_c = \left\langle e^{i(\theta_\mathbf{x}-\theta_\mathbf{0})} \right\rangle - \left\langle e^{i\theta_\mathbf{x}} \right\rangle \left\langle e^{i\theta_\mathbf{0}} \right\rangle,$$

is given by

$$\langle \vec{s}_\mathbf{x} \cdot \vec{s}_\mathbf{0} \rangle_c = e^{-\int \frac{d^dq}{(2\pi)^d} \frac{1}{K(\mathbf{q})}} \left\{ \exp\left[\int \frac{d^dq}{(2\pi)^d} \frac{\cos(\mathbf{q}\cdot\mathbf{x})}{K(\mathbf{q})} \right] - 1 \right\}.$$

In $d \geq 3$, the x dependent integral vanishes at $x \to \infty$. We can thus expand its exponential, for large x, obtaining

$$\langle \vec{s}_\mathbf{x} \cdot \vec{s}_\mathbf{0} \rangle_c \sim \int \frac{d^dq}{(2\pi)^d} \frac{\cos(\mathbf{q}\cdot\mathbf{x})}{K(\mathbf{q})} \approx \int \frac{d^dq}{(2\pi)^d} \frac{\cos(\mathbf{q}\cdot\mathbf{x})}{Kq^2} = \frac{1}{K} C_d(x) \sim \frac{1}{K|x|^{d-2}}.$$

Thus, the transverse susceptibility diverges as

$$\chi_t \propto \int d^dx \, \langle \vec{s}_\mathbf{x} \cdot \vec{s}_\mathbf{0} \rangle_c \sim \frac{L^2}{K}.$$

(h) In $d = 2$, show that χ_t only diverges for K larger than a critical value $K_c = 1/(4\pi)$.

- In $d = 2$, there is no long range order, $\langle \vec{s}_\mathbf{x} \rangle = 0$, and

$$\langle \vec{s}_\mathbf{x} \cdot \vec{s}_\mathbf{0} \rangle_c = \langle \vec{s}_\mathbf{x} \cdot \vec{s}_\mathbf{0} \rangle \sim |x|^{-1/(2\pi K)}.$$

The susceptibility

$$\chi_t \sim \int^L d^2x \, |x|^{-1/(2\pi K)},$$

thus converges for $1/(2\pi K) > 2$, for K below $K_c = 1/(4\pi)$. For $K > K_c$, the susceptibility diverges as

$$\chi_t \sim L^{2-2K_c/K}.$$

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2. *Capillary waves:* A reasonably flat surface in d–dimensions can be described by its height h, as a function of the remaining $(d-1)$ coordinates $\mathbf{x} = (x_1, ...x_{d-1})$. Convince yourself that the generalized "area" is given by $\mathcal{A} = \int d^{d-1}\mathbf{x}\sqrt{1+(\nabla h)^2}$. With a surface tension σ, the Hamiltonian is simply $\mathcal{H} = \sigma\mathcal{A}$.

(a) At sufficiently low temperatures, there are only slow variations in h. Expand the energy to quadratic order, and write down the partition function as a functional integral.

• For a surface parametrized by the height function

$$x_d = h\left(x_1, \ldots, x_{d-1}\right),$$

an area element can be calculated as

$$dA = \frac{1}{\cos\alpha} dx_1 \cdots dx_{d-1},$$

where α is the angle between the d^{th} direction and the normal

$$\vec{n} = \frac{1}{\sqrt{1+(\nabla h)^2}}\left(-\frac{\partial h}{\partial x_1}, \ldots, -\frac{\partial h}{\partial x_{d-1}}, 1\right)$$

to the surface ($n^2 = 1$). Since, $\cos\alpha = n_d = \left[1+(\nabla h)^2\right]^{-1/2} \approx 1 - \frac{1}{2}(\nabla h)^2$, we obtain

$$\mathcal{H} = \sigma\mathcal{A} \approx \sigma \int d^{d-1}x \left\{1 + \frac{1}{2}(\nabla h)^2\right\},$$

and, dropping a multiplicative constant,

$$Z = \int \mathcal{D}h(\mathbf{x}) \exp\left\{-\beta\frac{\sigma}{2}\int d^{d-1}x\,(\nabla h)^2\right\}.$$

(b) Use Fourier transformation to diagonalize the quadratic Hamiltonian into its normal modes $\{h_{\mathbf{q}}\}$ (capillary waves).

• After changing variables to the Fourier modes,

$$h(\mathbf{x}) = \int \frac{d^{d-1}q}{(2\pi)^{d-1}} h(\mathbf{q})\, e^{i\mathbf{q}\cdot\mathbf{x}},$$

the partition function is given by

$$Z=\int \mathcal{D}h\left(\mathbf{q}\right)\exp\left\{-\beta\frac{\sigma}{2}\int\frac{d^{d-1}q}{\left(2\pi\right)^{d-1}}q^{2}\left|h\left(\mathbf{q}\right)\right|^{2}\right\}.$$

(c) What symmetry breaking is responsible for these Goldstone modes?

• By selecting a particular height, the ground state breaks the translation symmetry in the d^{th} direction. The transformation $h\left(\mathbf{x}\right)\to h\left(\mathbf{x}\right)+\xi\left(\mathbf{x}\right)$ leaves the energy unchanged if $\xi\left(\mathbf{x}\right)$ is constant. By continuity, we can have an arbitrarily small change in the energy by varying $\xi\left(\mathbf{x}\right)$ arbitrarily slowly.

(d) Calculate the height–height correlations $\langle\left(h(\mathbf{x})-h(\mathbf{x}')\right)^{2}\rangle$.

• From

$$h\left(\mathbf{x}\right)-h\left(\mathbf{x}'\right)=\int\frac{d^{d-1}q}{\left(2\pi\right)^{d-1}}h\left(\mathbf{q}\right)\left(e^{i\mathbf{q}\cdot\mathbf{x}}-e^{i\mathbf{q}\cdot\mathbf{x}'}\right),$$

we obtain

$$\left\langle\left(h\left(\mathbf{x}\right)-h\left(\mathbf{x}'\right)\right)^{2}\right\rangle=\int\frac{d^{d-1}q}{\left(2\pi\right)^{d-1}}\frac{d^{d-1}q'}{\left(2\pi\right)^{d-1}}\left\langle h\left(\mathbf{q}\right)h\left(\mathbf{q}'\right)\right\rangle\left(e^{i\mathbf{q}\cdot\mathbf{x}}-e^{i\mathbf{q}\cdot\mathbf{x}'}\right)\left(e^{i\mathbf{q}'\cdot\mathbf{x}}-e^{i\mathbf{q}'\cdot\mathbf{x}'}\right).$$

The height-height correlations thus behave as

$$\begin{aligned}G\left(\mathbf{x}-\mathbf{x}'\right)&\equiv\left\langle\left(h\left(\mathbf{x}\right)-h\left(\mathbf{x}'\right)\right)^{2}\right\rangle\\&=\frac{2}{\beta\sigma}\int\frac{d^{d-1}q}{\left(2\pi\right)^{d-1}}\frac{1-\cos\left[\mathbf{q}\cdot\left(\mathbf{x}-\mathbf{x}'\right)\right]}{q^{2}}=\frac{2}{\beta\sigma}C_{d-1}\left(\mathbf{x}-\mathbf{x}'\right).\end{aligned}$$

(e) Comment on the form of the result (d) in dimensions $d=4$, 3, 2, and 1.

• We can now discuss the asymptotic behavior of the Coulomb Kernel for large $|\mathbf{x}-\mathbf{x}'|$, either using the results from problem 1(f), or the exact form given in lectures.

• In $d\geq 4$, $G\left(\mathbf{x}-\mathbf{x}'\right)\to$ constant, and the surface is *flat*.

• In $d=3$, $G\left(\mathbf{x}-\mathbf{x}'\right)\sim\ln|\mathbf{x}-\mathbf{x}'|$, and we come to the surprising conclusion that there are no asymptotically flat surfaces in three dimensions. While this is technically correct, since the logarithm grows slowly, very large surfaces are needed to detect appreciable fluctuations.

• In $d=2$, $G\left(\mathbf{x}-\mathbf{x}'\right)\sim|\mathbf{x}-\mathbf{x}'|$. This is easy to comprehend, once we realize that the interface $h(x)$ is similar to the path $x(t)$ of a random walker, and has similar ($x\sim\sqrt{t}$) fluctuations.

• In $d=1$, $G\left(\mathbf{x}-\mathbf{x}'\right)\sim|\mathbf{x}-\mathbf{x}'|^{2}$. The transverse fluctuation of the ‘point’ interface are very big, and the approximations break down as discussed next.

(f) By estimating typical values of ∇h, comment on when it is justified to ignore higher order terms in the expansion for $\mathcal{A}$.

• We can estimate $(\nabla h)^2$ as

$$\frac{\left\langle \left(h\left(\mathbf{x}\right)-h\left(\mathbf{x}'\right)\right)^2\right\rangle}{\left(\mathbf{x}-\mathbf{x}'\right)^2} \propto \left|\mathbf{x}-\mathbf{x}'\right|^{1-d}.$$

For dimensions $d \geq d_\ell = 1$, the typical size of the gradient decreases upon coarse-graining. The gradient expansion of the area used before is then justified. For dimensions $d \leq d_\ell$, the whole idea of the gradient expansion fails to be sensible.

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3. *Gauge fluctuations in superconductors:* The Landau–Ginzburg model of superconductivity describes a complex superconducting order parameter $\Psi(\mathbf{x}) = \Psi_1(\mathbf{x}) + i\Psi_2(\mathbf{x})$, and the electromagnetic vector potential $\vec{A}(\mathbf{x})$, which are subject to a Hamiltonian

$$\beta\mathcal{H} = \int d^3\mathbf{x}\left[\frac{t}{2}|\Psi|^2 + u|\Psi|^4 + \frac{K}{2}D_\mu\Psi D^*_\mu\Psi^* + \frac{L}{2}\left(\nabla\times A\right)^2\right].$$

The gauge-invariant derivative $D_\mu \equiv \partial_\mu - ieA_\mu(\mathbf{x})$, introduces the coupling between the two fields. (In terms of Cooper pair parameters, $e = e^*c/\hbar$, $K = \hbar^2/2m^*$.)

(a) Show that the above Hamiltonian is invariant under the *local gauge symmetry*:

$$\Psi(\mathbf{x}) \mapsto \Psi(x)\exp\left(i\theta(\mathbf{x})\right), \qquad \text{and} \qquad A_\mu(\mathbf{x}) \mapsto A_\mu(\mathbf{x}) + \frac{1}{e}\partial_\mu\theta.$$

• Under a local gauge transformation, $\beta\mathcal{H} \longmapsto$

$$\int d^3x\left\{\frac{t}{2}\left|\Psi\right|^2 + u\left|\Psi\right|^4 + \frac{K}{2}\left[\left(\partial_\mu - ieA_\mu - i\partial_\mu\theta\right)\Psi e^{i\theta}\right]\left[\left(\partial_\mu + ieA_\mu + i\partial_\mu\theta\right)\Psi^* e^{-i\theta}\right]\right.$$
$$\left. + \frac{L}{2}\left(\nabla\times\vec{A} + \nabla\times\frac{1}{e}\nabla\theta\right)^2\right\}.$$

But this is none other than $\beta\mathcal{H}$ again, since

$$\left(\partial_\mu - ieA_\mu - i\partial_\mu\theta\right)\Psi e^{i\theta} = e^{i\theta}\left(\partial_\mu - ieA_\mu\right)\Psi = e^{i\theta}D_\mu\Psi,$$

and

$$\nabla\times\frac{1}{e}\nabla\theta = 0.$$

(b) Show that there is a saddle point solution of the form $\Psi(\mathbf{x}) = \overline{\Psi}$, and $\vec{A}(\mathbf{x}) = 0$, and find $\overline{\Psi}$ for $t > 0$ and $t < 0$.

• The saddle point solutions are obtained from

$$\frac{\delta \mathcal{H}}{\delta \Psi^*} = 0, \quad \Longrightarrow \quad \frac{t}{2}\Psi + 2u\Psi\left|\Psi\right|^2 - \frac{K}{2}D_\mu D_\mu^* \Psi = 0,$$

and

$$\frac{\delta \mathcal{H}}{\delta A_\mu} = 0, \quad \Longrightarrow \quad \frac{K}{2}\left(-ie\Psi D_\mu^* \Psi^* + ie\Psi^* D_\mu \Psi\right) - L\epsilon_{\alpha\beta\mu}\epsilon_{\alpha\gamma\delta}\partial_\beta\partial_\gamma A_\delta = 0.$$

The ansatz $\Psi\left(\mathbf{x}\right) = \overline{\Psi}$, $\vec{A} = 0$, clearly solves these equations. The first equation then becomes

$$t\overline{\Psi} + 4u\overline{\Psi}\left|\overline{\Psi}\right|^2 = 0,$$

yielding (for $u > 0$) $\overline{\Psi} = 0$ for $t > 0$, whereas $\left|\overline{\Psi}\right|^2 = -t/4u$ for $t < 0$.

(c) For $t < 0$, calculate the cost of fluctuations by setting

$$\begin{cases} \Psi(\mathbf{x}) = \left(\overline{\Psi} + \phi(\mathbf{x})\right)\exp\left(i\theta(\mathbf{x})\right), \\ A_\mu(\mathbf{x}) = a_\mu(\mathbf{x}), \quad \text{(with } \partial_\mu a_\mu = 0 \text{ in the Coulomb gauge)} \end{cases}$$

and expanding $\beta\mathcal{H}$ to quadratic order in ϕ, θ, and $\vec{a}$.

• For simplicity, let us choose $\overline{\Psi}$ to be real. From the Hamiltonian term

$$D_\mu \Psi D_\mu^* \Psi^* = \left[\left(\partial_\mu - iea_\mu\right)\left(\overline{\Psi} + \phi\right)e^{i\theta}\right]\left[\left(\partial_\mu + iea_\mu\right)\left(\overline{\Psi} + \phi\right)e^{-i\theta}\right],$$

we get the following quadratic contribution

$$\overline{\Psi}^2\left(\nabla\theta\right)^2 + \left(\nabla\phi\right)^2 - 2e\overline{\Psi}^2 a_\mu \partial_\mu \theta + e^2\overline{\Psi}^2\left|\vec{a}\right|^2.$$

The third term in the above expression integrates to zero (as it can be seen by integrating by parts and invoking the Coulomb gauge condition $\partial_\mu a_\mu = 0$). Thus, the quadratic terms read

$$\beta\mathcal{H}^{(2)} = \int d^3x \Bigg\{ \left(\frac{t}{2} + 6u\overline{\Psi}^2\right)\phi^2 + \frac{K}{2}\left(\nabla\phi\right)^2 + \frac{K}{2}\overline{\Psi}^2\left(\nabla\theta\right)^2 + \frac{K}{2}e^2\overline{\Psi}^2\left|\vec{a}\right|^2 + \frac{L}{2}\left(\nabla\times\vec{a}\right)^2 \Bigg\}.$$

(d) Perform a Fourier transformation, and calculate the expectation values of $\left\langle|\phi(\mathbf{q})|^2\right\rangle$, $\left\langle|\theta(\mathbf{q})|^2\right\rangle$, and $\left\langle|\vec{a}(\mathbf{q})|^2\right\rangle$.

• In terms of Fourier transforms, we obtain

$$\beta\mathcal{H}^{(2)} = \sum_{\mathbf{q}} \left\{ \left(\frac{t}{2} + 6u\overline{\Psi}^2 + \frac{K}{2}q^2 \right) |\phi(\mathbf{q})|^2 + \frac{K}{2}\overline{\Psi}^2 q^2 |\theta(\mathbf{q})|^2 \right.$$
$$\left. + \frac{K}{2}e^2\overline{\Psi}^2 |\vec{a}(\mathbf{q})|^2 + \frac{L}{2}(\mathbf{q}\times\vec{a})^2 \right\}.$$

In the Coulomb gauge, $\mathbf{q} \perp \vec{a}(\mathbf{q})$, and so $[\mathbf{q}\times\vec{a}(\mathbf{q})]^2 = q^2 |\vec{a}(\mathbf{q})|^2$. This diagonal form then yields immediately (for $t<0$)

$$\left\langle |\phi(\mathbf{q})|^2 \right\rangle = \left(t + 12u\overline{\Psi}^2 + Kq^2 \right)^{-1} = \frac{1}{Kq^2 - 2t},$$
$$\left\langle |\theta(\mathbf{q})|^2 \right\rangle = \left(K\overline{\Psi}^2 q^2 \right)^{-1} = -\frac{4u}{Ktq^2},$$
$$\left\langle |\vec{a}(\mathbf{q})|^2 \right\rangle = 2\left(Ke^2\overline{\Psi}^2 + Lq^2 \right)^{-1} = \frac{2}{Lq^2 - Ke^2t/4u} \qquad (\vec{a} \text{ has 2 components}).$$

Note that the gauge field, "mass-less" in the original theory, acquires a "mass" $Ke^2t/4u$ through its coupling to the order parameter. This is known as the Higgs mechanism.

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4. *Fluctuations around a tricritical point:* As shown in a previous problem, the Hamiltonian

$$\beta\mathcal{H} = \int d^d\mathbf{x} \left[\frac{K}{2}(\nabla m)^2 + \frac{t}{2}m^2 + um^4 + vm^6 \right],$$

with $u=0$ and $v>0$ describes a tricritical point.

(a) Calculate the heat capacity singularity as $t \to 0$ by the saddle point approximation.

• As already calculated in a previous problem, the saddle point minimum of the free energy $\vec{m} = \overline{m}\hat{e}_\ell$, can be obtained from

$$\left.\frac{\partial\Psi}{\partial m}\right|_{\overline{m}} = \overline{m}\left(t + 6v\overline{m}^4 \right) = 0,$$

yielding,

$$\overline{m} = \begin{cases} 0 & \text{for} \quad t > \overline{t} = 0 \\ \left(-\dfrac{t}{6v} \right)^{1/4} & \text{for} \quad t < 0 \end{cases}.$$

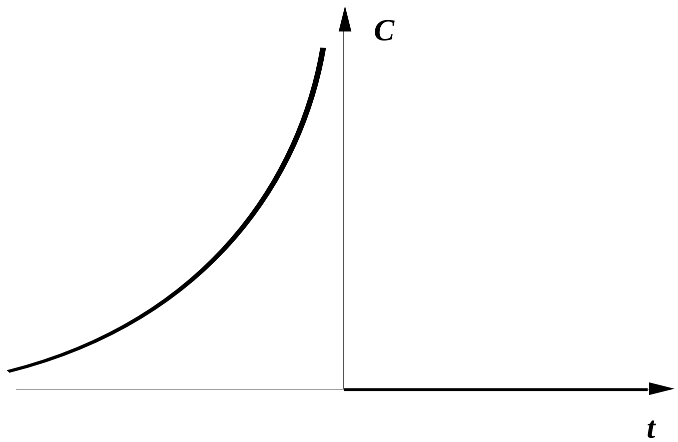
The corresponding free energy density equals to

$$\Psi(\overline{m}) = \frac{t}{2}\overline{m}^2 + v\overline{m}^6 = \begin{cases} 0 & \text{for} \quad t > 0 \\ -\dfrac{1}{3}\dfrac{(-t)^{3/2}}{(6v)^{1/2}} & \text{for} \quad t < 0 \end{cases}.$$

Therefore, the singular behavior of the heat capacity is given by

$$C = C_{s.p.} \sim -T_c \left.\frac{\partial^2 \Psi}{\partial t^2}\right|_{\overline{m}} = \begin{cases} 0 & \text{for} \quad t > 0 \\ \dfrac{T_c}{4}(-6vt)^{-1/2} & \text{for} \quad t < 0 \end{cases},$$

as sketched in the figure below.



(b) Include both longitudinal and transverse fluctuations by setting

$$\vec{m}(\mathbf{x}) = \big(\overline{m} + \phi_\ell(\mathbf{x})\big)\hat{e}_\ell + \sum_{\alpha=2}^{n} \phi_t^\alpha(\mathbf{x})\hat{e}_\alpha,$$

and expanding $\beta\mathcal{H}$ to quadratic order in ϕ.

- Let us now include both longitudinal and transversal fluctuations by setting

$$\vec{m}(\mathbf{x}) = (\overline{m} + \phi_\ell(\mathbf{x}))\hat{e}_\ell + \sum_{\alpha=2}^{n} \phi_t^\alpha(\mathbf{x})\hat{e}_\alpha,$$

where $\hat{e}_\ell$ and $\hat{e}_\alpha$ form an orthonormal set of n vectors. Consequently, the free energy $\beta\mathcal{H}$ is a function of ϕ_ℓ and ϕ_t. Since $\overline{m}\hat{e}_\ell$ is a minimum, there are no linear terms in the expansion of $\beta\mathcal{H}$ in ϕ. The contributions of each factor in the free energy to the quadratic term in the expansion are

$$(\nabla\vec{m})^2 \Longrightarrow (\nabla\phi_\ell)^2 + \sum_{\alpha=2}^{n}(\nabla\phi_t^\alpha)^2,$$

$$(\vec{m})^2 \Longrightarrow (\phi_\ell)^2 + \sum_{\alpha=2}^{n}(\phi_t^\alpha)^2,$$

$$(\vec{m})^6 = ((\vec{m})^2)^3 = (\overline{m}^2 + 2\overline{m}\phi_\ell + \phi_\ell^2 + \sum_{\alpha=2}^{n}(\phi_t^\alpha)^2)^3 \Longrightarrow 15\overline{m}^4(\phi_\ell)^2 + 3\overline{m}^4\sum_{\alpha=2}^{n}(\phi_t^\alpha)^2.$$

The expansion of $\beta\mathcal{H}$ to second order now gives

$$\beta\mathcal{H}(\phi_\ell, \phi_t^\alpha) = \beta\mathcal{H}(0,0) + \int d^d\mathbf{x} \left\{ \left[\frac{K}{2}(\nabla\phi_\ell)^2 + \frac{\phi_\ell^2}{2}\left(t + 30v\overline{m}^4\right) \right] \right. \\ \left. + \sum_{\alpha=2}^{n} \left[\frac{K}{2}(\nabla\phi_t^\alpha)^2 + \frac{(\phi_t^\alpha)^2}{2}\left(t + 6v\overline{m}^4\right) \right] \right\}.$$

We can formally rewrite it as

$$\beta\mathcal{H}(\phi_\ell, \phi_t^\alpha) = \beta\mathcal{H}(0,0) + \beta\mathcal{H}_\ell(\phi_\ell) + \sum_{\alpha=2}^{n} \beta\mathcal{H}_{t_\alpha}(\phi_t^\alpha),$$

where $\beta\mathcal{H}_i(\phi_i)$, with $i = \ell, t_\alpha$, is in general given by

$$\beta\mathcal{H}_i(\phi_i) = \frac{K}{2}\int d^d\mathbf{x} \left[(\nabla\phi_i)^2 + \frac{\phi_i^2}{\xi_i^2} \right],$$

with the inverse correlation lengths

$$\xi_\ell^{-2} = \begin{cases} \dfrac{t}{K} & \text{for} \quad t > 0 \\ \dfrac{-4t}{K} & \text{for} \quad t < 0 \end{cases},$$

and

$$\xi_{t_\alpha}^{-2} = \begin{cases} \dfrac{t}{K} & \text{for} \quad t > 0 \\ 0 & \text{for} \quad t < 0 \end{cases}.$$

As shown in the lectures for the critical point of a magnet, for $t > 0$ there is no difference between longitudinal and transverse components, whereas for $t < 0$, there is no restoring force for the Goldstone modes ϕ_t^α due to the rotational symmetry of the *ordered* state.

(c) Calculate the longitudinal and transverse correlation functions.

• Since in the harmonic approximation $\beta\mathcal{H}$ turns out to be a sum of the Hamiltonians of the different fluctuating components ϕ_ℓ, ϕ_t^α, these quantities are independent of each other, i.e.

$$\langle \phi_\ell \phi_t^\alpha \rangle = 0, \qquad \text{and} \qquad \langle \phi_t^\gamma \phi_t^\alpha \rangle = 0 \quad \text{for } \alpha \neq \gamma.$$

To determine the longitudinal and transverse correlation functions, we first express the free energy in terms of Fourier modes, so that the probability of a particular fluctuation configuration is given by

$$\mathcal{P}(\{\phi_\ell, \phi_t^\alpha\}) \propto \prod_{\mathbf{q},\alpha} \exp\left\{ -\frac{K}{2}\left(q^2 + \xi_\ell^{-2}\right)|\phi_{\ell,\mathbf{q}}|^2 \right\} \cdot \exp\left\{ -\frac{K}{2}\left(q^2 + \xi_{t_\alpha}^{-2}\right)|\phi_{t,\mathbf{q}}^\alpha|^2 \right\}.$$

Thus, as it was also shown in the lectures, the correlation function is

$$\langle \phi_\alpha(\mathbf{x})\phi_\beta(0)\rangle = \frac{\delta_{\alpha,\beta}}{VK}\sum_{\mathbf{q}} \frac{e^{i\mathbf{q}\cdot\mathbf{x}}}{\left(q^2+\xi_\alpha^{-2}\right)} = -\frac{\delta_{\alpha,\beta}}{K} I_d(\mathbf{x}, \xi_\alpha),$$

therefore,

$$\langle \phi_\ell(\mathbf{x})\phi_\ell(0)\rangle = -\frac{1}{K} I_d(\mathbf{x}, \xi_\ell),$$

and

$$\langle \phi_t^\alpha(\mathbf{x})\phi_t^\beta(0)\rangle = -\frac{\delta_{\alpha,\beta}}{K} I_d(\mathbf{x}, \xi_{t_\alpha}).$$

(d) Compute the first correction to the saddle point free energy from fluctuations.

• Let us calculate the first correction to the saddle point free energy from fluctuations. The partition function is

$$\begin{aligned} Z &= e^{-\beta\mathcal{H}(0,0)} \int \mathcal{D}\phi(\mathbf{x}) \, \exp\left\{-\frac{K}{2}\int d^d\mathbf{x}\left[(\nabla\phi)^2 + \xi^{-2}\phi^2\right]\right\} \\ &= e^{-\beta\mathcal{H}(0,0)} \int \prod_{\mathbf{q}} d\phi_{\mathbf{q}} \, \exp\left\{-\frac{K}{2}\sum_{\mathbf{q}}\left(q^2+\xi^{-2}\right)\phi_{\mathbf{q}}\phi_{\mathbf{q}}^*\right\} \\ &= \prod_{\mathbf{q}}\left[K\left(q^2+\xi^{-2}\right)\right]^{-1/2} = \exp\left\{-\frac{1}{2}\sum_{\mathbf{q}}\ln\left(Kq^2+K\xi^{-2}\right)\right\} \end{aligned},$$

and the free energy density equals to

$$\beta f = \frac{\beta\mathcal{H}(0,0)}{V} + \begin{cases} \frac{n}{2}\int \frac{d^d\mathbf{q}}{(2\pi)^d}\ln\left(Kq^2+t\right) & \text{for} \quad t>0 \\ \frac{1}{2}\int \frac{d^d\mathbf{q}}{(2\pi)^d}\ln\left(Kq^2-4t\right) + \frac{n-1}{2}\int\frac{d^d\mathbf{q}}{(2\pi)^d}\ln\left(Kq^2\right) & \text{for} \quad t<0 \end{cases}.$$

Note that the first term is the saddle point free energy, and that there are n contributions to the free energy from fluctuations.

(e) Find the fluctuation correction to the heat capacity.

• As $C = -T(d^2f/dT^2)$, the fluctuation corrections to the heat capacity are given by

$$C - C_{s.p.} \propto \begin{cases} \frac{n}{2}\int\frac{d^d\mathbf{q}}{(2\pi)^d}\left(Kq^2+t\right)^{-2} & \text{for} \quad t>0 \\ \frac{16}{2}\int\frac{d^d\mathbf{q}}{(2\pi)^d}\left(Kq^2-4t\right)^{-2} & \text{for} \quad t<0 \end{cases}.$$

These integrals change behavior at $d = 4$. For $d > 4$, the integrals diverge at large $\mathbf{q}$, and are dominated by the upper cutoff $\Delta \simeq 1/a$. That is why fluctuation corrections to the heat capacity add just a constant term on each side of the transition, and the saddle point solution keeps its qualitative form. On the other hand, for $d < 4$, the integrals are proportional to the corresponding correlation length ξ^{4-d}. Due to the divergence of ξ, the fluctuation corrections diverge as

$$C_{fl.} = C - C_{s.p.} \propto K^{-d/2}|t|^{d/2-2}.$$

(f) By comparing the results from parts (a) and (e) *for* $t < 0$ obtain a Ginzburg criterion, and the upper critical dimension for validity of mean–field theory at a tricritical point.

• To obtain a Ginzburg criterion, let us consider $t < 0$. In this region, the saddle point contribution already diverges as $C_{s.p.} \propto (-vt)^{-1/2}$, so that

$$\frac{C_{fl.}}{C_{s.p.}} \propto (-t)^{\frac{d-3}{2}} \left(\frac{v}{K^d}\right)^{1/2}.$$

Therefore at $t < 0$, the saddle point contribution dominates the behavior of this ratio provided that $d > 3$. For $d < 3$, the mean field result will continue being dominant far enough from the critical point, i.e. if

$$(-t)^{d-3} \gg \left(\frac{K^d}{v}\right), \qquad \text{or} \qquad |t| \gg \left(\frac{K^d}{v}\right)^{1/(d-3)}.$$

Otherwise, i.e. if

$$|t| < \left(\frac{K^d}{v}\right)^{1/(d-3)},$$

the fluctuation contribution to the heat capacity becomes dominant. The upper critical dimension for the tricritical point is then $d = 3$.

(g) A generalized multicritical point is described by replacing the term vm^6 with $u_{2n}m^{2n}$. Use simple power counting to find the upper critical dimension of this multicritical point.

• If instead of the term vm^6 we have a general factor of the form $u_{2n}m^{2n}$, we can easily generalize our results to

$$\overline{m} \propto (-t)^{1/(2n-2)}, \qquad \Psi(\overline{m}) \propto (-t)^{n/(n-1)}, \qquad C_{s.p.} \propto (-t)^{n/(n-1)-2}.$$

Moreover, the fluctuation correction to the heat capacity for any value of n is the same as before

$$C_{fl.} \propto (-t)^{d/2-2}.$$

Hence the upper critical dimension is, in general, determined by the equation

$$\frac{d}{2} - 2 = \frac{n}{n-1} - 2, \qquad \text{or} \qquad d_u = \frac{2n}{n-1}.$$

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