Problems & Solutions

1. *Migdal–Kadanoff method:* Consider Potts spins $s_i = (1, 2, \cdots, q)$, on sites i of a hypercubic lattice, interacting with their nearest neighbors via a Hamiltonian

$$-\beta\mathcal{H} = K \sum_{<ij>} \delta_{s_i, s_j} \ .$$

(a) In $d = 1$ find the exact recursion relations by a $b = 2$ renormalization/decimation process. Indentify all fixed points and note their stability.

• In $d = 1$, if we average over the q possible values of s_1, we obtain

$$\sum_{s_1=1}^{q} e^{K(\delta_{\sigma_1 s_1} + \delta_{s_1 \sigma_2})} = \begin{cases} q - 1 + e^{2K} & \text{if} \quad \sigma_1 = \sigma_2 \\ q - 2 + 2e^{K} & \text{if} \quad \sigma_1 \neq \sigma_2 \end{cases} = e^{g' + K' \delta_{\sigma_1 \sigma_2}},$$

from which we arrive at the exact recursion relations:

$$e^{K'} = \frac{q - 1 + e^{2K}}{q - 2 + 2e^{K}}, \qquad e^{g'} = q - 2 + 2e^{K}.$$

To find the fixed points we set $K' = K = K^*$. As in the previous problem, let us introduce the variable $x = e^{K^*}$. Hence, we have to solve the equation

$$x = \frac{q - 1 + x^2}{q - 2 + 2x}, \qquad \text{or} \qquad x^2 + (q - 2)x - (q - 1) = 0,$$

whose only meaningful solution is $x = 1$, resulting in $K^* = 0$. To check its stability, we consider $K \ll 1$, so that

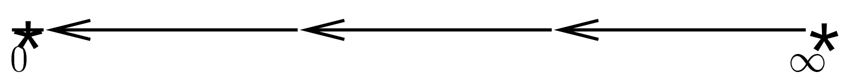
$$K' \simeq \ln\left(\frac{q + 2K + 2K^2}{q + 2K + K^2}\right) \simeq \frac{K^2}{q} \ll K,$$

which indicates that this fixed point is stable.

In addition, $K^* \to \infty$ is also a fixed point. If we consider $K \gg 1$,

$$e^{K'} \simeq \frac{1}{2} e^{K}, \quad \Longrightarrow \quad K' = K - \ln 2 < K,$$

which implies that this fixed point is unstable.



(b) Write down the recursion relation $K'(K)$ in d–dimensions for $b = 2$, using the Migdal–Kadanoff bond moving scheme.

• In the Migdal-Kadanoff approximation, moving bonds strengthens the remaining bonds by a factor 2^{d-1}. Therefore, in the decimated lattice we have

$$e^{K'} = \frac{q - 1 + e^{2\times 2^{d-1}K}}{q - 2 + 2e^{2^{d-1}K}}.$$

(c) By considering the stability of the fixed points at zero and infinite coupling, prove the existence of a non–trivial fixed point at finite K^* for $d > 1$.

• In the vicinity of the fixed point $K^* = 0$, i.e. for $K \ll 1$,

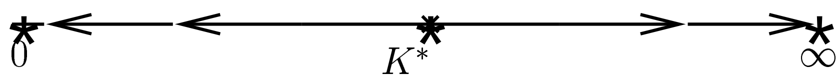
$$K' \simeq \frac{2^{2d-2}K^2}{q} \ll K,$$

and consequently, this point is again stable. However, for $K^* \to \infty$, we have

$$e^{K'} \simeq \frac{1}{2}\exp\left[\left(2^d - 2^{d-1}\right)K\right], \quad \Longrightarrow \quad K' = 2^{d-1}K - \ln 2 \gg K,$$

which implies that this fixed point is now stable provided that $d > 1$.

• As a result, there must be a finite K^* fixed point, which separates the flows to the other fixed points.



(d) For $d = 2$, obtain K^* and y_t, for $q = 3$, 1, and 0.

• Let us now discuss a few particular cases in $d = 2$. For instance, if we consider $q = 3$, the non-trivial fixed point is a solution of the equation

$$x = \frac{2 + x^4}{1 + 2x^2}, \qquad \text{or} \qquad x^4 - 2x^3 - x + 2 = (x-2)(x^3 - 1) = 0,$$

which clearly yields a non-trivial fixed point at $K^* = \ln 2 \simeq 0.69$. The thermal exponent for this point

$$\left.\frac{\partial K'}{\partial K}\right|_{K^*} = 2^{y_t} = 2\left[\frac{e^{4K^*}}{e^{4K^*} + 2} - \frac{e^{2K^*}}{1 + 2e^{2K^*}}\right] = \frac{16}{9}, \quad \Longrightarrow \quad y_t \simeq 0.83,$$

which can be compared to the exact values, $K^* = 1.005$, and $y_t = 1.2$.

By analytic continuation for $q \to 1$, we obtain

$$e^{K'} = \frac{e^{4K}}{-1 + 2e^{2K}}.$$

The non-trivial fixed point is a solution of the equation

$$x = \frac{x^4}{-1+2x^2}, \qquad \text{or} \qquad (x^3 - 2x^2 + 1) = (x-1)(x^2 - x - 1) = 0,$$

whose only non-trivial solution is $x = (1+\sqrt{5})/2 = 1.62$, resulting in $K^* = 0.48$. The thermal exponent for this point

$$\left.\frac{\partial K'}{\partial K}\right|_{K^*} = 2^{y_t} = 4\left[1 - e^{-K^*}\right], \qquad \Longrightarrow \qquad y_t \simeq 0.61.$$

As discussed in the next problem set, the Potts model for $q \to 1$ can be mapped onto the problem of *bond percolation*, which despite being a purely geometrical phenomenon, shows many features completely analogous to those of a continuous thermal phase transition.

And finally for $q \to 0$, relevant to *lattice animals* (see PS#9), we obtain

$$e^{K'} = \frac{-1+e^{4K}}{-2+2e^{2K}},$$

for which we have to solve the equation

$$x = \frac{-1+x^4}{-2+2x^2}, \qquad \text{or} \qquad x^4 - 2x^3 + 2x - 1 = (x-1)^3(x+1) = 0,$$

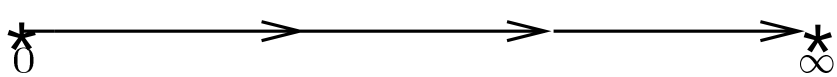
whose only finite solution is the trivial one, $x = 1$. For $q \to 0$, if $K \ll 1$, we obtain

$$K' \simeq K + \frac{K^2}{2} > K,$$

indicating that this fixed point is now unstable. Note that the first correction only indicates marginal stability ($y_t = 0$). Nevertheless, for $K^* \to \infty$, we have

$$e^{K'} \simeq \frac{1}{2}\exp[2K], \quad \Longrightarrow \quad K' = 2K - \ln 2 \gg K,$$

which implies that this fixed point is stable.



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2. *The Potts model:* The *transfer matrix* procedure can be extended to Potts model, where the spin s_i on each site takes q values $s_i = (1, 2, \cdots, q)$; and the Hamiltonian is $-\beta\mathcal{H} = K\sum_{i=1}^{N}\delta_{s_i,s_{i+1}} + K\delta_{s_N,s_1}$.

(a) Write down the transfer matrix and diagonalize it. Note that you do not have to solve a q^{th} order secular equation as it is easy to guess the eigenvectors from the symmetry of the matrix.

• The partition function is

$$Z = \sum_{\{s_i\}} < s_1|T|s_2 >< s_2|T|s_3 > \cdots < s_{N-1}|T|s_N >< s_N|T|s_1 >= \mathrm{tr}(T^N),$$

where $< s_i|T|s_j >= \exp\left(K\delta_{s_i,s_j}\right)$ is a $q \times q$ transfer matrix. The diagonal elements of the matrix are e^K, while the off-diagonal elements are unity. The eigenvectors of the matrix are easily found by inspection. There is one eigenvectors with all elements equal; the corresponding eigenvalue is $\lambda_1 = e^K + q - 1$. There are also $(q-1)$ eigenvectors orthogonal to the first, i.e. the sum of whose elements is zero. This corresponding eigenvalues are degenerate and equal to $e^K - 1$. Thus

$$Z = \sum_\alpha \lambda_\alpha^N = \left(e^K + q - 1\right)^N + (q-1)\left(e^K - 1\right)^N .$$

(b) Calculate the free energy per site.

• Since the largest eigenvalue dominates for $N \gg 1$,

$$\frac{\ln Z}{N} = \ln\left(e^K + q - 1\right).$$

(c) Give the expression for the correlation length ξ (you don't need to provide a detailed derivation), and discuss its behavior as $T = 1/K \to 0$.

• Correlations decay as the ratio of the eigenvalues to the power of the separation. Hence the correlation length is

$$\xi = \left[\ln\left(\frac{\lambda_1}{\lambda_2}\right)\right]^{-1} = \left[\ln\left(\frac{e^K + q - 1}{e^K - 1}\right)\right]^{-1}.$$

In the limit of $K \to \infty$, expanding the above result gives

$$\xi \simeq \frac{e^K}{q} = \frac{1}{q}\exp\left(\frac{1}{T}\right).$$

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3. *Transition probability matrix:* Consider a system of two Ising spins with a coupling K, which can thus be in one of four states.

(a) Explicitly write the 4×4 transition matrix corresponding to single spin flips for a Metropolis algorithm. Verify that the equilibrium weights are indeed a left eigenvector of this matrix.

• The transition matrix for the Metropolis algorithm is

$$\begin{pmatrix} 1-x^2 & x^2/2 & x^2/2 & 0 \\ 1/2 & 0 & 0 & 1/2 \\ 1/2 & 0 & 0 & 1/2 \\ 0 & x^2/2 & x^2/2 & 1-x^2 \end{pmatrix},$$

where the states are, from top to bottom and left to right, ++, +-, -+, −, and $x = e^{-\beta K}$. The equilibrium weights (x^{-1}, x, x, x^{-1}) form a left eigenvector of the transition matrix.

(b) Repeat the above exercise if both single spin and double spin flips are allowed. The two types of moves are chosen randomly with probabilities p and $q = 1 - p$.

• Double spin flips do not change energy, and have weighted probability $q = 1 - p$. The transition matrix is now

$$\begin{pmatrix} p(1-x^2) & px^2/2 & px^2/2 & 1-p \\ p/2 & 0 & 1-p & p/2 \\ p/2 & 1-p & 0 & p/2 \\ 1-p & px^2/2 & px^2/2 & p(1-x^2) \end{pmatrix}.$$

The equilibrium weights still form a left eigenvector.

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